Next-to-eikonal/soft corrections in QCD

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in collaboration with

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[arXiv 1410.6406, arXiv:1503.05156]

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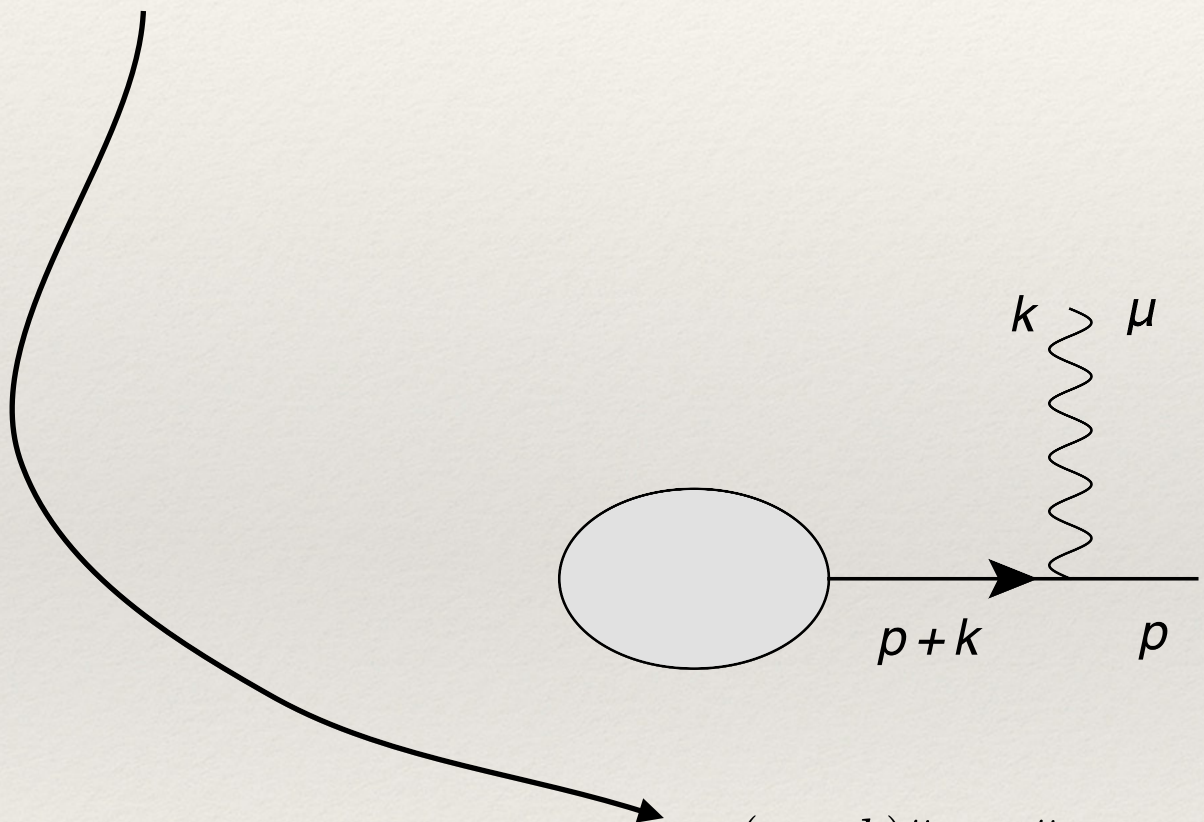
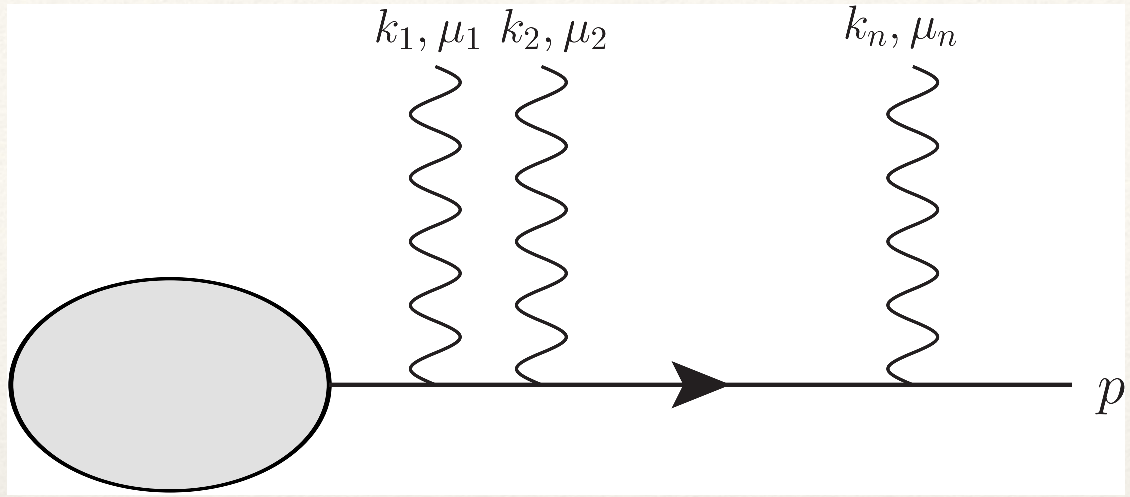
Basics of eikonal approximation: QED

- Charged particle emits soft photon
  - Propagator: expand numerator & denominator in soft momentum, keep lowest order
  - Vertex: expand in soft momentum, keep lowest order



$$\frac{(p+k)^\mu + p^\mu}{2p\cdot k + k^2} \longrightarrow \frac{2p^\mu}{2p\cdot k}$$

Basics of eikonal approximation in QED



Exact: $$\frac{1}{(p+K_1)^2}(2p+K_2+K_1)^{\mu_1}\dots\frac{1}{(p+K_n)^2}(2p+K_n)^{\mu_n}, \quad K_i=\sum_{m=i}^{n}k_m.$$

Approx: $$\frac{1}{2pK_1}2p^{\mu_1}\dots\frac{1}{2pK_n}2p^{\mu_n}$$

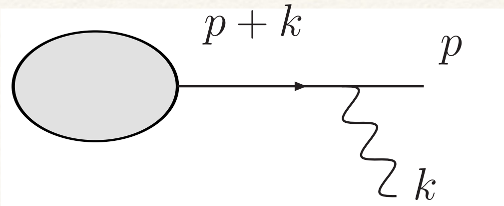
Eikonal identity: $$\frac{1}{p\cdot(k_1+k_2)\,p\cdot k_2}+\frac{1}{p\cdot(k_1+k_2)\,p\cdot k_1}=\frac{1}{p\cdot k_1\,p\cdot k_2}$$

Sum over all perm's: $$\prod_i \frac{p^{\mu_i}}{p\cdot k_i}.$$

Independent, uncorrelated emissions, Poisson process

Eikonal approximation: no dependence on emitter spin

- Emitter spin becomes irrelevant in eikonal approximation
  - Fermion



$$M \frac{i(\not{p}+\not{k})}{(p+k)^2}(-ig_s\gamma^\mu)u(p)$$

  - Approximate, and use Dirac equation $\not{p}u(p) = 0$
  - Result same as scalar case

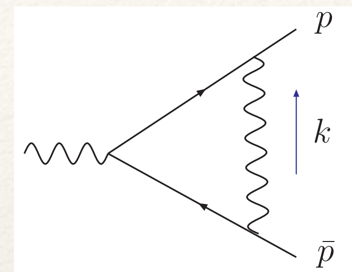
$$g\,(M\,u(p)) \times \frac{p^\mu}{p\cdot k}$$

  - Two things have happened
    - ✓ No sign of emitter spin anymore
    - ✓ Coupling of photon proportional to p^μ

Eikonal exponentiation

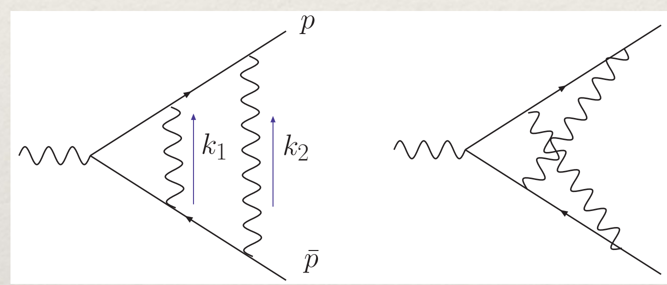
- In the eikonal approximation, interesting patterns emerge

One loop vertex correction, in eikonal approximation



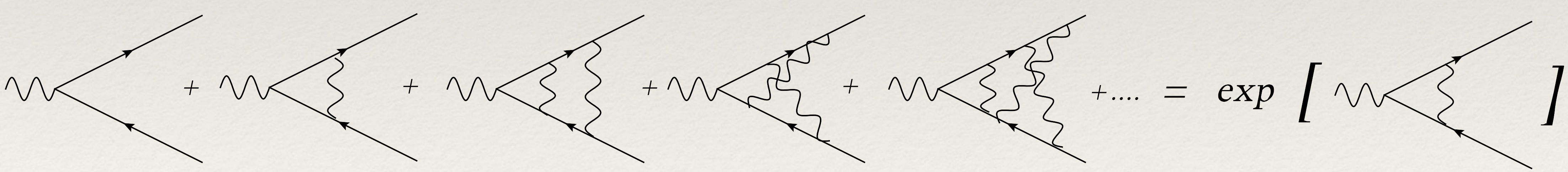
$$\mathcal{A}_0 \int d^n k \frac{1}{k^2} \frac{p \cdot \bar{p}}{(p \cdot k)(\bar{p} \cdot k)}$$

Two loop vertex correction, in eikonal approximation



$$\mathcal{A}_0 \frac{1}{2} \left(\int d^n k \frac{1}{k^2} \frac{p \cdot \bar{p}}{(p \cdot k)(\bar{p} \cdot k)} \right)^2$$

Exponential series!



Yennie, Frautschi, Suura

Exponentiation using path integrals

EL, Stavenga, White

Textbook result

$$\text{Sum of all diagrams} = \exp\Big(\text{Connected diagrams}\Big)$$

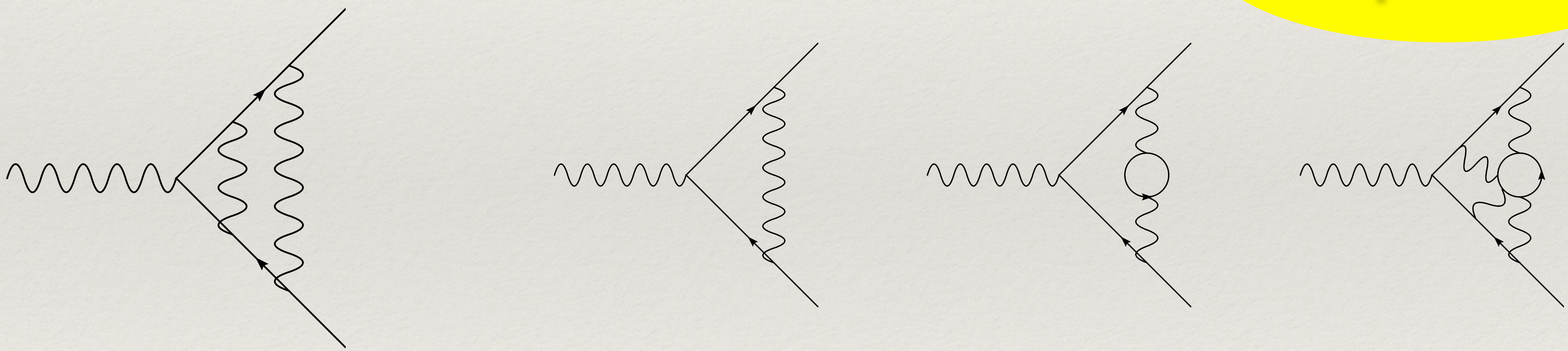
$$f = e^{i\int dt(\frac{1}{2}\dot{x}^2 + p\cdot A + ..)}$$

Can write scattering amplitude as nested path integral

$$M(p_1, p_2, \{k\}) = \int \mathcal{D}A_s\, \mathcal{D}x(t)\, H[x]\, f_1[A_s, x(t)]\, f_2[A_s, x(t)]\, e^{iS[A_s]}$$

x(t): path of charged particle

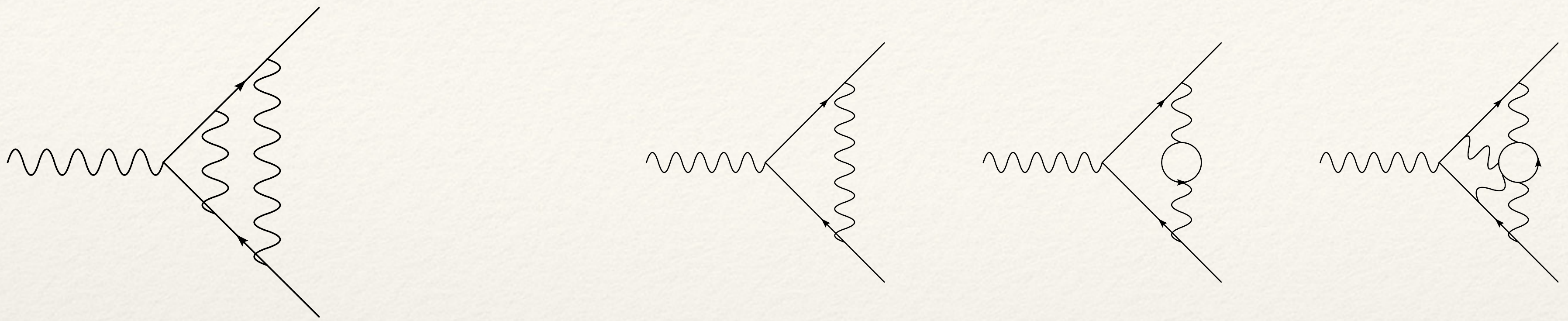
Eikonal vertices: sources for gauge bosons living on lines



Disconnected

Connected

Path integral method, non-abelian



EL, Stavenga, White

- Not immediately obvious how this could work (the path integral must be an actual exponential), since
    - Source terms have non-abelian charges, so don't commute
    - External line factors are path-ordered exponentials
    - Nevertheless

$$\sum_D \mathcal{F}_D C_D = \exp\left[\sum_i \bar{C}_i w_i\right]$$

Gatheral; Frenkel, Taylor; Sterman

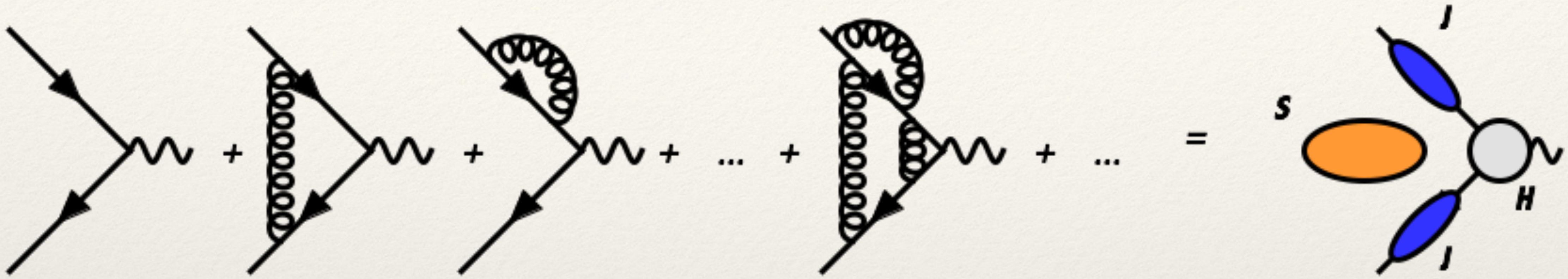
Modified color factors

Webs

- Proof uses replica trick from statistical physics

More than eikonal: resummation for quark form factor

- Consider all corrections to the quark form factor



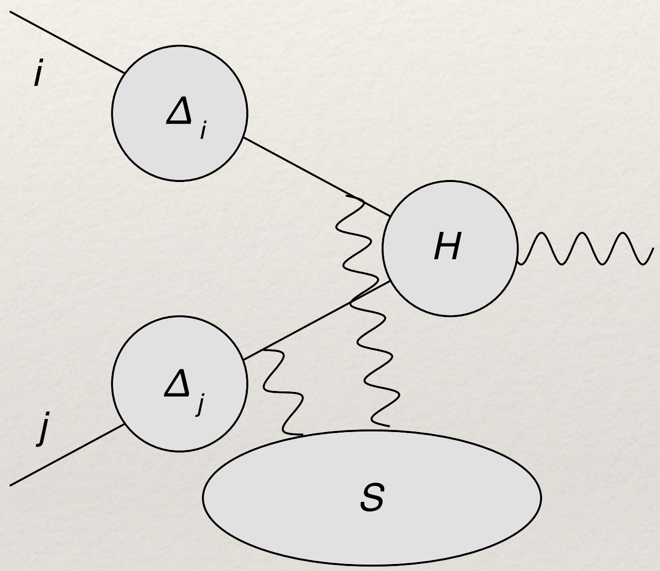
  - a diagrammatic analysis shows that it factorizes into a product of functions:
    - ✓ A soft function "S" (only IR/eikonal modes of loop momenta)
    - ✓ 2 jets functions "J" (collinear modes)
    - ✓ A hard functions "H" (off-shell, hard modes)
- These are also all the virtual diagrams for the Drell-Yan process
- This factorization also implies a resummation

A. Sen; Collins; Magnea, Sterman

Factorization and resummation for Drell-Yan

$$\sigma(N) = \Delta(N, \mu, \xi_1)\Delta(N, \mu, \xi_2)S(N, \mu, \xi_1, \xi_2)H(\mu)$$

- Now with Mellin moment "N" dependence (i.e., with radiation)
- Near threshold, cross section is equivalent to product of 4 well-defined functions
- Demand independence of
  - renormalization scale μ
  - gauge dependence parameter ξ
    - ✓ find exponent of double logarithm



$$0 = \mu\frac{d}{d\mu}\sigma(N) = \xi_1\frac{d}{d\xi_1}\sigma(N) = \xi_2\frac{d}{d\xi_2}\sigma(N)$$

Contopanagos, EL, Sterman
Forte, Ridolfi

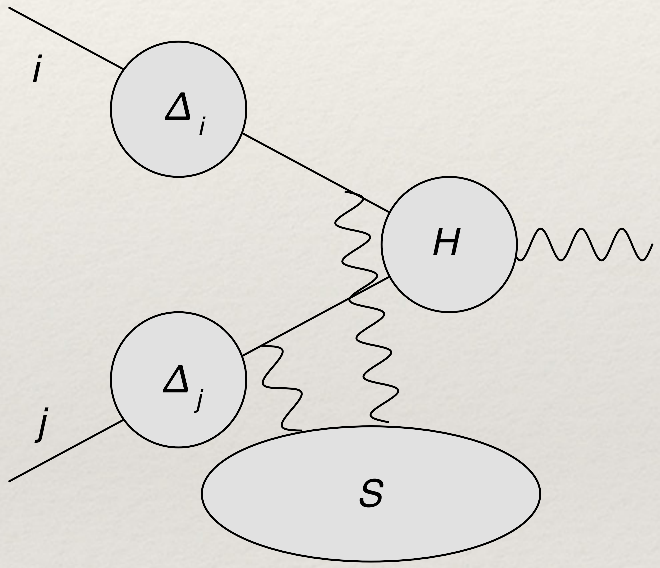
$$\Delta = \exp[\int\frac{d\mu}{\mu}\int\frac{d\xi}{\xi}..]$$

Factorization and threshold resummation

- $\Delta_i(N)$: initial state soft+collinear radiation effects
  - real+virtual
  - $\alpha_s^n \ln^{2n} N$

$$\sigma(N) = \sum_{ij} \phi_i(N)\phi_j(N) \times \underbrace{\left[\Delta_i(N)\Delta_j(N)S_{ij}(N)\, H_{ij}\right]}_{\hat{\sigma}_{ij}(N)}$$

- $S_{ij}(N)$: soft, non-collinear radiation effects
  - $\alpha_s^n \ln^n N$
- H: hard function, no soft and collinear effects



$$\Delta_i(N) = \exp\left[\ln N \frac{C_F}{2\pi b_0 \lambda}\{2\lambda + (1-2\lambda)\ln(1-2\lambda)\} + ..\right]$$
$$= \exp\left[\frac{2\alpha_s C_F}{\pi}\ln^2 N + ..\right]$$

Soft Collinear Effective Theory

Bauer, Fleming, Pirjol, Stewart,...

Becher, Neubert

- Previous "(d)QCD" analysis was diagram based
- Effective field theory approach: SCET
  - Distinguish separate **fields** for soft, collinear, hard partons, and ultrasoft gluons

$$\mathcal{L}_{SCET,qq} = \bar{\xi}_n (in \cdot D + i\not{D}_{c,\perp} \frac{1}{i\bar{n} \cdot D_c} i\not{D}_{c,\perp}) \frac{\not{\bar{n}}}{2} \xi_n - \frac{1}{4} \mathrm{Tr}\{G^c_{\mu\nu} G^{c,\mu\nu}\}$$

  - ✓ Powerful power counting. Using +,-,T notation

$$p_h \sim Q(1,1,1) \quad p_c \sim Q(\lambda, 1, \sqrt{\lambda}) \quad p_s \sim Q(\lambda, \lambda, \lambda)$$

  - ✓ Fields scale similarly:

$$\xi_n \sim \lambda \quad \xi_{\bar{n}} \sim \lambda^2 \quad A_s \sim \lambda \quad \bar{n} \cdot A_c \sim \lambda^0$$

- Resummation via renormalization group

Generic large x behavior

- For DY, DIS, Higgs, singular behavior when x→ 1

$$\delta(1-x) \qquad \left[\frac{\ln^i(1-x)}{1-x}\right]_+ \qquad \ln^i(1-x)$$

  - singularity structure for plus distributions is organizable to all orders, perhaps also for divergent logarithms?

- After Mellin transform $\quad \text{Constants} \qquad \ln^i(N) \qquad \dfrac{\ln^k(N)}{N}$

- We know a lot about logs and constants, very little about 1/N
- Can we learn about such “next-to-eikonal/soft” corrections?
- “Zurich” method of regions allows computation (for NNNLO Higgs production)

$$(1-x)^p \ln^q(1-x)$$

  - ✓ at least to p=37

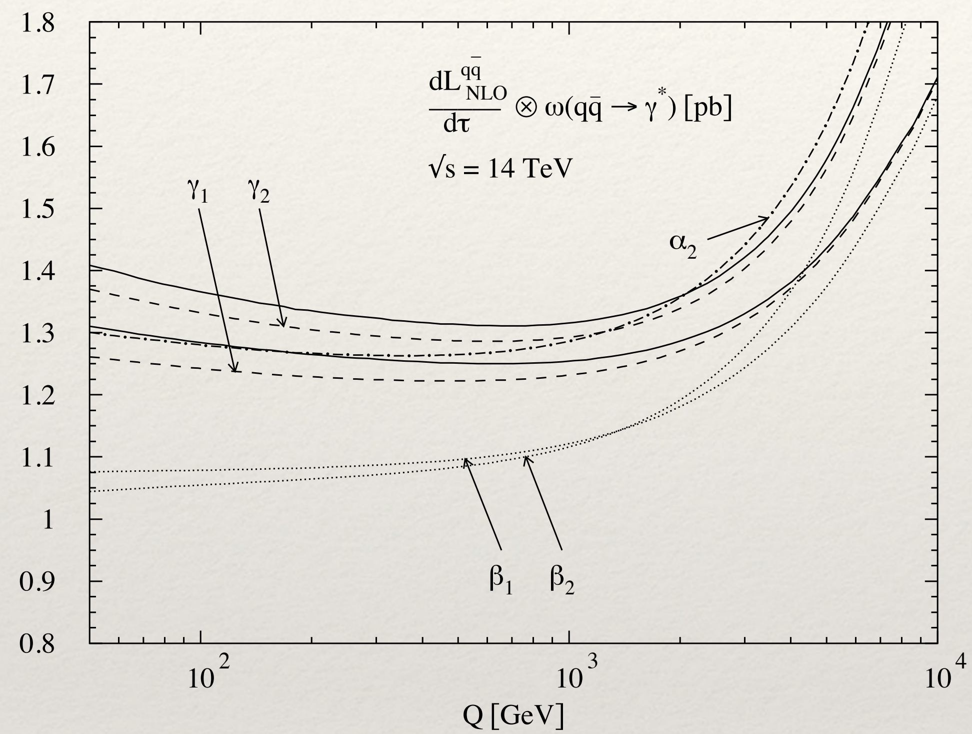
Anasthasiou, Duhr, Dulat, Furlan, Gehrmann, Herzog, Mistlberger

ln(N)/N terms

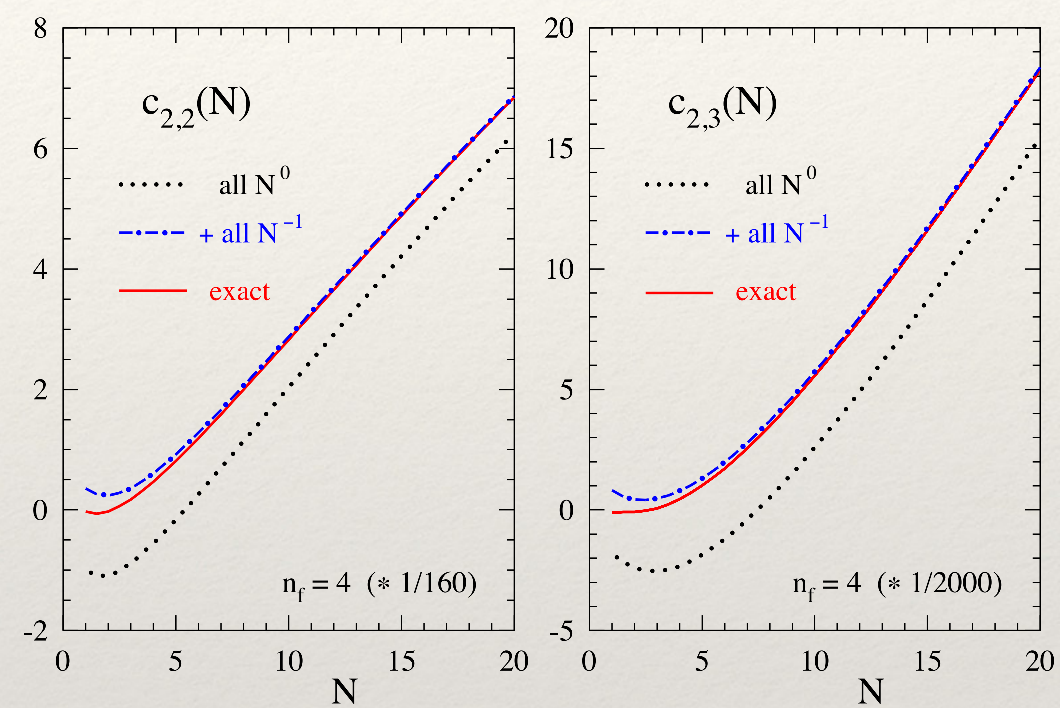
- Can be numerically important

Kraemer, EL, Spira; Catani, De Florian, Grazzini; Kilgore, Harlander

Kraemer, EL, Spira



Moch,Vogt



- We know that the leading series $\ln^i(N)/N$ exponentiates
  - by replacing in resummation formula

$$\exp\left[\int_0^1 dz\,(z^{N-1}-1)\frac{1+z^2}{1-z}\int_{\mu_F}^{Q(1-z)}\dots\right]$$

$$\frac{1+z^2}{1-z} \longrightarrow \frac{2}{1-z} - 2$$

Extended Drell-Yan threshold resummation

EL, Magnea, Stavenga   Gruenberg

Ball, Bonvini, Forte, Marzani, Ridolfi

Ansatz: modified resummed expression

$$\ln\Big[\sigma(N)\Big] = \mathcal{F}_{\rm DY}\left(\alpha_s(Q^2)\right) + \int_0^1 dz\, z^{N-1}\left\{\frac{1}{1-z} D\left[\alpha_s\left(\frac{(1-z)^2 Q^2}{z}\right)\right]\right.$$
$$\left. + 2\int_{Q^2}^{(1-z)^2 Q^2/z} \frac{dq^2}{q^2} P_s\Big[z, \alpha_s(q^2)\Big]\right\}_+$$

where

$$P_s^{(n)}(z) = \frac{z}{1-z} A^{(n)} + C_\gamma^{(n)} \ln(1-z) + \overline{D}_\gamma^{(n)}$$

(We constructed a similar expression for DIS). Structure:

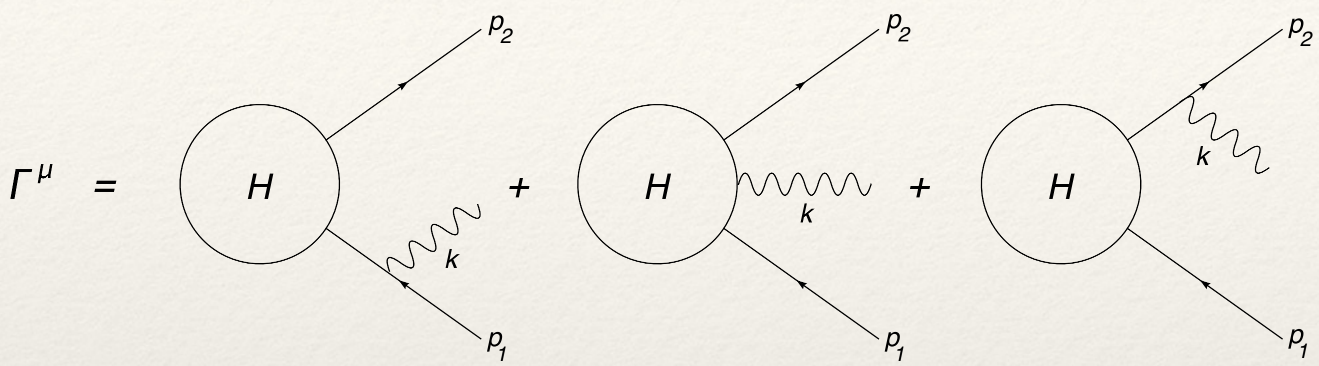
$$\sigma(N) = \sum_{n=0}^{\infty} \left(g^2\right)^n \left[\sum_{m=0}^{2n} a_{nm} \ln^m N + \sum_{m=0}^{2n-1} b_{nm} \frac{\ln^m N}{N}\right] + \mathcal{O}\left(N^{-2}\right)$$

	C_F^2		$C_A C_F$		$n_f C_F$	
b_{23}	4	4	0	0	0	0
b_{22}	$\frac{7}{2}$	4	$\frac{11}{6}$	$\frac{11}{6}$	$-\frac{1}{3}$	$-\frac{1}{3}$
b_{21}	$8\zeta_2 - \frac{43}{4}$	$8\zeta_2 - 11$	$-\zeta_2 + \frac{239}{36}$	$-\zeta_2 + \frac{133}{18}$	$-\frac{11}{9}$	$-\frac{11}{9}$
b_{20}	$-\frac{1}{2}\zeta_2 - \frac{3}{4}$	$4\zeta_2$	$-\frac{7}{4}\zeta_3 + \frac{275}{216}$	$\frac{7}{4}\zeta_3 + \frac{11}{3}\zeta_2 - \frac{101}{54}$	$-\frac{19}{27}$	$-\frac{2}{3}\zeta_2 + \frac{7}{27}$

Close, but no cigar..

Classic result: Low's theorem

- So far we only looked at emissions from external lines. At next-to-eikonal/soft order, also 1 "internal" emission contributes



- Low's theorem (scalars, generalization to spinors by Burnett-Kroll, to massless particles by Del Duca → LBKD theorem)
    - ✓ Work to order k, and use Ward identity

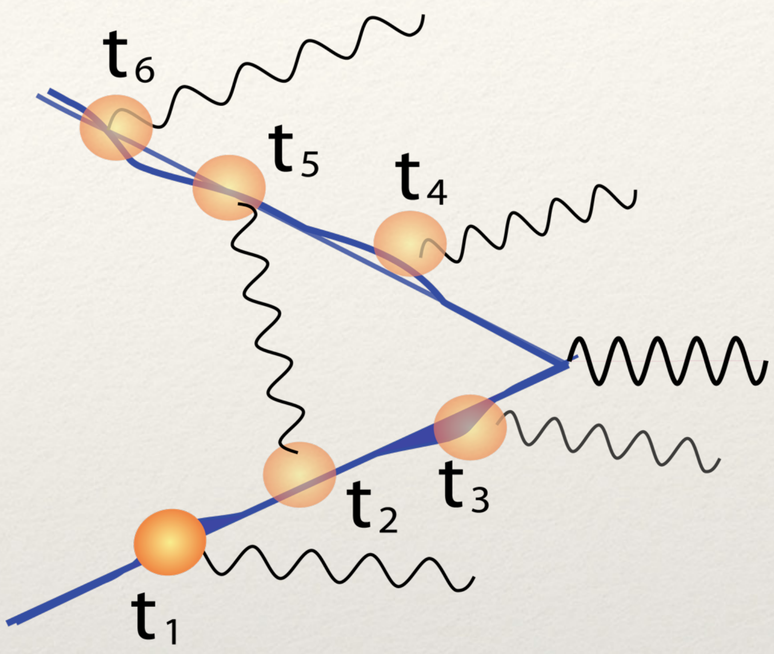
$$\Gamma^\mu = \left[\frac{(2p_1 - k)^\mu}{-2p_1\cdot k} + \frac{(2p_2 + k)^\mu}{2p_2\cdot k}\right]\Gamma + \left[\frac{p_1^\mu(k\cdot p_2 - k\cdot p_1)}{p_1\cdot k} + \frac{p_2^\mu(k\cdot p_1 - k\cdot p_2)}{p_2\cdot k}\right]\frac{\partial\Gamma}{\partial p_1\cdot p_2}$$

- Non-emitting amplitude determines the emission to NE accuracy,
    - with its derivative
    - but no detailed knowledge of internals needed

Next-to-eikonal exponentiation via path integral

EL, Magnea, Stavenga, White

- Wilson lines are classical solutions of path integral
- Fluctuations around classical path are NE corrections
    - All NE corrections from external lines exponentiate
    - Keep track via scaling variable λ $\quad p^\mu = \lambda n^\mu$

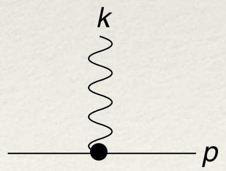


$$f(\infty) = \int_{x(0)=0} \mathcal{D}x \, \exp\left[i \int_0^\infty dt \left(\frac{\lambda}{2}\dot{x}^2 + (n+\dot{x})\cdot A(x_i + nt + x) + \frac{i}{2\lambda}\partial \cdot A(x_i + p_f t + x) \right) \right]$$

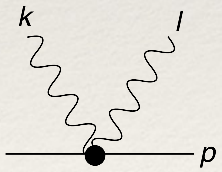
Use 1-D field theory propagator

$$\langle x(t)x(t')\rangle = G(t,t') = \frac{i}{\lambda}\min(t,t')$$

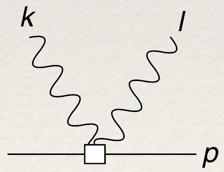
NE Feynman rules



$$\frac{k^\mu}{2p\cdot k} - k^2 \frac{p^\mu}{2(p\cdot k)^2}$$



$$+\frac{\eta^{\mu\nu}}{p\cdot(k+l)}$$

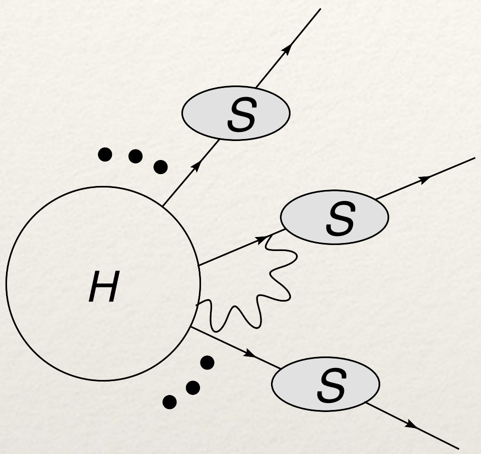


$$-\frac{l^\mu p^\nu p\cdot k + k^\nu p^\mu p\cdot l}{p\cdot(k+l)p\cdot k p\cdot l}$$

Low-Burnett-Kroll and path integral

Path integral method provides elegant way to derive Low's theorem

$$S(p_1,\ldots,p_n) = \int \mathcal{D}A_s H(x_1,\ldots,x_n;A_s) e^{-ip_1x_1} f(x_1,p_1;A_s)\ldots e^{-ip_nx_n} f(x_1,p_1;A_s) e^{iS[A_s]}$$



Gauge transformation must cancel between f's and H

$$f(x_i,p_f;A) \to f(x_i,p_f;A+\partial\Lambda) = e^{-iq\Lambda(x_i)} f(x_i,p_f;A)$$

Opposite transformation in H, expand to first order in A and Λ

Low contribution is then:

$$S(p_1,\ldots,p_n) = \int \mathcal{D}A \Big[\int \frac{d^dk}{(2\pi)^d} \sum_j^n q_j \Big(\frac{n_j^\mu}{n_j\cdot k} k_\nu \frac{\partial}{\partial p_{j_\nu}} - \frac{\partial}{\partial p_{j_\mu}} \Big) H(p_1,\ldots,p_n) A_\mu(k) \Big]$$
$$\times f(0,p_1;A)\ldots f(0,p_n;A)$$

First term is due to displacement of f(x,p,A)

Missing: careful treatment of collinear radiation. Back to basics

Next-to-eikonal corrections

- Keep 1 term more in k expansion beyond eikonal approximation

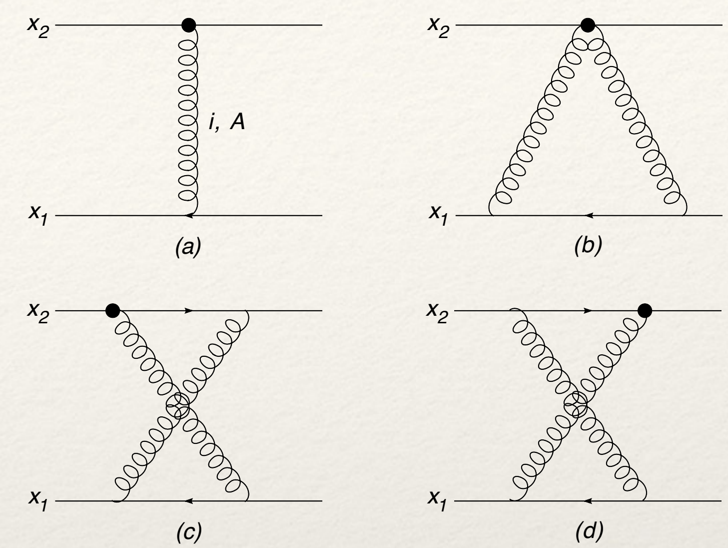
$$\text{scalar}: \quad \frac{2p^\mu + k^\mu}{2p\cdot k + k^2} \longrightarrow \frac{2p^\mu}{2p\cdot k} + \frac{k^\mu}{2p\cdot k} - k^2 \frac{2p^\mu}{(2p\cdot k)^2}$$

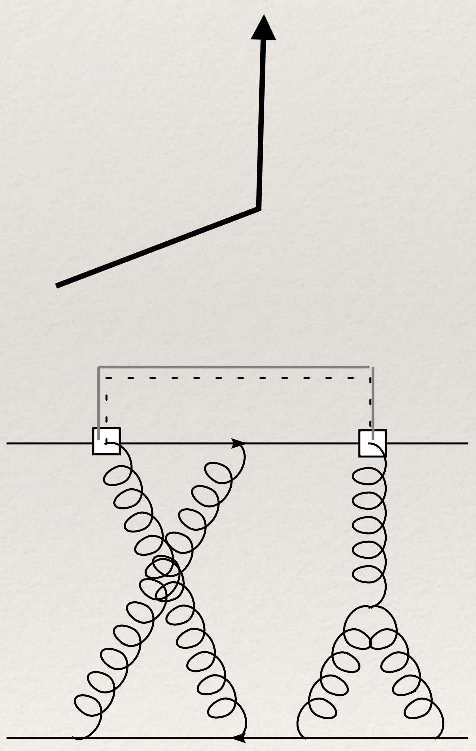
$$\text{fermion}: \quad \frac{\not{p} + \not{k}}{2p\cdot k + k^2}\gamma^\mu u(p) \longrightarrow \left[\frac{2p^\mu}{2p\cdot k} + \frac{\not{k}\gamma^\mu}{2p\cdot k} - k^2 \frac{2p^\mu}{(2p\cdot k)^2}\right] u(p)$$

  - Now emitter-spin dependent, and has recoil
  - Decorrelation not obvious
  - Can we still make systematic statements (exponentiation, factorization) about next-to-eikonal/soft corrections?

Next-to-eikonal diagrammar

- As for eikonal case earlier
  - identify next-to-eikonal vertices
  - show that they "decorrelate"
    - ✓ as eikonal webs (2 eik. line irreducible), but now with new vertices
    - ✓ they become spin-sensitive
- New 2-gluon correlations between eikonal webs → NE webs





Exponentiation for NE corrections

- Upshot: one can define NE webs, using such NE Feynman rules.

$$\sum C(D)\mathcal{F}(D) = \exp\left[\bar{C}(D)W_{\rm E}(D) + \bar{C}'(D)W_{\rm NE}(D)\right]$$

- They exponentiate too, no new proof needed
  - but they are not the *only* source of next-to-soft corrections

Next-to-eikonal logarithms

Vernazza, Bonocore, EL, Magnea, Melville, White

- Our approach: understand NE corrections at amplitude level, then construct cross section
- Use Drell-Yan as testbed
- Goal: combine NE matrix elements with phase space to predict NE (=NLP) logs for NNLO Drell-Yan
  - Leading power done

$$\log^3(1-z)$$

  - Next-to-leading powers?

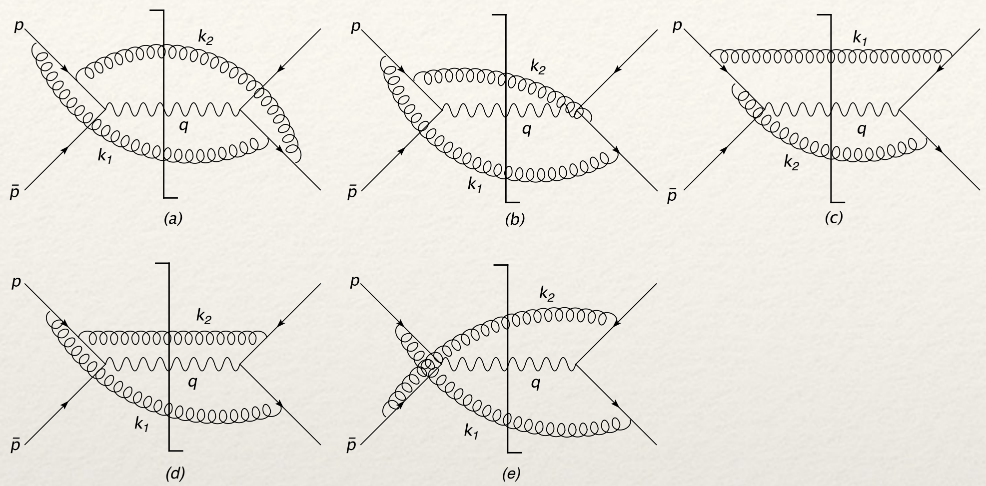
$$\log^i(1-z), \quad i = 2, 1, 0$$

    - ✓ They come from double real emission, and one-real + one-virtual

NE logs in DY: double real

EL, Magnea, Stavenga, White

- Check NE Feynman rules for NNLO Drell-Yan *double real* emission (only C_F^2 terms)



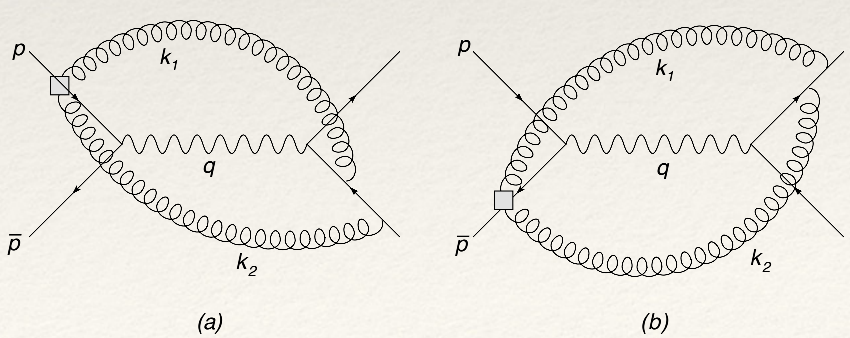
- Result at NE level (agrees with equivalent exact result)

$$K^{(2)}_{\rm NE}(z) = \left(\frac{\alpha_s}{4\pi}C_F\right)^2 \left[-\frac{32}{\epsilon^3}\mathcal{D}_0(z) + \frac{128}{\epsilon^2}\mathcal{D}_1(z) - \frac{128}{\epsilon^2}\log(1-z) - \frac{256}{\epsilon}\mathcal{D}_2(z) + \frac{256}{\epsilon}\log^2(1-z) - \frac{320}{\epsilon}\log(1-z) + \frac{1024}{3}\mathcal{D}_3(z) - \frac{1024}{3}\log^3(1-z) + 640\log^2(1-z)\right],$$

$$\mathcal{D}_i = \left[\frac{\log^i(1-z)}{1-z}\right]_+$$

- Special vertex (2-gluon correlation)

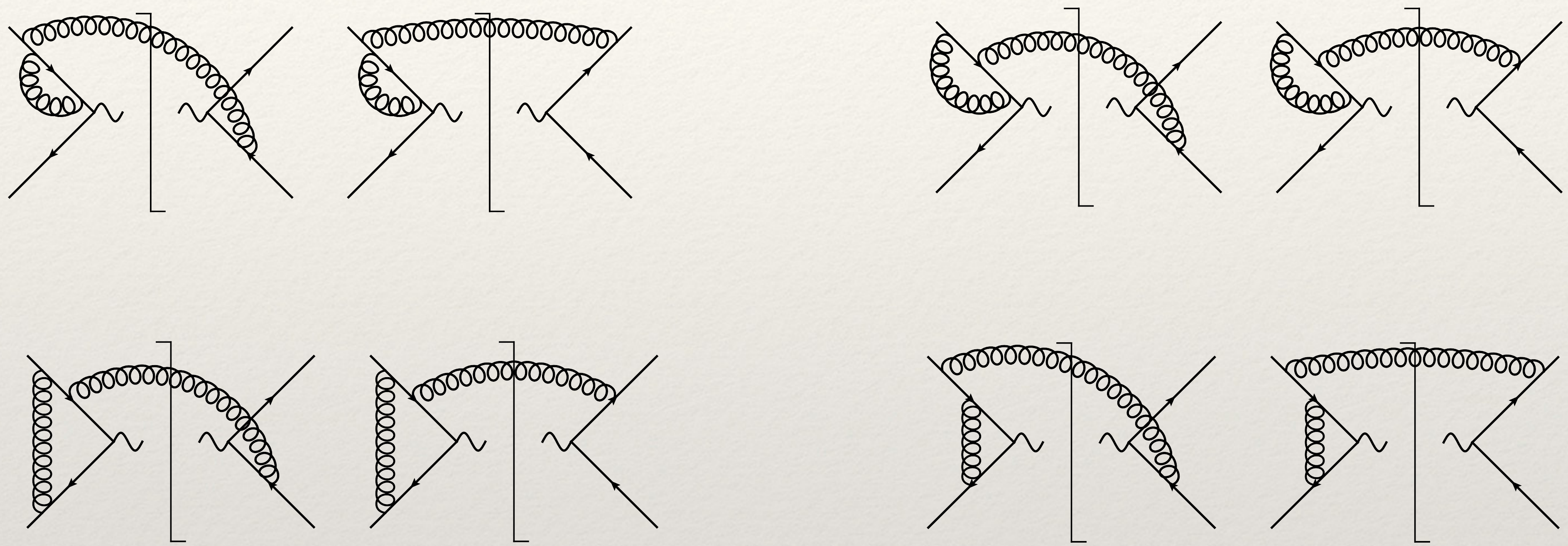
$$R^{\mu\nu}(p;k_1,k_2) = -\frac{(p\cdot k_2)p^\mu k_1^\nu + (p\cdot k_1)k_2^\mu p^\nu - (p\cdot k_1)(p\cdot k_2)g^{\mu\nu} - (k_1\cdot k_2)p^\mu p^\nu}{p\cdot(k_1+k_2)}$$



✓ gives zero after azimuthal integration

NE logs in Drell-Yan: one real - one virtual

- For the complete set of subleading NE logarithms, we must also consider also 1-real plus 1-virtual contributions



  - More subtle, virtual momenta are not always (next-to)-soft. We follow two approaches:
    - method of regions
    - factorization

1 Real plus 1 Virtual

- We redid exact calculation, keeping only C_F^2 terms
  - only the full result was known in the literature Matsuura, van Neerven
  - result, up to constants (dropped higher powers of 1-z)

$$K^{(1)}_{1\mathrm{r},1\mathrm{v}} = \frac{32\mathcal{D}_0 - 32}{\epsilon^3} + \frac{-64\mathcal{D}_1 + 48\mathcal{D}_0 + 64L_1 - 96}{\epsilon^2} + \frac{64\mathcal{D}_2 - 96\mathcal{D}_1 + 128\mathcal{D}_0 - 196 - 64L_1^2 + 208L_1}{\epsilon}$$
$$- \frac{128}{3}\mathcal{D}_3 + 96\mathcal{D}_2 - 256\mathcal{D}_1 + 256\mathcal{D}_0 + \frac{128}{3}L_1^3 - 232L_1^2 + 412L_1 - 408, \qquad (4.12)$$

$$\mathcal{D}_i = \left[\frac{\log^i(1-z)}{1-z}\right]_+ \qquad L_1 = \log(1-z)$$

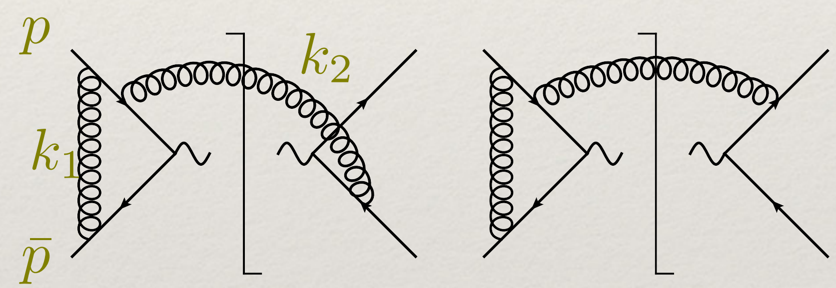
  - bare results, no renormalization or factorization counterterms
  - Can we reproduce (some of) this using method of regions?

Method of regions approach

Bonocore, EL, Magnea, Melville, Vernazza, White

- Method of region approach, extended to next power
  - Should allow treatment of (next-to-)soft and (next-to-)collinear on equal footing
- How does it work? Beneke, Smirnov; Jantzen
  - Divide up k_1 (=loop-momentum) integral into hard, 2 collinear and a soft region, by appropiate scaling

$$\text{Hard}: \quad k_1 \sim \sqrt{\hat{s}}\,(1,1,1)\ ; \qquad \text{Soft}: \quad k_1 \sim \sqrt{\hat{s}}\,(\lambda^2,\lambda^2,\lambda^2)\ ;$$
$$\text{Collinear}: \quad k_1 \sim \sqrt{\hat{s}}\,(1,\lambda,\lambda^2)\ ; \quad \text{Anticollinear}: \quad k_1 \sim \sqrt{\hat{s}}\,(\lambda^2,\lambda,1)\ .$$

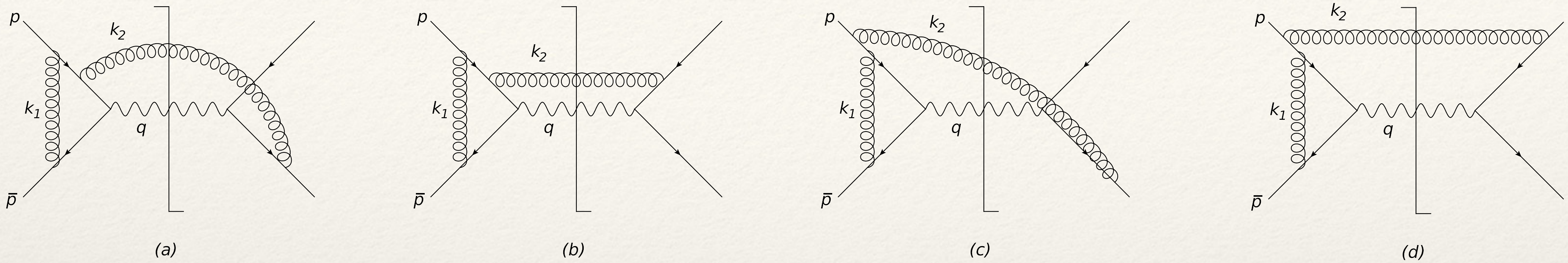


  - expand integrand in λ, to leading and next-to-leading order
  - but then integrate over *all* k_1 anyway
  - Treat emitted momentum as soft and incoming momenta as hard

$$k_2^\mu = (\lambda^2,\lambda^2,\lambda^2) \qquad p^\mu = \tfrac{1}{2}\sqrt{s}n_+^\mu \qquad \bar{p}^\mu = \tfrac{1}{2}\sqrt{s}n_-^\mu$$

MoR: collinear region

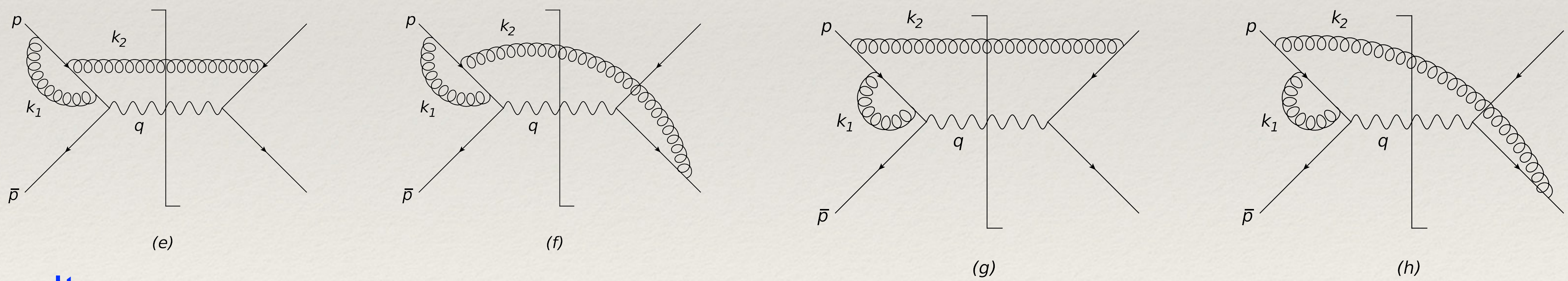
- Soft emission from triangle graphs:



  - Result

$$K^{(2),a-d}_{\mathrm{NE},c+\bar{c}} = \left(\frac{\alpha_s C_F}{4\pi}\right)^2 \left[\frac{-8}{\epsilon^2} + \frac{24}{\epsilon}\log(1-z) - 36\log^2(1-z) + 16\right]$$

- Soft emission from self-energy diagrams



  - Result

$$K^{(2),e-h}_{\mathrm{NE},c+\bar{c}} = \left(\frac{\alpha_s C_F}{4\pi}\right)^2 \left[\frac{-8}{\epsilon^2} - \frac{20}{\epsilon} + 60\log(1-z) + \frac{24}{\epsilon}\log(1-z) - 36\log^2(1-z) - 40\right]$$

Collinear(+anti-collinear) region

- Note
  - Only NLP logarithms (intermediate LP logs cancel)
  - Terms after loop integral contain

$$\frac{(-2p\cdot k_2)^{-\epsilon}}{\epsilon}, \qquad \frac{(-2\bar{p}\cdot k_2)^{-\epsilon}}{\epsilon}$$

  - When integrated over k_2, give the right log(1-z) terms, so
    - expand in ϵ before expanding in k_2!
    - illustrates again breakdown of original LBK theorem

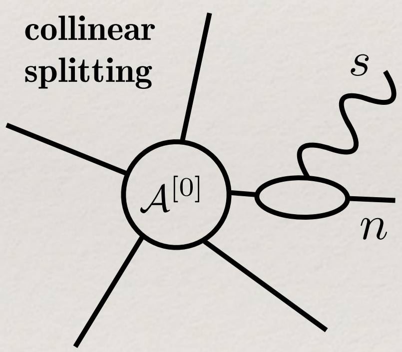
Method of regions results

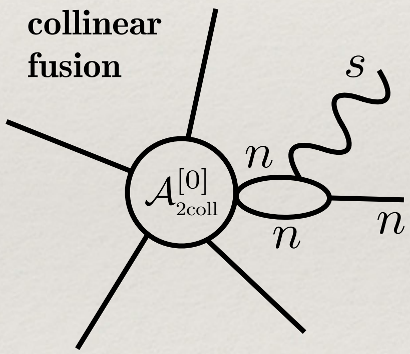
- One finds
  - Hard region (expansion in λ^2)
    - ✓ reproduces already all plus-distributions, and some NLP logarithms
  - Soft region (expansion in λ^2)
    - ✓ all integrals are scale-less, hence all zero in dimensional regularization
  - (anti-)collinear regions (expansion in λ)
    - ✓ only give NLP logarithms, once all diagrams in set are summed
- Nice:
  - the full $K^{(1)}_{1r,1v}$ is reproduced, including constants $\longrightarrow$ 4 powers of NLP logs
- MoR gives diagnostic of next-to-soft power logs, but doesn't give predictive power
- For this, we need a factorization approach

Next-to-soft in SCET

- Early SCET results beyond leading power in heavy-to-light currents
  - need for multi-pole expansions for appropiate scaling — Beneke, Diehl, Feldmann; Chapovsky
  - application underway for Drell-Yan current operator
- Analysis of LBKD theorem at one-loop level in SCET
  - general approach, has collinear splitting and collinear fusion terms

Larkoski, Neill, Stewart

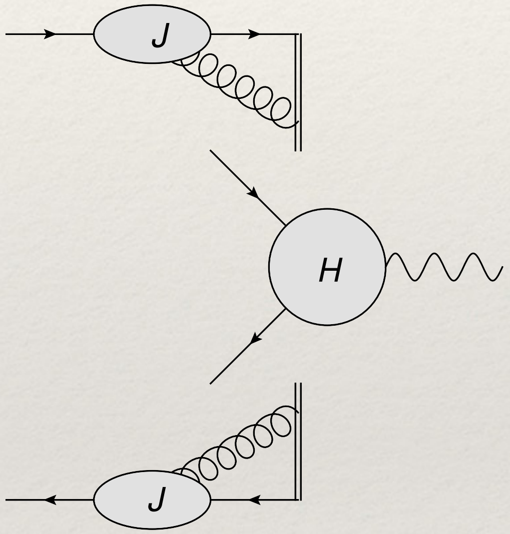




A factorization approach to next-to-soft

Bonocore, EL, Magnea, Melville, Vernaza, White
arXiv:1503.05156

- Can we *predict* the log(1-z) logarithms?
  - For both we need to factorize the cross section, as we did earlier
    - ✓ H contains both the hard and the soft function (non-collinear factors)
    - ✓ J: incoming jet functions



- Next, add one extra soft emission, as in Low's theorem. Let every blob radiate!
    - ✓ Can we compute each new "blob + radiation?", and put it together?

Del Duca, 1991

(Next-to-)Soft currents

- Eikonal/soft approximation for gauge theories and gravity long known — White

$$A_{n+1}(\{p_i\},k) = S_n^{(0)} A_n(\{p_i\}), \quad S_n^{(0)} = \sum_{i=1}^{n} \frac{\epsilon_\mu(k) p_i^\mu}{p_i \cdot k}$$

$$M_{n+1}(\{p_i\},k) = S_{n,grav}^{(0)} A_n(\{p_i\}), \quad S_{n,grav}^{(0)} = \sum_{i=1}^{n} \frac{\epsilon_{\mu\nu}(k) p_i^\mu p_i^\nu}{p_i \cdot k}$$

Weinberg's soft theorem

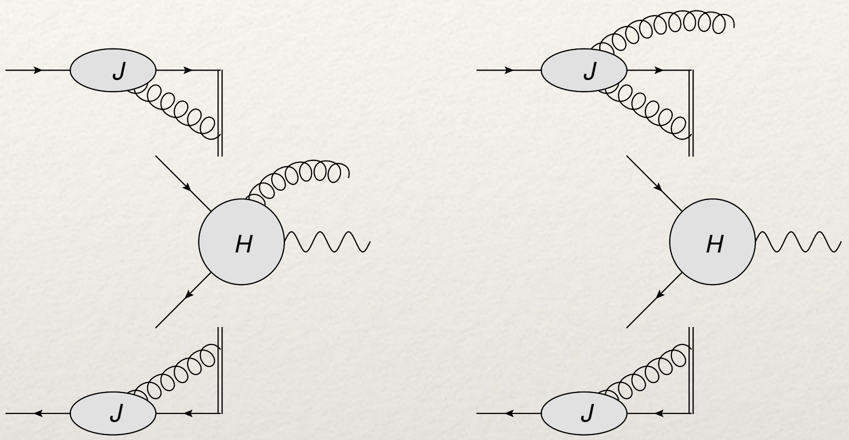
- Generalization to next-to-eikonal/soft

$$S_n^{(1)} = \sum_{i=1}^{n} \frac{\epsilon_\mu(k) k_\rho J^{(i)\mu\rho}}{p_i \cdot k} \qquad S_{n,grav}^{(1)} = \sum_{i=1}^{n} \frac{\epsilon_{\mu\nu}(k) p_i^\mu k_\rho J^{(i)\rho\nu}}{p_i \cdot k}$$

- ✓ Very generally true for soft spin 0,1/2,1,2 emissions, abelian and non-abelian
  - Includes emissions from inside hard function
- ✓ Coupling to full Lorentz generator (where spin part is included, e.g. for fermions)
  - Much recent work — Bern, Davies, Di Vecchia, Nohle; Broedel, Plefka, de Leeuw, Rosso
  - Breaking at loop level

A factorization approach

- Work at amplitude level, again only C_F^2 terms
  - Later: contract with c.c. amplitude and integrate over phase space
- Emission can occur from either H or J's



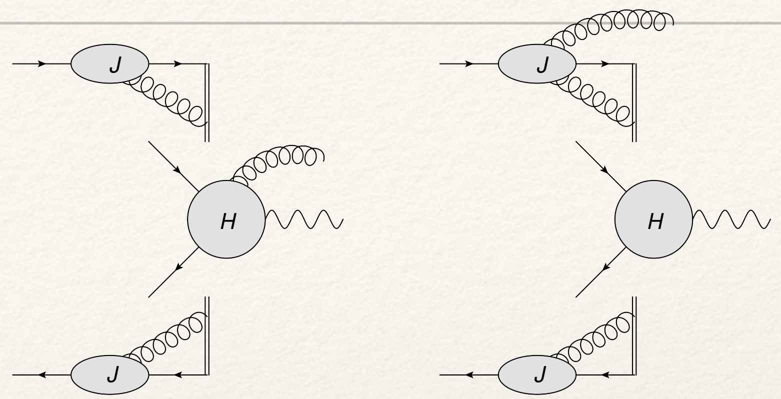
$$\mathcal{A}_\mu \, \epsilon^\mu(k) \, = \, \mathcal{A}_\mu^J \, \epsilon^\mu(k) + \mathcal{A}_\mu^H \, \epsilon^\mu(k)$$

  - For emission from jet function, define

$$J_\mu \left(p, n, k_2\right) u(p) = \left\langle 0 \left| \int d^d y e^{-i(p+k_2)\cdot y} \Phi_n(y,\infty)\psi(y) j_\mu(0) \right| p \right\rangle$$

    - ✓ "radiative jet function", universal

Ward identity



- For emission from H, use Ward identity

$$k^\mu \mathcal{A}_\mu = 0 \qquad\qquad k^\mu \mathcal{A}^H_\mu = -k^\mu \mathcal{A}^J_\mu$$

  - where for the radiative jet function there is the simple WI

$$k^\mu J_\mu(\ldots, k, \epsilon) = q\, J(\ldots, \epsilon)\,, \qquad q = \pm 1$$

  - Then hard function emission is just derivative

$$\mathcal{A}^H_\mu(p_i, k) = \sum_{i=1}^{2} q_i \left(\frac{\partial}{\partial p_i^\mu} H(p_i; p_j, n_j)\right) \prod_{j=1}^{2} J(p_j, n_j)$$

- Split polarization sum of emitted gluon/photon using “K” and “G” projectors

$$\eta^{\mu\nu} = G^{\mu\nu} + K^{\mu\nu}\,, \qquad K^{\mu\nu}(p;k) = \frac{(2p-k)^\nu}{2p\cdot k - k^2}\, k^\mu$$

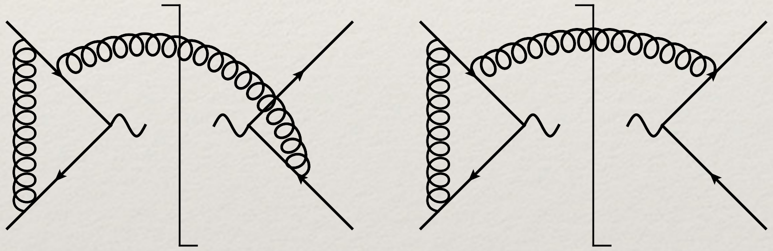
  - Useful: K is leading, G gives subleading terms

Factorization approach: main formula

- Upshot: a factorization formula for the emission amplitude (C_F^2 terms) Del Duca, 1991

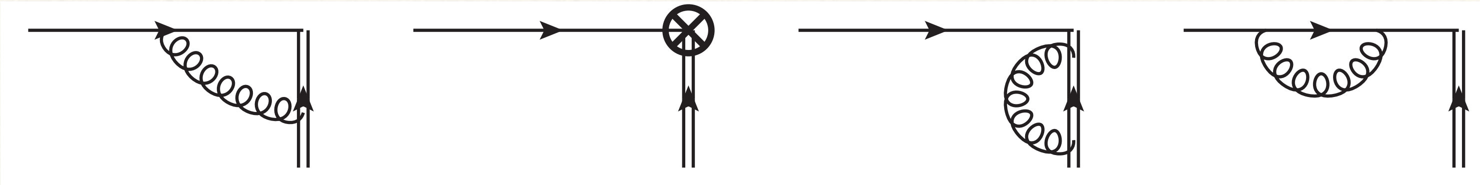
$$\mathcal{A}^\mu(p_j,k) = \sum_{i=1}^{2}\Bigg[q_i \left(\frac{(2p_i-k)^\mu}{2p_i\cdot k - k^2} + G_i^{\nu\mu}\frac{\partial}{\partial p_i^\nu} \right)\mathcal{A}(p_i;p_j)$$
$$+\, \mathcal{H}(p_j,n_j)\,\overline{\mathcal{S}}(\beta_j,n_j)\,G_i^{\nu\mu}\left(J_\nu(p_i,k,n_i) - q_i\frac{\partial}{\partial p_i^\nu}J(p_i,n_i) \right)\prod_{j\neq i} J(p_j,n_j)\Bigg]$$

- Remarks
  - for logs: to be contracted with cc amplitude
  - only process dependent terms are H and A
  - J_μ is important, need it at loop level



LBKD theorem, simplified

- For the non-radiative jet we would need to compute



  - double line is Wilson line in n^μ direction
- We choose $n^\mu = p^\mu$, so $n^2 = 0$. In dimensional regularization we have then

$$J(p_i, n_i) = 1$$

  - Yields simple expression for emission amplitude

$$\mathcal{A}^\mu(p_j, k) = \sum_{i=1}^{2} \left(q_i \, \frac{(2p_i - k)^\mu}{2p_i \cdot k - k^2} + q_i \, G_i^{\nu\mu} \frac{\partial}{\partial p_i^\nu} + G_i^{\nu\mu} J_\nu(p_i, k) \right) \mathcal{A}(p_i; p_j)$$

  - A is known, so need
    - factorized (external) contributions
    - derivative contribution
    - J_μ contribution

External contribution

- Fairly straightforward

$$K^{(2)}_{\rm ext}(z) = \left(\frac{\alpha_s}{4\pi}C_F\right)^2 \left\{ \frac{32}{\varepsilon^3}\Big[\mathcal{D}_0(z) - 1\Big] + \frac{8}{\varepsilon^2}\Big[-8\mathcal{D}_1(z) + 6\mathcal{D}_0(z) + 8L(z) - 14\Big] \right.$$
$$+ \frac{16}{\varepsilon}\Big[4\mathcal{D}_2(z) - 6\mathcal{D}_1(z) + 8\mathcal{D}_0(z) - 4L^2(z) + 14L(z) - 14\Big]$$
$$- \frac{128}{3}\mathcal{D}_3(z) + 96\mathcal{D}_2(z) - 256\mathcal{D}_1(z) + 256\mathcal{D}_0(z)$$
$$\left. + \frac{128}{3}L^3(z) - 224L^2(z) + 448L(z) - 512 \right\}. \quad (5.62)$$

  - Reproduces all LP logs (plus-distributions)
  - Agrees with factorization of eikonal radiation, and NE Feynman rules

Derivative contribution

- Not through effective Feynman rules, but still not too hard

$$K^{(2)}_{\partial\mathcal{A}}(z) = \left(\frac{\alpha_s}{4\pi}C_F\right)^2\left\{\frac{32}{\varepsilon^2} + \frac{16}{\varepsilon}\Big[-4L(z)+3\Big] + 64L^2(z) - 96L(z) + 128\right\}.$$

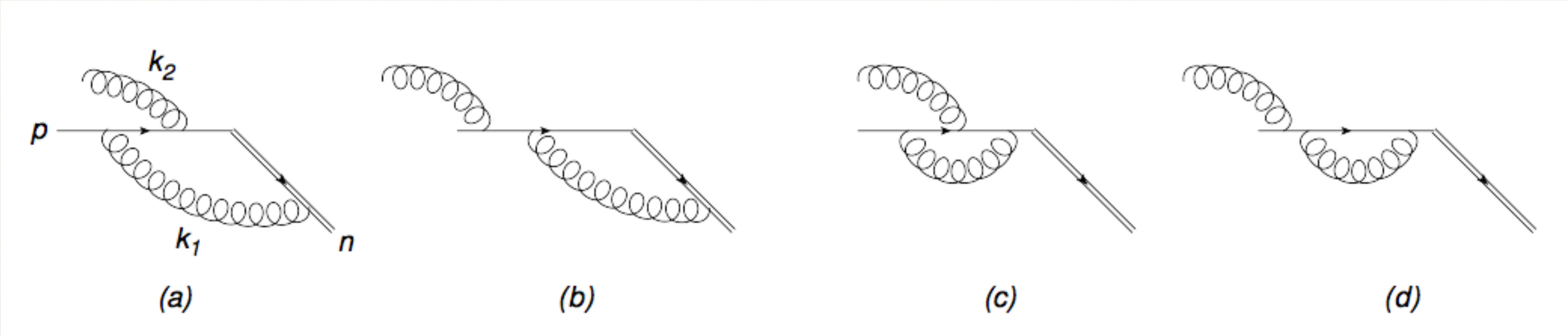
  - NLP terms only
  - Sum of external and derivative contributions corresponds precisely to MoR hard region contribution

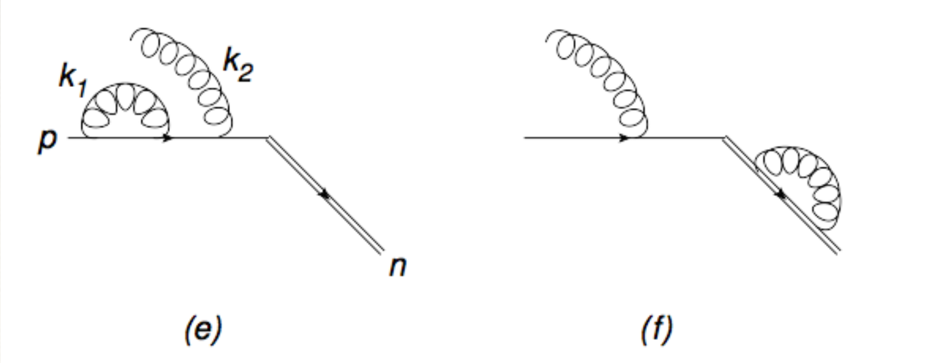
Radiative jet function contribution

- Formal definition

$$J_\mu\left(p,n,k_2\right)u(p)=\left\langle 0\left|\int d^dy e^{-i(p+k_2)\cdot y}\Phi_n(y,\infty)\psi(y)j_\mu(0)\right|p\right\rangle$$

- Diagrams:





Radiative jet function contribution

- Find

$$J^{\nu(1)}(p,n,k\,;\epsilon) = (2p\cdot k)^{-\epsilon}\Big[\left(\frac{2}{\epsilon}+4+8\epsilon\right)\left(\frac{n\cdot k}{p\cdot k}\frac{p^\nu}{p\cdot n}-\frac{n^\nu}{p\cdot n}\right)-(1+2\epsilon)\frac{\mathrm{i}\,k_\alpha\Sigma^{\alpha\nu}}{p\cdot k} + \left(\frac{1}{\epsilon}-\frac{1}{2}-3\epsilon\right)\frac{k^\nu}{p\cdot k}+(1+3\epsilon)\left(\frac{\gamma^\nu \not{n}}{p\cdot n}-\frac{p^\nu \not{k}\not{n}}{p\cdot k\, p\cdot n}\right)\Big]+\ldots$$

- Occurs with G-tensor: filters spin-dependent part. At lowest order $J^{\nu(0)}$:

$$G^{\nu\mu}\left(-\frac{p_\nu}{p\cdot k_2}+\frac{\not{k}_2\gamma_\nu}{2p\cdot k_2}\right)=\frac{k_{2\,\nu}\,[\gamma^\nu,\gamma^\mu]}{4p\cdot k_2}$$

- One-loop terms breaks next-to-soft theorem. Interestingly it is an eigenstate of $G^{\mu\nu}$

$$G^{\nu\mu}J^{(1)}_\nu(p,n,k)=J^{(1)}_\nu(p,n,k)$$

- Find after phase space (k_2) integral (chosing n=p)

$$K^{(2)}_{\mathrm{radJ}}=\left(\frac{\alpha_s C_F}{4\pi}\right)^2\left[\frac{-16}{\epsilon^2}-\frac{20}{\epsilon}+60\log(1-z)+\frac{48}{\epsilon}\log(1-z)-72\log^2(1-z)-24\right]$$

  - Precise correspondence with collinear region

From amplitudes to logarithms

- Now put it all together, contract with cc amplitude and integrate over phase space

$$d\sigma \; = \; d\Phi_{3,\mathrm{LP}} \left(\mathcal{P}_{\mathrm{LP}} + \mathcal{P}_{\mathrm{NLP}}\right) + d\Phi_{3,\mathrm{NLP}} \mathcal{P}_{\mathrm{LP}}$$

- Find also here perfect agreement with exact NLP result (and of course MoR result), for 4 powers of logarithms

Next steps

- Recent
  - January 2016 workshop at Higgs Centre, Edinburgh
- First on deck
  - non-abelian terms (DY, Higgs..)
    - new regions, also captured by radiative function
  - Resummation
    - Effective field theory operators (many!) known, now compute anomalous dimensions
    - Using next-to-eikonal webs for exponential form

Summary

- Next-to-soft corrections
  - approach through NLP terms in SCET
  - here: factorization approach
- Obey extended non-abelian exponentiation (new webs)
- Governed by LBKD theorem; collinear loop momenta key
  - understood through method of regions
  - established predictive power through factorized expression
    - ✓ clear correspondence to MoR terms
- Expect non-abelian extension soon