On the Evaluation and Reduction of Generalized Polylogarithms

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On the reduction of generalized polylogarithms to Li_n and $\mathrm{Li}_{2,2}$ and on the evaluation thereof

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Abstract: We give expressions for all generalized polylogarithms up to weight four in terms of the functions log, Li_n, and $\mathrm{Li}_{2,2}$, valid for arbitrary complex variables. Furthermore we provide algorithms for manipulation and numerical evaluation of Li_n and $\mathrm{Li}_{2,2}$, and add codes in Mathematica and C++ implementing the results. With these results we calculate a number of previously unknown integrals, which we add in App. C.

Keywords: Generalized polylogarithms, Multiple polylogarithms, Higher orders, Feynman integrals, Computer algebra.

Introduction

Generalized Polylogarithms (GPLs) are a class of mathematical functions defined recurcively as

$$G(a_1, \ldots, a_n; x) = \int_0^x \frac{\mathrm{d}t}{t - a_1} G(a_2, \ldots, a_n; t)$$

with $G(; x) \equiv 1$.

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$$G(\underbrace{0, \ldots, 0}_{n}; x) \equiv G(\bar{0}_n; x) = \frac{\log^n(x)}{n!}$$

$$G(0;x) = \log(x) \qquad G(a;x) = \log\left(1 - \tfrac{x}{a}\right) \qquad G(\bar{0}_{n-1}, a; x) = -\mathrm{Li}_n\left(\tfrac{x}{a}\right)$$

The shuffle rule: $$G(\bar{a};x)G(\bar{b};x) = \sum_{\bar{c} \in \bar{a} ⧢ \bar{b}} G(\bar{c};x)$$

Motivation (mathematical)

$$G(a_1,\ldots,a_n;x) = \int_0^x \frac{\mathrm{d}t}{t-a_1} G(a_2,\ldots,a_n;t) \qquad G(0;x) = \log(x)$$

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$$\int_0^x \frac{\log(t)\log(1-t)\log\left(1-\frac{t}{a}\right)}{t-b}\,\mathrm{d}t$$

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$$=\int_0^x \frac{G(0;t)G(1;t)G(a;t)}{t-b}\,\mathrm{d}t$$

$$=\int_0^x \frac{G(0,1,a;t)+G(0,a,1;t)+\dots}{t-b}\,\mathrm{d}t$$

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$$\begin{aligned}
&\int_0^x \frac{\log(t)\log(1-t)\log\left(1-\frac{t}{a}\right)}{t-b}\,\mathrm{d}t \\
&= \int_0^x \frac{G(0;t)G(1;t)G(a;t)}{t-b}\,\mathrm{d}t \\
&= \int_0^x \frac{G(0,1,a;t)+G(0,a,1;t)+\ldots}{t-b}\,\mathrm{d}t \\
&= G(b,0,1,a;x)+G(b,0,a,1;x)+\ldots
\end{aligned}$$

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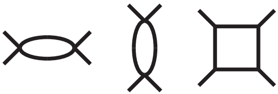
For loop-corrections to scattering amplitude calculations,
a minimal set of Feynman integrals (master integrals) are needed.

Example: The one-loop corection to mass-less $2 \to 2$ scattering
(e.g. QCD contribution to 2-jet production at the LHC)

Three master integrals: two 'bubbles', one 'box'.



$$I_{s-\text{bubble}} = \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{1}{k^2\ (k+p_1+p_2)^2}$$

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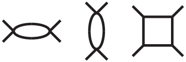

We define the vector $f = \{I_{s-\mathrm{bubble}}, I_{t-\mathrm{bubble}}, I_{s-\mathrm{box}}\}$ and $x = s/t$.
Then $\frac{df}{dx} = A(x,\epsilon)f$.

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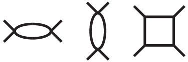

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A smart choice of f gives the 'canonical' form

$$df = \epsilon \sum_i d\log(q_i(x)) \tilde{A}_i f$$

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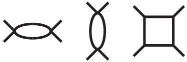

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If we expand $f = \sum \epsilon^i f^{(i)}$, and if $q_1 = x - a_1$, then

$$\frac{df^{(n)}}{dx} = \frac{1}{x-a_1} \tilde{A}_1 f^{(n-1)} + \ldots$$

Reduction

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We want to express GPLs in terms of better known objects.

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It was conjectured by Goncharov [arXiv:1110.0458] that all GPLs up to weight 4 (all that is needed at two-loop) can be expressed in terms of

$$\log(x)\ ,\quad \mathrm{Li}_2(x)\ ,\quad \mathrm{Li}_3(x)\ ,\quad \mathrm{Li}_4(x)\ ,\quad \mathrm{Li}_{2,2}(x,y)$$

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Let's show that...

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$$= G(1-a,0;1-x) - G(1-a,0;1)$$

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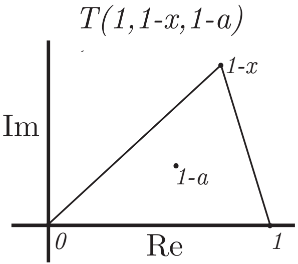

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$$= G(1-a;1-x)G(0,1-x) - G(0,1-a;1-x) - G(1-a;1)G(0,1)$$

$$+ G(0,1-a;1) + 2\pi i T(1,1-x,1-a)G(0;1-a)$$

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$$= G(1-a;1-x)G(0,1-x) - G(0,1-a;1-x) - G(1-a;1)G(0,1)$$
$$+\, G(0,1-a;1) + 2\pi i T(1,1-x,1-a)G(0;1-a)$$

$$= \log\left(\tfrac{x-a}{1-a}\right)\log(1-x) + \mathrm{Li}_2\left(\tfrac{1-x}{1-a}\right) - \mathrm{Li}_2\left(\tfrac{1}{1-a}\right) + 2\pi i T(1,1-x,1-a)\log(1-a)$$

Goal acheaved: $G(a,b;x)$ in terms of $\log$, Li_2.

Reduction

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The reduction consists of two parts:

First map a letter to zero,
then do the remaining integrals.

$$G(A_1,\ldots,A_n;X) \to G(a_1,\ldots,a_{n-1},0,1)$$
$$\to \log, \mathrm{Li}_n, \mathrm{Li}_{2,2} \text{ at weights } \leq 4$$

The first step can be done at all weights,
The second gets much harder for each weight.

Reduction

$$
\begin{aligned}
G(a,b,c,x) \;=\;& \tfrac{1}{6}\log\left(\tfrac{c-b}{a}\right)^3 - \tfrac{1}{6}\log\left(\tfrac{(a-c)(c-b)}{(a-b)c}\right)^3 - \tfrac{1}{6}\log\left(\tfrac{c-b}{a-x}\right)^3 + \tfrac{1}{6}\log\left(\tfrac{(a-c)(c-b)}{(a-b)(c-x)}\right)^3 - \tfrac{1}{2}\log\left(\tfrac{a}{a-b}\right)\log\left(\tfrac{b}{b-c}\right)^2 + \mathrm{Li}_2\left(\tfrac{x-b}{a-b}\right)\log\left(\tfrac{b-x}{b-c}\right) \\
& -\tfrac{1}{2}\log\left(1-\tfrac{b}{a}\right)\log\left(\tfrac{b}{b-c}\right)^2 + \tfrac{1}{2}\log\left(\tfrac{a-b}{a-x}\right)\log\left(\tfrac{b-x}{b-c}\right)^2 + \tfrac{1}{2}\log\left(\tfrac{a-x}{a-b}\right)\log\left(\tfrac{b-x}{b-c}\right)^2 + \mathrm{Li}_3\left(\tfrac{b}{a}\right) + \mathrm{Li}_3\left(\tfrac{b}{b-a}\right) - \mathrm{Li}_3\left(\tfrac{b(a-c)}{a(b-c)}\right) - \mathrm{Li}_3\left(\tfrac{b(a-c)}{(a-b)c}\right) \\
& -\mathrm{Li}_3\left(-\tfrac{c}{b-c}\right) + \mathrm{Li}_3\left(1-\tfrac{c}{a}\right) - \mathrm{Li}_3\left(\tfrac{a-c}{a-x}\right) - \mathrm{Li}_3\left(\tfrac{b-x}{a-x}\right) + \mathrm{Li}_3\left(\tfrac{(a-c)(b-x)}{(b-c)(a-x)}\right) + \mathrm{Li}_3\left(\tfrac{(a-c)(b-x)}{(a-b)(c-x)}\right) - \mathrm{Li}_3\left(\tfrac{x-b}{a-b}\right) - \mathrm{Li}_3\left(\tfrac{x-c}{a-c}\right) + \mathrm{Li}_3\left(\tfrac{x-c}{b-c}\right) \\
& -\mathrm{Li}_2\left(\tfrac{a-c}{b-c}\right)\log\left(\tfrac{a}{a-c}\right) + \mathrm{Li}_2\left(\tfrac{-c}{b-c}\right)\log\left(\tfrac{a}{a-c}\right) - \mathrm{Li}_2\left(\tfrac{b}{a}\right)\log\left(\tfrac{b}{b-c}\right) - \mathrm{Li}_2\left(\tfrac{b}{b-a}\right)\log\left(\tfrac{b}{b-c}\right) + \mathrm{Li}_2\left(\tfrac{b(a-c)}{a(b-c)}\right)\log\left(\tfrac{b}{b-c}\right) + \mathrm{Li}_2\left(\tfrac{b(a-c)}{(a-b)c}\right)\log\left(\tfrac{b}{b-c}\right) \\
& +\tfrac{1}{2}\log\left(\tfrac{b}{b-c}\right)^2\log\left(\tfrac{(b-a)c}{a(b-c)}\right) + \tfrac{1}{2}\log\left(\tfrac{b}{b-c}\right)^2\log\left(\tfrac{a(c-b)}{(a-b)c}\right) + \mathrm{Li}_2\left(\tfrac{a-c}{b-c}\right)\log\left(\tfrac{a-x}{a-c}\right) - \mathrm{Li}_2\left(-\tfrac{c}{b-c}\right)\log\left(\tfrac{a-x}{a-c}\right) + \mathrm{Li}_2\left(\tfrac{b-x}{a-x}\right)\log\left(\tfrac{b-x}{b-c}\right) \\
& -\mathrm{Li}_2\left(\tfrac{(a-c)(b-x)}{(b-c)(a-x)}\right)\log\left(\tfrac{b-x}{b-c}\right) - \mathrm{Li}_2\left(\tfrac{(a-c)(b-x)}{(a-b)(c-x)}\right)\log\left(\tfrac{b-x}{b-c}\right) - \tfrac{1}{2}\log\left(\tfrac{b-x}{b-c}\right)^2\log\left(-\tfrac{(b-c)(a-x)}{(a-b)(c-x)}\right) - \tfrac{1}{2}\log\left(\tfrac{b-x}{b-c}\right)^2\log\left(\tfrac{(b-a)(c-x)}{(b-c)(a-x)}\right) + \mathrm{Li}_3\left(-\tfrac{c}{a-c}\right) \\
& -\mathrm{Li}_2\left(\tfrac{c-b}{a-b}\right)\log\left(1-\tfrac{x}{c}\right) + \mathrm{Li}_2\left(\tfrac{x-b}{a-b}\right)\log\left(1-\tfrac{x}{c}\right) + \log\left(\tfrac{a-x}{a-b}\right)\log\left(\tfrac{b-x}{b-c}\right)\log\left(1-\tfrac{x}{c}\right) - \pi^2\Big(-\tfrac{1}{6}\log\left(\tfrac{c-b}{a}\right) + \tfrac{1}{6}\log\left(\tfrac{(a-c)(c-b)}{(a-b)c}\right) + \tfrac{1}{6}\log\left(\tfrac{c-b}{a-x}\right) \\
& -\tfrac{1}{6}\log\left(\tfrac{(a-c)(c-b)}{(a-b)(c-x)}\right) + 4\log\left(1-\tfrac{b}{c}\right)\mathsf{T}\left(1,1-\tfrac{x}{c},1-\tfrac{b}{c}\right)\mathsf{T}\left(\mathsf{P}\left(\tfrac{b}{c},1-\tfrac{x}{c}\right),1-\tfrac{x}{c},1-\tfrac{a}{c}\right)\Big) + i\pi\Big(\mathsf{T}\left(1,\tfrac{b}{b-c},\tfrac{a}{a-c}\right)\mathsf{sgn}\left(\mathsf{Im}\left(\tfrac{c}{a-c}\right)\right)\log\left(\tfrac{a}{a-c}\right)^2 \\
& +\log\left(\tfrac{b-a}{b-c}\right)^2\mathsf{T}\left(1,\tfrac{b}{b-c},\tfrac{b-a}{b-c}\right)\mathsf{sgn}\left(\mathsf{Im}\left(\tfrac{a-c}{b-c}\right)\right) - \log\left(\tfrac{b-a}{b-c}\right)^2\mathsf{T}\left(1,\tfrac{b-x}{b-c},\tfrac{b-a}{b-c}\right)\mathsf{sgn}\left(\mathsf{Im}\left(\tfrac{a-c}{b-c}\right)\right) - 2\mathrm{Li}_2\left(\tfrac{a-c}{b-c}\right)\mathsf{T}\left(1,1-\tfrac{x}{c},1-\tfrac{a}{c}\right)\mathsf{sgn}\left(\mathsf{Im}\left(\tfrac{a}{c}\right)\right) \\
& +2\mathrm{Li}_2\left(-\tfrac{c}{b-c}\right)\mathsf{T}\left(1,1-\tfrac{x}{c},1-\tfrac{a}{c}\right)\mathsf{sgn}\left(\mathsf{Im}\left(\tfrac{a}{c}\right)\right) - 2\log\left(1-\tfrac{a}{c}\right)\log\left(\tfrac{b-a}{b-c}\right)\mathsf{T}\left(1,1-\tfrac{x}{c},1-\tfrac{a}{c}\right)\mathsf{sgn}\left(\mathsf{Im}\left(\tfrac{a}{c}\right)\right) \\
& -\mathcal{H}_1\left(1-\tfrac{a}{c},1-\tfrac{b}{c}\right)\log\left(\tfrac{(b-a)c}{(a-c)(c-b)}\right)^2\mathsf{sgn}\left(\mathsf{Im}\left(\tfrac{b}{c}\right)\right) + 2\log\left(1-\tfrac{b}{c}\right)\log\left(\tfrac{a-b}{a-c}\right)\mathsf{T}\left(1,1-\tfrac{x}{c},1-\tfrac{b}{c}\right)\mathsf{sgn}\left(\mathsf{Im}\left(\tfrac{b}{c}\right)\right) \\
& -2\log\left(1-\tfrac{b}{c}\right)\log\left(\tfrac{a-x}{a-c}\right)\mathsf{T}\left(1,1-\tfrac{x}{c},1-\tfrac{b}{c}\right)\mathsf{sgn}\left(\mathsf{Im}\left(\tfrac{b}{c}\right)\right) + \mathcal{H}_1\left(\tfrac{b-c}{a-c},\tfrac{c-b}{c-x}\right)\log\left(\tfrac{a-x}{b-c}\right)^2\mathsf{sgn}\left(\mathsf{Im}\left(\tfrac{c-b}{c-x}\right)\right) \\
& -\mathcal{H}_1\left(\tfrac{c-a}{c-x},\tfrac{c-b}{c-x}\right)\log\left(-\tfrac{(a-b)(c-x)}{(a-c)(c-b)}\right)^2\mathsf{sgn}\left(\mathsf{Im}\left(\tfrac{c-b}{c-x}\right)\right) + 2\log\left(\tfrac{b-a}{b-c}\right)\log\left(1-\tfrac{x}{c}\right)\mathsf{T}\left(1,\tfrac{b-x}{b-c},\tfrac{b-a}{b-c}\right)\mathsf{sgn}\left(\mathsf{Im}\left(\tfrac{c-b}{c-x}\right)\right) \\
& +\log\left(\tfrac{a-x}{a-c}\right)^2\mathsf{T}\left(1,\tfrac{b-x}{b-c},\tfrac{a-x}{a-c}\right)\mathsf{sgn}\left(\mathsf{Im}\left(\tfrac{x-c}{a-c}\right)\right) + \mathcal{H}_1\left(\tfrac{b-c}{a-c},1-\tfrac{b}{c}\right)\log\left(\tfrac{a}{b-c}\right)^2\mathsf{sgn}\left(\mathsf{Im}\left(\tfrac{b}{c}\right)\right)\Big)\,.
\end{aligned}
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Evaluation

We need to evaluate $\mathrm{Li}_n(x)$ and $\mathrm{Li}_{2,2}(x)$

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Outside we may use the inversion relation $\mathrm{Li}_n(x) = \pm\mathrm{Li}_n(\frac{1}{x}) + \mathrm{easier}$.

$$\mathrm{Li}_3(x) = \mathrm{Li}_3\left(\frac{1}{x}\right) - \frac{1}{6}\log^3(-x) - \frac{\pi^2}{6}\log(-x)$$

(There are better ways to evaluate $\mathrm{Li}_n(x)$).

Evaluation

$\mathrm{Li}_{2,2}(x,y)$ is given by a similarly looking sum

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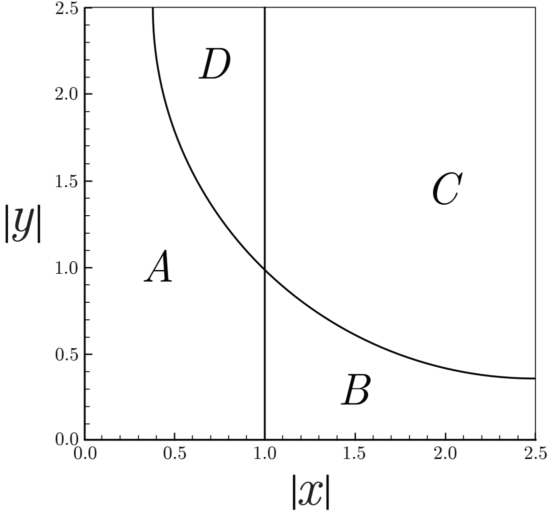
If $|xy| > 1$ we may use the inversion relation

$$\begin{aligned}\mathrm{Li}_{2,2}(x,y) = {} & \mathrm{Li}_{2,2}\left(\tfrac{1}{x},\tfrac{1}{y}\right) - \mathrm{Li}_4(xy) + 3\Big(\mathrm{Li}_4\left(\tfrac{1}{x}\right) + \mathrm{Li}_4(y)\Big) \\ & + 2\Big(\mathrm{Li}_3\left(\tfrac{1}{x}\right) - \mathrm{Li}_3(y)\Big)\log(-xy) + \mathrm{Li}_2\left(\tfrac{1}{x}\right)\left(\tfrac{\pi^2}{6} + \tfrac{1}{2}\log^2(-xy)\right) \\ & + \tfrac{1}{2}\mathrm{Li}_2(y)\Big(\log^2(-xy) - \log^2(-x)\Big)\end{aligned}$$

And then if $|x| > 1$ we may use the 'stuffle relation'

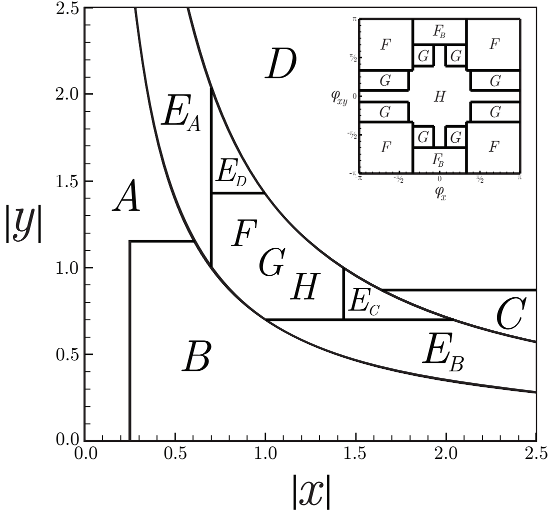
$$\mathrm{Li}_{2,2}(x,y) = -\mathrm{Li}_{2,2}(y,x) - \mathrm{Li}_4(xy) + \mathrm{Li}_2(x)\mathrm{Li}_2(y)$$

Evaluation



- A: Direct summation
- B: Stuffle: $x \leftrightarrow y$
- C: Inversion: $z \leftrightarrow z^{-1}$
- D: Stuffle and inversion

Evaluation



A: Direct summation

B: Stuffle: $x \leftrightarrow y$

C: Inversion: $z \leftrightarrow z^{-1}$

D: Stuffle and inversion

E: Diagonal sum

F: Hölder relation

G: $\log(1-z)$ expansion

H: $\log(z)$ expansion

The added code

With the paper [ArXiv:1601.02649] we have added code implemeting the results.

The reductions are implemented in Mathematica as the replacement rule gtolrules.

```
In[1]:= G(0,a,0,b,x)/.gtolrules
Out[1]= MyLi22(x/a,a/b)
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Please try our code, and send us any critique/bug reports.

Discussion

The goal of reducing all GPLs up to weight four to

$$\log(x)\ ,\ \ \mathrm{Li}_2(x)\ ,\ \ \mathrm{Li}_3(x)\ ,\ \ \mathrm{Li}_4(x)\ ,\ \ \mathrm{Li}_{2,2}(x,y)$$

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Thank you for listening...