Precision calculations for LHC physics

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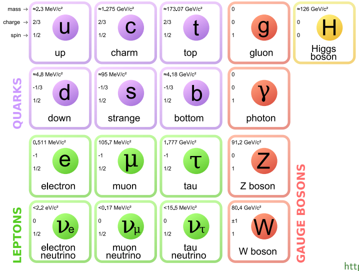
INPP, NCSR "Demokritos"

HEP2016, Thessaloniki, May 13, 2016

Outline

Develop theoretical knowledge, algorithms and tools ...

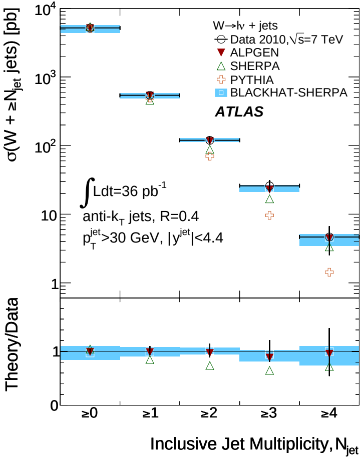
Particle physics today

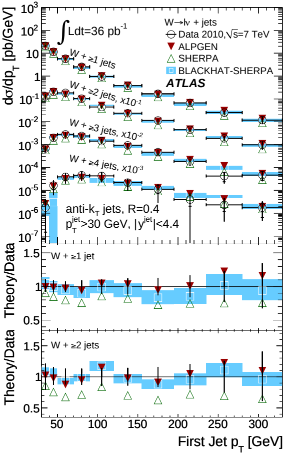


http://en.wikipedia.org

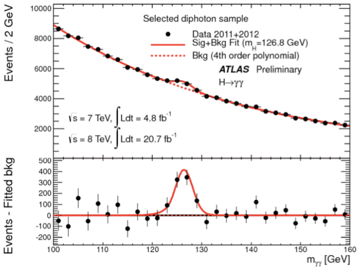
$$
\begin{aligned}
\mathcal{L}_{QCD} &= i\bar{\psi}_i \left(\left(\gamma^\mu D_\mu\right)_{ij} - m_i \delta_{ij} \right) \psi_j \\
&- \frac{1}{4} G^a_{\mu\nu} G^{a\mu\nu}
\end{aligned}
$$

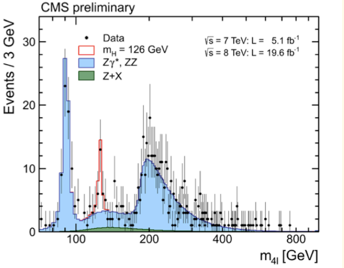
in order to analyse experimental data ...

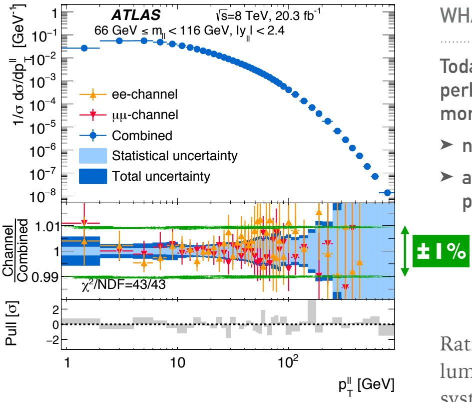




so that discoveries (Higgs) become possible!







WHAT'S POSSIBLE EXPERIMENTALLY?

Today's most precise results are perhaps for the Z transverse momentum

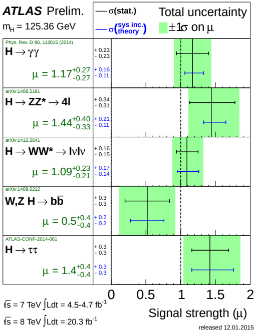
- normalised to Z fiducal σ
- achieves <1%, from p_T = 1 to 200 GeV

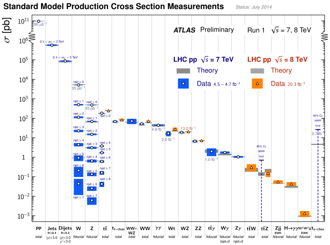
Ratio to total cross section cancels lumi & some lepton-efficiency systematics.

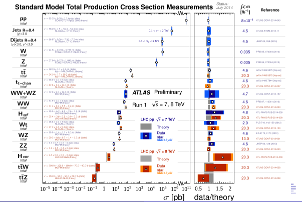
7

LHC: PRECISION

Forthcoming experimental precision vs theoretical predictions





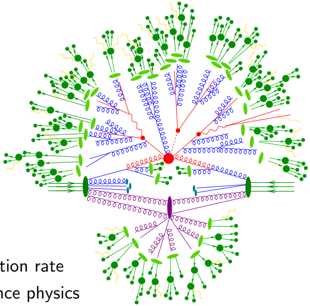


Factorization

Factorization

Collins,Soper,Sterman'85–'89

- Calculate
  - Scattering probability
  - Gluon emission probability
- Measure
  - Long distance interactions
  - Particle decay rates



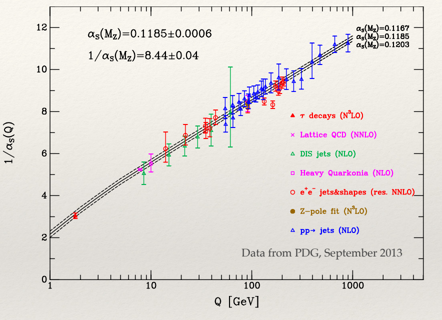
Divide et Impera

- Quantity of interest: Total interaction rate
- Convolution of short & long distance physics

$$\sigma_{p_1 p_2 \to X} = \sum_{i,j \in \{q,g\}} \int \mathrm{d}x_1 \mathrm{d}x_2 \underbrace{f_{p_1,i}(x_1,\mu_F^2) f_{p_2,j}(x_2,\mu_F^2)}_{\text{long distance physics}} \underbrace{\hat{\sigma}_{ij \to X}(x_1 x_2, \mu_F^2)}_{\text{short distance physics}}$$

QCD as a perturbative quantum field theory: Fixed-order calculations

Expected theoretical precision



Also some outliers from thrust and C-parameter, Parton fits
Abate et al, 1060.3080, αs(MZ)=0.1135+0.0010
Hoang et al, 1501.04111,1501.04753,αs(MZ)=0.1123±0.0015
Alekhin et al, 0908.2766,αs(MZ)=0.1135±0.0014

- QCD is asymptotically free!
- $\alpha_s = 0.1181 \pm 0.0013 (1.1\%)$
- NLO: 10%
- NNLO: 1%

Anastasiou, Loops&Legs, April 2016

Composition of the inclusive cross-section

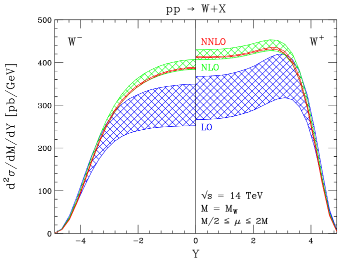
48.58 pb =	16.00 pb	(+32.9%)	(LO, rEFT)
	+ 20.84 pb	(+42.9%)	(NLO, rEFT)
	− 2.05 pb	(−4.2%)	((t, b, c), exact NLO)
	+ 9.56 pb	(+19.7%)	(NNLO, rEFT)
	+ 0.34 pb	(+0.7%)	(NNLO, $1/m_t$)
	+ 2.40 pb	(+4.9%)	(EW, QCD-EW)
	+ 1.49 pb	(+3.1%)	(N^3LO, rEFT)

- N3LO QCD for infinite Mtop limit — CA, Duhr, Dulat, Furlan, Gehrmann, Herzog, Lazopoulos, Mistlberger
- Finite quark-mass corrections at
  - NLO exact — Dawson; Djouadi, Gtaudenz, Spira, Zerwas; Harlander, Kant; CA,Beerli, Bucherer, Daleo, Kunszt; Bonciani, Degrassi, Vicini
  - NNLO 1/mtop expansion — Harlander, Mantler, Marzani, Ozeren; Pak, Rogal, Steinhauser
- Two-loop electroweak corrections — Actis, Passarino, Sturm, Uccirati; Aglietti, Bonciani, Degrassi, Vicini
- Mixed QCD-electroweak corrections — CA, Boughezal, Petriello

QCD (N)NLO

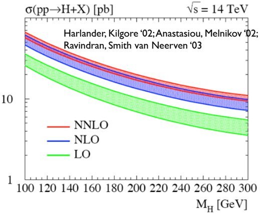
(N)NLO needed in order to properly interpret the data at the LHC

- LO: shape
- NLO: shape+normalization
- NNLO: shape+normalization+uncertainty

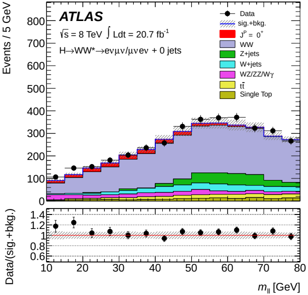


C. Anastasiou, L. J. Dixon, K. Melnikov and F. Petriello, Phys. Rev. D **69** (2004) 094008 [hep-ph/0312266].

QCD (N)NLO



Signal estimation

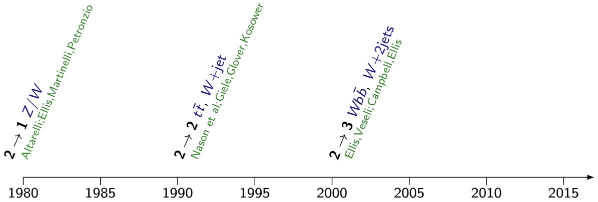


Precise background knowledge

(N)NLO corrections: impressive impact on theoretical uncertainties and differential shapes

G. P. Salam, PoS ICHEP **2010**, 556 (2010) [arXiv:1103.1318 [hep-ph]]

The NLO revolution



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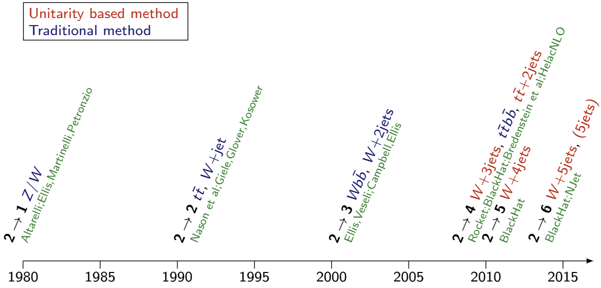

BlackHat → Berger,Bern,Dixon,Febres Cordero,Forde,Ita,Kosower,Mâitre
HelacNLO → Bevilacqua,Czakon,Papadopoulos,Pittau,Worek
NJet → Badger,Biedermann,Uwer,Yundin
Rocket → Ellis,Melnikov,Zanderighi

The NLO wishlist

Process ($V \in \{Z, W, \gamma\}$)	Status
1. $pp \to VV$ jet	WW jet completed by Dittmaier/Kallweit/Uwer; Campbell/Ellis/Zanderighi ZZ jet completed by Binoth/Gleisberg/Karg/Kauer/Sanguinetti WZ jet, $W\gamma$ jet completed by Campanario et al.
2. $pp \to$ Higgs+2 jets	NLO QCD to the gg channel completed by Campbell/Ellis/Zanderighi NLO QCD+EW to the VBF channel completed by Ciccolini/Denner/Dittmaier Interference QCD-EW in VBF channel
3. $pp \to V\,V\,V$	ZZZ completed by Lazopoulos/Melnikov/Petriello and WWZ by Hankele/Zeppenfeld see also Binoth/Ossola/Papadopoulos/Pittau VBFNLOmeanwhile also contains $WWW, ZZW, ZZZ, WW\gamma, ZZ\gamma, WZ\gamma, W\gamma\gamma, Z\gamma\gamma, \gamma\gamma\gamma, W\gamma\gamma j$
4. $pp \to t\bar{t}\,b\bar{b}$	relevant for $t\bar{t}H$, computed by Bredenstein/Denner/Dittmaier/Pozzorini and Bevilacqua/Czakon/Papadopoulos/Pittau/Worek
5. $pp \to V$+3 jets	W+3 jets calculated by the Blackhat/Sherpa and Rocket collaborations Z+3jets by Blackhat/Sherpa
6. $pp \to t\bar{t}$+2jets	relevant for $t\bar{t}H$, computed by Bevilacqua/Czakon/Papadopoulos/Worek
7. $pp \to VV\,b\bar{b}$,	Pozzorini et al.Bevilacqua et al.
8. $pp \to VV$+2jets	W^+W^++2jets, W^+W^-+2jets, relevant for VBF $H \to VV$ VBF contributions by (Bozzi/)Jäger/Oleari/Zeppenfeld
9. $pp \to b\bar{b}b\bar{b}$	Binoth et al.
10. $pp \to V+4$ jets	top pair production, various new physics signatures Blackhat/Sherpa: W+4jets, Z+4jets see also HEJfor $W + n$jets
11. $pp \to Wb\bar{b}j$	top, new physics signatures, Reina/Schutzmeier
12. $pp \to t\bar{t}t\bar{t}$	various new physics signatures, Bevilacqua/Worek
$pp \to W\gamma\gamma$ jet	Campanario/Englert/Rauch/Zeppenfeld
$pp \to 4/5$ jets	Blackhat+Sherpa/NJets

- NLO calculations requested by LHC experimenters
- List constructed in 2005

The NLO wishlist

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1. $pp \to VV$ jet	WW jet completed by Dittmaier/Kallweit/Uwer; Campbell/Ellis/Zanderighi ZZ jet completed by Binoth/Gleisberg/Karg/Kauer/Sanguinetti WZ jet, $W\gamma$ jet completed by Campanario et al.
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- NLO calculations requested by LHC experimenters
- List constructed in 2005
- Calculations completed 2012

Realistic wishlist

- 4 top final state

Constraining BSM Physics at the LHC: Four top final states with NLO accuracy in perturbative QCD

arXiv:1206.3064v1 [hep-ph] 14 Jun 2012

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[b] *Theoretische Physik, Fachbereich C, Bergische Universität Wuppertal, Gauss Str. 20, D-42097 Wuppertal, Germany*
E-mail: bevilacqua@physik.rwth-aachen.de, worek@physik.uni-wuppertal.de

Abstract: Many theories, from Supersymmetry to models of Strong Electroweak Symmetry Breaking, look at the production of four top quarks as an interesting channel to evidentiate signals of new physics beyond the Standard Model. The production of four-top final states requires large partonic energies, above the $4m_t$ threshold, that are available at the CERN Large Hadron Collider and will become more and more accessible with increasing energy and luminosity of the proton beams. A good theoretical control on the Standard Model background is a fundamental prerequisite for a correct interpretation of the possible signals of new physics that may arise in this channel. In this paper we report on the calculation of the next-to-leading order QCD corrections to the Standard Model process $pp \to t\bar{t}t\bar{t} + X$. As it is customary for such studies, we present results for both integrated and differential cross sections. A judicious choice of a dynamical scale allows us to obtain nearly constant $\mathcal{K}$-factors in most distributions.

Keywords: NLO Computations, Heavy Quark Physics, Standard Model, Beyond Standard Model

WUB/12-12, TTK-12-22

NNLO QCD+NLO EW wishlist

Higgs

Process	known	desired	details
H	$d\sigma$ @ NNLO QCD $d\sigma$ @ NLO EW finite quark mass effects @ NLO	$d\sigma$ @ NNNLO QCD + NLO EW MC@NNLO finite quark mass effects @ NNLO	H branching ratios and couplings
H + j	$d\sigma$ @ NNLO QCD (g only) $d\sigma$ @ NLO EW finite quark mass effects @ LO	$d\sigma$ @ NNLO QCD + NLO EW finite quark mass effects @ NLO	H p_T
H + 2j	$\sigma_{\rm tot}$(VBF) @ NNLO(DIS) QCD $d\sigma$(gg) @ NLO QCD $d\sigma$(VBF) @ NLO EW	$d\sigma$ @ NNLO QCD + NLO EW	H couplings
H + V	$d\sigma$ @ NNLO QCD $d\sigma$ @ NLO EW	with $H \to b\bar{b}$ @ same accuracy	H couplings
$t\bar{t}$H	$d\sigma$(stable tops) @ NLO QCD	$d\sigma$(top decays) @ NLO QCD + NLO EW	top Yukawa coupling
HH	$d\sigma$ @ LO QCD (full m_t dependence) $d\sigma$ @ NLO QCD (infinite m_t limit)	$d\sigma$ @ NLO QCD (full m_t dependence) $d\sigma$ @ NNLO QCD (infinite m_t limit)	Higgs self coupling

Table 1: Wishlist part 1 – Higgs (V = W, Z)

→ justify the requested precision based on current/extrapolated experimental errors

S. Dittmaier, N. Glover, J. Huston

NNLO QCD + NLO EWK wishlist

heavy quarks, photons, jets

Process	known	desired	details
$t\bar{t}$	σ_{tot} @ NNLO QCD $d\sigma$(top decays) @ NLO QCD $d\sigma$(stable tops) @ NLO EW	$d\sigma$(top decays) @ NNLO QCD + NLO EW	precision top/QCD, gluon PDF, effect of extra radiation at high rapidity, top asymmetries
$t\bar{t} + j$	$d\sigma$(NWA top decays) @ NLO QCD	$d\sigma$(NWA top decays) @ NNLO QCD + NLO EW	precision top/QCD top asymmetries
single-top	$d\sigma$(NWA top decays) @ NLO QCD	$d\sigma$(NWA top decays) @ NNLO QCD (t channel)	precision top/QCD, V_{tb}
dijet	$d\sigma$ @ NNLO QCD (g only) $d\sigma$ @ NLO weak	$d\sigma$ @ NNLO QCD + NLO EW	Obs.: incl. jets, dijet mass $\rightarrow$ PDF fits (gluon at high x) $\rightarrow \alpha_s$ CMS http://arxiv.org/abs/1212.6660
3j	$d\sigma$ @ NLO QCD	$d\sigma$ @ NNLO QCD + NLO EW	Obs.: $R3/2$ or similar $\rightarrow \alpha_s$ at high scales dom. uncertainty: scales CMS http://arxiv.org/abs/1304.7498
$\gamma + j$	$d\sigma$ @ NLO QCD $d\sigma$ @ NLO EW	$d\sigma$ @ NNLO QCD +NLO EW	gluon PDF $\gamma + b$ for bottom PDF

Table 2: Wishlist part 2 – jets and heay quarks

NNLO QCD + NLO EWK wishlist

Vector bosons

Process	known	desired	details
V	$d\sigma$(lept. V decay) @ NNLO QCD $d\sigma$(lept. V decay) @ NLO EW	$d\sigma$(lept. V decay) @ NNNLO QCD + NLO EW MC@NNLO	precision EW, PDFs
$V + j$	$d\sigma$(lept. V decay) @ NLO QCD $d\sigma$(lept. V decay) @ NLO EW	$d\sigma$(lept. V decay) @ NNLO QCD + NLO EW	$Z + j$ for gluon PDF $W + c$ for strange PDF
$V + jj$	$d\sigma$(lept. V decay) @ NLO QCD	$d\sigma$(lept. V decay) @ NNLO QCD + NLO EW	study of systematics of $H + jj$ final state
VV'	$d\sigma$(V decays) @ NLO QCD $d\sigma$(stable V) @ NLO EW	$d\sigma$(V decays) @ NNLO QCD + NLO EW	off-shell leptonic decays TGCs
$gg \to VV$	$d\sigma$(V decays) @ LO QCD	$d\sigma$(V decays) @ NLO QCD	bkg. to $H \to VV$ TGCs
$V\gamma$	$d\sigma$(V decay) @ NLO QCD $d\sigma$(PA, V decay) @ NLO EW	$d\sigma$(V decay) @ NNLO QCD + NLO EW	TGCs
$Vb\bar{b}$	$d\sigma$(lept. V decay) @ NLO QCD massive b	$d\sigma$(lept. V decay) @ NNLO QCD massless b	bkg. for $VH \to b\bar{b}$
$VV'\gamma$	$d\sigma$(V decays) @ NLO QCD	$d\sigma$(V decays) @ NLO QCD + NLO EW	QGCs
$VV'V''$	$d\sigma$(V decays) @ NLO QCD	$d\sigma$(V decays) @ NLO QCD + NLO EW	QGCs, EWSB
$VV' + j$	$d\sigma$(V decays) @ NLO QCD	$d\sigma$(V decays) @ NLO QCD + NLO EW	bkg. to H, BSM searches
$VV' + jj$	$d\sigma$(V decays) @ NLO QCD	$d\sigma$(V decays) @ NLO QCD + NLO EW	QGCs, EWSB
$\gamma\gamma$	$d\sigma$ @ NNLO QCD		bkg to $H \to \gamma\gamma$

Table 3: Wishlist part 3 – EW gauge bosons ($V = W, Z$)

LO - Dyson-Schwinger Recursive Equations

From Feynman Diagrams to recursive equations: taming the n!

- 1999 HELAC: The first code to calculate recursively tree-order amplitudes for (practically) arbitrary number of particles
- For QCD colour connection representation: revival of the 't Hooft ideas ('71) in the modern era.

LO - Dyson-Schwinger Recursive Equations

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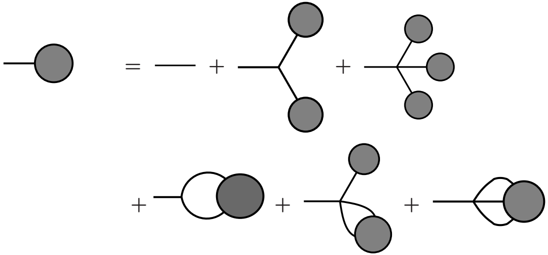
A. Kanaki and C. G. Papadopoulos, Comput. Phys. Commun. **132** (2000) 306 [arXiv:hep-ph/0002082].

F. A. Berends and W. T. Giele, Nucl. Phys. B **306** (1988) 759.

F. Caravaglios and M. Moretti, Phys. Lett. B **358** (1995) 332.



Unfortunately not so much on the second line !

- For QCD colour connection representation: revival of the 't Hooft ideas ('71) in the modern era.

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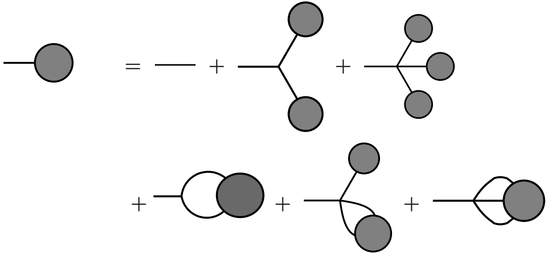

Unfortunately not so much on the second line !

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-• Colour flow or colour connection representation

$$\mathcal{M}^{a_1,i_2,\ldots,i_k}_{j_2,\ldots,j_k} t^{a_1}_{i_1 j_1} \to \mathcal{M}^{i_1,i_2,\ldots,i_k}_{j_1,j_2,\ldots,j_k}$$

$$\mathcal{M}^{i_1,i_2,\ldots,i_k}_{j_1,j_2,\ldots,j_k} = \sum_{\sigma} \delta_{i_{\sigma_1},j_1} \delta_{i_{\sigma_2},j_2} \ldots \delta_{i_{\sigma_k},j_k} A_\sigma \to n!$$

gluons $\to (i,j)$, quark $\to (i,0)$, anti-quark $\to (0,j)$, other $\to (0,0)$

$$\sum_{\sigma,\sigma'} A^*_\sigma \mathcal{C}_{\sigma,\sigma'} A_{\sigma'}$$

$$\mathcal{C}_{\sigma,\sigma\prime} \equiv \sum_{\{i\},\{j\}} \delta_{i_{\sigma_1},j_1} \delta_{i_{\sigma_2},j_2} \ldots \delta_{i_{\sigma_k},j_k} \delta_{i_{\sigma'_1},j_1} \delta_{i_{\sigma'_2},j_2} \ldots \delta_{i_{\sigma'_k},j_k} = N_c^{m(\sigma,\sigma\prime)}$$

HELAC Color treatment

- Colour configuration representation (Monte Carlo integration)

$$\sum_{\{i\},\{j\}} |\mathcal{M}^{i_1,i_2,\dots,i_k}_{j_1,j_2,\dots,j_k}|^2 \to \beta^n$$

Partial solution $n < 6 - 7$

$$\mathcal{M}^{i_1,i_2,\dots,i_k}_{j_1,j_2,\dots,j_k} = \sum A_\sigma$$

From Feynman graphs ...

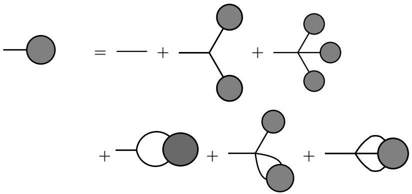
$gg \to ng$	2	3	4	5	6	7	8	9
# FG	4	25	220	2,485	34,300	559,405	10,525,900	224,449,225

Taming the beast ...

From Feynman graphs ...

$gg \to ng$	2	3	4	5	6	7	8	9
# FG	4	25	220	2,485	34,300	559,405	10,525,900	224,449,225

to Dyson-Schwinger recursion! `Helac-Phegas`



$gg \to ng$	2	3	4	5	6	7	8	9
#	5	15	35	70	126	210	330	495

TAMING THE BEAST ...

VOLUME 56, NUMBER 23 PHYSICAL REVIEW LETTERS 9 JUNE 1986

Amplitude for *n*-Gluon Scattering

Stephen J. Parke and T. R. Taylor
Fermi National Accelerator Laboratory, Batavia, Illinois 60510
(Received 17 March 1986)

A nontrivial squared helicity amplitude is given for the scattering of an arbitrary number of gluons to lowest order in the coupling constant and to leading order in the number of colors.

PACS numbers: 12.38.Bx

Computations of the scattering amplitudes for the vector gauge bosons of non-Abelian gauge theories, besides being interesting from a purely quantum-field-theoretical point of view (determination of the *S* matrix), have a wide range of important applications. In particular, within the framework of quantum chromodynamics (QCD), the scattering of vector gauge bosons (gluons) gives rise to experimentally observable multijet production at high-energy hadron colliders. The knowledge of cross sections for the gluon scattering is crucial for any reliable phenomenology of jet physics, which holds great promise for testing QCD as well as for the discovery of new physics at present (CERN $Sp\bar{p}S$ and Fermilab Tevatron) and future (Superconducting Super Collider) hadron colliders.[1]

In this short Letter, we give a nontrivial squared helicity amplitude for the scattering of an arbitrary number of gluons to lowest order in the coupling constant and to leading order in the number of colors. To our knowledge this is the first time in a non-Abelian gauge theory that a nontrivial, on-mass-shell, squared Green's function has been written down for an arbitrary number of external points. Our result can be used to improve the existing numerical programs for the QCD jet production, and in particular for the studies of the four-jet production for which no analytic results have been available so far. Before presenting the helicity amplitude, let us make it clear that this result is an educated guess which we have compared to the existing computations and verified by a series of highly nontrivial and nonlinear consistency checks.

For the *n*-gluon scattering amplitude, there are $(n+2)/2$ independent helicity amplitudes. At the tree level, the two helicity amplitudes which most violate the conservation of helicity are zero. This is easily seen by the embedding of the Yang-Mills theory in a supersymmetric theory.[2,3] Here we give an expression for the next helicity amplitude, also at tree level, to leading order in the number of colors in SU(N) Yang-Mills theory.

If the helicity amplitude for gluons $1,\ldots,n$, of momenta $p_1,\ldots,p_n$ and helicities $\lambda_1,\ldots,\lambda_n$, is $\mathcal{M}_n(\lambda_1,\ldots,\lambda_n)$, where the momenta and helicities are labeled as though all particles are outgoing, then the three helicity amplitudes of interest, squared and summed over color, are

$$|\mathcal{M}_n(+++++\cdots)|^2 = c_n(g,N)[0+O(g^4)], \tag{1}$$

$$|\mathcal{M}_n(-++++\cdots)|^2 = c_n(g,N)[0+O(g^4)], \tag{2}$$

$$|\mathcal{M}_n(--+++\cdots)|^2 = c_n(q,N)[(p_1\cdot p_2)^4 \times \textstyle\sum_P[(p_1\cdot p_2)(p_2\cdot p_3)(p_3\cdot p_4)\cdots(p_n\cdot p_1)]^{-1}+O(N^{-2})+O(g^2)], \tag{3}$$

where $c_n(g,N)=g^{2n-4}N^{n-2}(N^2-1)/2^{n-4}n$. The sum is over all permutations P of $1,\ldots,n$.

Equation (3) has the correct dimensions and symmetry properties for this *n*-particle scattering amplitude squared. Also it agrees with the known results[4,5] for $n=4$, 5, and 6. The agreement for $n=6$ is numerical.[5,6] More importantly, this set of amplitudes is consistent with the Altarelli and Parisi[7] relationship for all *n*, when two of the gluons are made parallel. This is trivial for the first two helicity amplitudes but is a highly nontrivial statement for the last amplitude, as shown here:

$$|\mathcal{M}_n(--+++\cdots)|^2 \underset{1\parallel 2}{\longrightarrow} 0, \tag{4}$$

$$|\mathcal{M}_n(--+++\cdots)|^2 \underset{2\parallel 3}{\longrightarrow} 2g^2N\frac{z^4}{z(1-z)}\frac{1}{s}|\mathcal{M}_{n-1}(--++\cdots)|^2, \tag{5}$$

$$|\mathcal{M}_n(--+++\cdots)|^2 \underset{3\parallel 4}{\longrightarrow} 2g^2N\frac{1}{z(1-z)}\frac{1}{s}|\mathcal{M}_{n-1}(--++\cdots)|^2, \tag{6}$$

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What do we need for an NLO calculation ?

$$p_1, p_2 \to p_3, \ldots, p_{m+2}$$

$$\begin{aligned}\sigma_{NLO} &= \int_m d\Phi_m |M_m^{(0)}|^2 J_m(\Phi) \\ &+ \int_m d\Phi_m 2Re(M_m^{(0)*} M_m^{(1)}(\epsilon_{UV}, \epsilon_{IR})) J_m(\Phi) \\ &+ \int_{m+1} d\Phi_{m+1} |M_{m+1}^{(0)}|^2 J_{m+1}(\Phi)\end{aligned}$$

$J_m(\Phi)$ jet function: Infrared safeness $J_{m+1} \to J_m$

Perturbative QCD at NLO

What do we need for an NLO calculation ?

$$p_1, p_2 \to p_3, \ldots, p_{m+2}$$

$$\begin{aligned}\sigma_{NLO} &= \int_m d\Phi_m^{D=4}\left(|M_m^{(0)}|^2 + 2Re(M_m^{(0)*} M_m^{(CT)}(\epsilon_{UV}))\right)J_m(\Phi)\\ &+ \int_m d\Phi_m^{D=4} 2Re(M_m^{(0)*} M_m^{(1)}(\epsilon_{UV}, \epsilon_{IR}))J_m(\Phi)\\ &+ \int_{m+1} d\Phi_{m+1}^{D=4-2\epsilon_{IR}} |M_{m+1}^{(0)}|^2 J_{m+1}(\Phi)\end{aligned}$$

IR and UV divergencies, Four-Dimensional-Helicity scheme; scale dependence μ_R

Perturbative QCD at NLO

What do we need for an NLO calculation ?

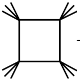
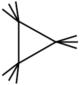
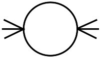
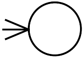
$$p_1, p_2 \to p_3, \ldots, p_{m+2}$$

$$\begin{aligned}
\sigma_{NLO} &= \int_m d\Phi_m J_m(\Phi) \\
&+ \int_m d\Phi_m 2Re(M_m^{(0)*} M_m^{(1)}(\epsilon_{UV}, \epsilon_{IR})) J_m(\Phi) \\
&+ \int_{m+1} d\Phi_{m+1} |M_{m+1}^{(0)}|^2 J_{m+1}(\Phi)
\end{aligned}$$

QCD factorization$-\mu_F$ Collinear counter-terms when PDF are involved basis of scalar integrals:

G. Passarino and M. J. G. Veltman, Nucl. Phys. B **160** (1979) 151.

Z. Bern, L. J. Dixon, D. C. Dunbar and D. A. Kosower, Nucl. Phys. B **425** (1994) 217 [arXiv:hep-ph/9403226].

$\mathcal{A} = \sum d_{i_1 i_2 i_3 i_4}$  $+ \sum c_{i_1 i_2 i_3}$  $+ \sum b_{i_1 i_2}$  $+ \sum a_{i_1}$  $+ R$

$a, b, c, d \rightarrow$ cut-constructible part $\qquad R \rightarrow$ rational terms

$$\mathcal{A} = \sum_{I \subset \{0,1,\cdots,m-1\}} \int \frac{\mu^{(4-d)} d^d q}{(2\pi)^d} \frac{\bar{N}_I(\bar{q})}{\prod_{i \in I} \bar{D}_i(\bar{q})}$$

THE OLD "MASTER" FORMULA

$$\int \frac{N(q)}{\bar{D}_0 \bar{D}_1 \cdots \bar{D}_{m-1}} = \sum_{i_0<i_1<i_2<i_3}^{m-1} d(i_0 i_1 i_2 i_3) \int \frac{1}{\bar{D}_{i0} \bar{D}_{i1} \bar{D}_{i2} \bar{D}_{i2}}$$

$$+ \sum_{i_0<i_1<i_2}^{m-1} c(i_0 i_1 i_2) \int \frac{1}{\bar{D}_{i0} \bar{D}_{i1} \bar{D}_{i2}}$$

$$+ \sum_{i_0<i_1}^{m-1} b(i_0 i_1) \int \frac{1}{\bar{D}_{i0} \bar{D}_{i1}}$$

$$+ \sum_{i_0}^{m-1} a(i_0) \int \frac{1}{\bar{D}_{i0}}$$

$+$ rational terms

G. 't Hooft and M. J. G. Veltman, Nucl. Phys. B **153** (1979) 365.

Z. Bern, L. J. Dixon and D. A. Kosower, Nucl. Phys. B **412** (1994) 751

The new "master" formula

$$
\begin{aligned}
\frac{N(q)}{\bar{D}_0 \bar{D}_1 \cdots \bar{D}_{m-1}} &= \sum_{i_0<i_1<i_2<i_3}^{m-1} \frac{d(i_0 i_1 i_2 i_3) + \tilde{d}(q; i_0 i_1 i_2 i_3)}{\bar{D}_{i0} \bar{D}_{i1} \bar{D}_{i2} \bar{D}_{i2}} \\
&+ \sum_{i_0<i_1<i_2}^{m-1} \frac{c(i_0 i_1 i_2) + \tilde{c}(q; i_0 i_1 i_2)}{\bar{D}_{i0} \bar{D}_{i1} \bar{D}_{i2}} \\
&+ \sum_{i_0<i_1}^{m-1} \frac{b(i_0 i_1) + \tilde{b}(q; i_0 i_1)}{\bar{D}_{i0} \bar{D}_{i1}} \\
&+ \sum_{i_0}^{m-1} \frac{a(i_0) + \tilde{a}(q; i_0)}{\bar{D}_{i0}} \\
&+ \text{rational terms}
\end{aligned}
$$

G. Ossola, C. G. Papadopoulos and R. Pittau, Nucl. Phys. B **763**, 147 (2007)

OPP "MASTER" FORMULA - I

General expression for the 4-dim $N(q)$ at the integrand level in terms of D_i

$$
\begin{aligned}
N(q) &= \sum_{i_0<i_1<i_2<i_3}^{m-1} \left[d(i_0 i_1 i_2 i_3) + \tilde{d}(q; i_0 i_1 i_2 i_3) \right] \prod_{i \neq i_0, i_1, i_2, i_3}^{m-1} D_i \\
&+ \sum_{i_0<i_1<i_2}^{m-1} \left[c(i_0 i_1 i_2) + \tilde{c}(q; i_0 i_1 i_2) \right] \prod_{i \neq i_0, i_1, i_2}^{m-1} D_i \\
&+ \sum_{i_0<i_1}^{m-1} \left[b(i_0 i_1) + \tilde{b}(q; i_0 i_1) \right] \prod_{i \neq i_0, i_1}^{m-1} D_i \\
&+ \sum_{i_0}^{m-1} \left[a(i_0) + \tilde{a}(q; i_0) \right] \prod_{i \neq i_0}^{m-1} D_i
\end{aligned}
$$

OPP "MASTER" FORMULA - II

$$
\begin{aligned}
N(q) &= \sum_{i_0<i_1<i_2<i_3}^{m-1}\left[d(i_0 i_1 i_2 i_3)+\tilde{d}(q;i_0 i_1 i_2 i_3)\right]\prod_{i\neq i_0,i_1,i_2,i_3}^{m-1} D_i + \sum_{i_0<i_1<i_2}^{m-1}\left[c(i_0 i_1 i_2)+\tilde{c}(q;i_0 i_1 i_2)\right]\prod_{i\neq i_0,i_1,i_2}^{m-1} D_i \\
&+ \sum_{i_0<i_1}^{m-1}\left[b(i_0 i_1)+\tilde{b}(q;i_0 i_1)\right]\prod_{i\neq i_0,i_1}^{m-1} D_i + \sum_{i_0}^{m-1}\left[a(i_0)+\tilde{a}(q;i_0)\right]\prod_{i\neq i_0}^{m-1} D_i
\end{aligned}
$$

- The quantities $d(i_0 i_1 i_2 i_3)$ are the coefficients of 4-point functions with denominators labeled by i_0, i_1, i_2, and i_3.
- $c(i_0 i_1 i_2)$, $b(i_0 i_1)$, $a(i_0)$ are the coefficients of all possible 3-point, 2-point and 1-point functions, respectively.

What is the explicit expression of the spurious term?

OPP "MASTER" FORMULA - II

$$
\begin{aligned}
N(q) &= \sum_{i_0<i_1<i_2<i_3}^{m-1}\left[d(i_0 i_1 i_2 i_3) + \tilde{d}(q; i_0 i_1 i_2 i_3)\right] \prod_{i\neq i_0,i_1,i_2,i_3}^{m-1} D_i + \sum_{i_0<i_1<i_2}^{m-1}\left[c(i_0 i_1 i_2) + \tilde{c}(q; i_0 i_1 i_2)\right] \prod_{i\neq i_0,i_1,i_2}^{m-1} D_i \\
&+ \sum_{i_0<i_1}^{m-1}\left[b(i_0 i_1) + \tilde{b}(q; i_0 i_1)\right] \prod_{i\neq i_0,i_1}^{m-1} D_i + \sum_{i_0}^{m-1}\left[a(i_0) + \tilde{a}(q; i_0)\right] \prod_{i\neq i_0}^{m-1} D_i
\end{aligned}
$$

The quantities $\tilde{d}$, $\tilde{c}$, $\tilde{b}$, $\tilde{a}$ are the "spurious" terms

- They still depend on q (integration momentum)
- They should vanish upon integration

What is the explicit expression of the spurious term?

- Not only tensor integrals need reduction!

$$\int \frac{1}{D_0 D_1 D_2 D_3 \dots D_{m-1}}$$

$$1 = \sum \left[d(i_0 i_1 i_2 i_3) + \tilde{d}(q; i_0 i_1 i_2 i_3) \right] D_{i_4} D_{i_5} \dots D_{i_{m-1}}$$

Hilbert's Nullstellensatz theorem

$$\int \frac{1}{D_0 D_1 D_2 D_3 \dots D_{m-1}} \sum \left[d(i_0 i_1 i_2 i_3) + \tilde{d}(q; i_0 i_1 i_2 i_3) \right] D_{i_4} D_{i_5} \dots D_{i_{m-1}}$$

$$\int \frac{1}{D_0 D_1 D_2 D_3 \dots D_{m-1}} = \sum d(i_0 i_1 i_2 i_3) D_0(i_0 i_1 i_2 i_3)$$

$$d(i_0 i_1 i_2 i_3) = \frac{1}{2} \left(\prod_{j \neq i_0, i_1, i_2, i_3} \frac{1}{D_j(q^+)} + \prod_{j \neq i_0, i_1, i_2, i_3} \frac{1}{D_j(q^-)} \right)$$

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$$\int \frac{1}{D_0 D_1 D_2 D_3 \ldots D_{m-1}}$$

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A next to simple example

- Not only tensor integrals need reduction!

$$\int \frac{1}{D_0 D_1 D_2 D_3 \ldots D_{m-1}}$$

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Hilbert's Nullstellensatz theorem

$$\int \frac{1}{D_0 D_1 D_2 D_3 \ldots D_{m-1}} \sum \left[d(i_0 i_1 i_2 i_3) + \tilde{d}(q; i_0 i_1 i_2 i_3) \right] D_{i_4} D_{i_5} \ldots D_{i_{m-1}}$$

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Hilbert's Nullstellensatz theorem

$$\int \frac{1}{D_0 D_1 D_2 D_3 \ldots D_{m-1}} \sum \left[d(i_0 i_1 i_2 i_3) + \tilde{d}(q; i_0 i_1 i_2 i_3) \right] D_{i_4} D_{i_5} \ldots D_{i_{m-1}}$$

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$$d(i_0 i_1 i_2 i_3) = \frac{1}{2} \left(\prod_{j \neq i_0, i_1, i_2, i_3} \frac{1}{D_j(q^+)} + \prod_{j \neq i_0, i_1, i_2, i_3} \frac{1}{D_j(q^-)} \right)$$

A next to simple example

- Not only tensor integrals need reduction!

$$\int \frac{1}{D_0 D_1 D_2 D_3 \ldots D_{m-1}}$$

$$1 = \sum \left[d(i_0 i_1 i_2 i_3) + \tilde{d}(q; i_0 i_1 i_2 i_3) \right] D_{i_4} D_{i_5} \ldots D_{i_{m-1}}$$

Hilbert's Nullstellensatz theorem

$$\int \frac{1}{D_0 D_1 D_2 D_3 \ldots D_{m-1}} \sum \left[d(i_0 i_1 i_2 i_3) + \tilde{d}(q; i_0 i_1 i_2 i_3) \right] D_{i_4} D_{i_5} \ldots D_{i_{m-1}}$$

$$\int \frac{1}{D_0 D_1 D_2 D_3 \ldots D_{m-1}} = \sum d(i_0 i_1 i_2 i_3) D_0(i_0 i_1 i_2 i_3)$$

$$d(i_0 i_1 i_2 i_3) = \frac{1}{2} \left(\prod_{j \neq i_0, i_1, i_2, i_3} \frac{1}{D_j(q^+)} + \prod_{j \neq i_0, i_1, i_2, i_3} \frac{1}{D_j(q^-)} \right)$$

Rational Terms

Expand in D-dimensions ?

$$\bar{D}_i = D_i + \tilde{q}^2$$

$$\begin{aligned} N(q) &= \sum_{i_0<i_1<i_2<i_3}^{m-1} \left[d(i_0 i_1 i_2 i_3; \tilde{q}^2) + \tilde{d}(q; i_0 i_1 i_2 i_3; \tilde{q}^2) \right] \prod_{i \neq i_0, i_1, i_2, i_3}^{m-1} \bar{D}_i \\ &+ \sum_{i_0<i_1<i_2}^{m-1} \left[c(i_0 i_1 i_2; \tilde{q}^2) + \tilde{c}(q; i_0 i_1 i_2; \tilde{q}^2) \right] \prod_{i \neq i_0, i_1, i_2}^{m-1} \bar{D}_i \\ &+ \sum_{i_0<i_1}^{m-1} \left[b(i_0 i_1; \tilde{q}^2) + \tilde{b}(q; i_0 i_1; \tilde{q}^2) \right] \prod_{i \neq i_0, i_1}^{m-1} \bar{D}_i \\ &+ \sum_{i_0}^{m-1} \left[a(i_0; \tilde{q}^2) + \tilde{a}(q; i_0; \tilde{q}^2) \right] \prod_{i \neq i_0}^{m-1} \bar{D}_i + \tilde{P}(q) \prod^{m-1} \bar{D}_i \end{aligned}$$

Expand in D-dimensions ?

$$
\begin{aligned}
N(q) &= \sum_{i_0<i_1<i_2<i_3}^{m-1}\left[d(i_0 i_1 i_2 i_3;\tilde{q}^2)+\tilde{d}(q;i_0 i_1 i_2 i_3;\tilde{q}^2)\right]\prod_{i\neq i_0,i_1,i_2,i_3}^{m-1}\bar{D}_i \\
&+ \sum_{i_0<i_1<i_2}^{m-1}\left[c(i_0 i_1 i_2;\tilde{q}^2)+\tilde{c}(q;i_0 i_1 i_2;\tilde{q}^2)\right]\prod_{i\neq i_0,i_1,i_2}^{m-1}\bar{D}_i \\
&+ \sum_{i_0<i_1}^{m-1}\left[b(i_0 i_1;\tilde{q}^2)+\tilde{b}(q;i_0 i_1;\tilde{q}^2)\right]\prod_{i\neq i_0,i_1}^{m-1}\bar{D}_i \\
&+ \sum_{i_0}^{m-1}\left[a(i_0;\tilde{q}^2)+\tilde{a}(q;i_0;\tilde{q}^2)\right]\prod_{i\neq i_0}^{m-1}\bar{D}_i+\tilde{P}(q)\prod_{i}^{m-1}\bar{D}_i
\end{aligned}
$$

$$m_i^2 \to m_i^2 - \tilde{q}^2$$

Rational Terms

Expand in D-dimensions ?

$$
\begin{aligned}
N(q) \;=\; & \sum_{i_0<i_1<i_2<i_3}^{m-1} \left[d(i_0 i_1 i_2 i_3;\tilde{q}^2) + \tilde{d}(q;i_0 i_1 i_2 i_3;\tilde{q}^2)\right] \prod_{i\neq i_0,i_1,i_2,i_3}^{m-1} \bar{D}_i \\
+ & \sum_{i_0<i_1<i_2}^{m-1} \left[c(i_0 i_1 i_2;\tilde{q}^2) + \tilde{c}(q;i_0 i_1 i_2;\tilde{q}^2)\right] \prod_{i\neq i_0,i_1,i_2}^{m-1} \bar{D}_i \\
+ & \sum_{i_0<i_1}^{m-1} \left[b(i_0 i_1;\tilde{q}^2) + \tilde{b}(q;i_0 i_1;\tilde{q}^2)\right] \prod_{i\neq i_0,i_1}^{m-1} \bar{D}_i \\
+ & \sum_{i_0}^{m-1} \left[a(i_0;\tilde{q}^2) + \tilde{a}(q;i_0;\tilde{q}^2)\right] \prod_{i\neq i_0}^{m-1} \bar{D}_i + \tilde{P}(q) \prod_{i}^{m-1} \bar{D}_i
\end{aligned}
$$

$$
m_i^2 \rightarrow m_i^2 - \tilde{q}^2
$$

Rational Terms

In practice, once the 4-dimensional coefficients have been determined, one can redo the fits for different values of $\tilde{q}^2$, in order to determine $b^{(2)}(ij)$, $c^{(2)}(ijk)$ and $d^{(2m-4)}$.

$$
\begin{aligned}
\mathrm{R}_1 &= -\frac{i}{96\pi^2} d^{(2m-4)} - \frac{i}{32\pi^2} \sum_{i_0<i_1<i_2}^{m-1} c^{(2)}(i_0 i_1 i_2) \\
&- \frac{i}{32\pi^2} \sum_{i_0<i_1}^{m-1} b^{(2)}(i_0 i_1) \left(m_{i_0}^2 + m_{i_1}^2 - \frac{(p_{i_0} - p_{i_1})^2}{3} \right) .
\end{aligned}
$$

G. Ossola, C. G. Papadopoulos and R. Pittau,arXiv:0802.1876 [hep-ph]

Rational Terms - R_2

A different source of Rational Terms, called R_2, can also be generated from the ϵ-dimensional part of $N(q)$

$$\bar{N}(\bar{q}) = N(q) + \tilde{N}(\tilde{q}^2, \epsilon; q)$$

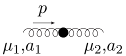
$$\mathrm{R}_2 \equiv \frac{1}{(2\pi)^4} \int d^n\,\bar{q} \frac{\tilde{N}(\tilde{q}^2, \epsilon; q)}{\bar{D}_0 \bar{D}_1 \cdots \bar{D}_{m-1}} \equiv \frac{1}{(2\pi)^4} \int d^n\,\bar{q}\,\mathcal{R}_2$$

$$\begin{aligned} \bar{q} &= q + \tilde{q}\,, \\ \bar{\gamma}_{\bar{\mu}} &= \gamma_\mu + \tilde{\gamma}_{\tilde{\mu}}\,, \\ \bar{g}^{\bar{\mu}\bar{\nu}} &= g^{\mu\nu} + \tilde{g}^{\tilde{\mu}\tilde{\nu}}\,. \end{aligned}$$

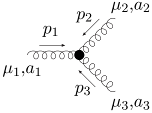
New vertices/particles or GKMZ-approach

HELAC R2 TERMS

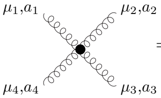
Contribution from $d-$dimensional parts in numerators:



$$= \frac{ig^2 N_{col}}{48\pi^2}\,\delta_{a_1a_2}\left[\frac{p^2}{2}g_{\mu_1\mu_2} + \lambda_{HV}\left(g_{\mu_1\mu_2}p^2 - p_{\mu_1}p_{\mu_2}\right) + \frac{N_f}{N_{col}}\,(p^2 - 6\,m_q^2)\,g_{\mu_1\mu_2}\right]$$



$$= -\frac{g^3 N_{col}}{48\pi^2}\left(\frac{7}{4} + \lambda_{HV} + 2\frac{N_f}{N_{col}}\right) f^{a_1a_2a_3}\,V_{\mu_1\mu_2\mu_3}(p_1,p_2,p_3)$$



$$= -\frac{ig^4 N_{col}}{96\pi^2}\sum_{P(234)}\left\{\left[\frac{\delta_{a_1a_2}\delta_{a_3a_4} + \delta_{a_1a_3}\delta_{a_4a_2} + \delta_{a_1a_4}\delta_{a_2a_3}}{N_{col}}\right.\right.$$

$$+\,4\,Tr(t^{a_1}t^{a_3}t^{a_2}t^{a_4} + t^{a_1}t^{a_4}t^{a_2}t^{a_3})\,(3+\lambda_{HV})$$

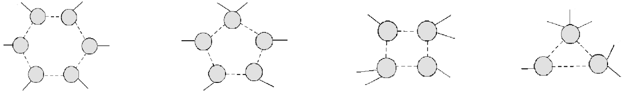
$$\left.-\,Tr(\{t^{a_1}t^{a_2}\}\{t^{a_3}t^{a_4}\})\,(5+2\lambda_{HV})\right]g_{\mu_1\mu_2}g_{\mu_3\mu_4}$$

$$\left.+12\frac{N_f}{N_{col}}Tr(t^{a_1}t^{a_2}t^{a_3}t^{a_4})\left(\frac{5}{3}g_{\mu_1\mu_3}g_{\mu_2\mu_4} - g_{\mu_1\mu_2}g_{\mu_3\mu_4} - g_{\mu_2\mu_3}g_{\mu_1\mu_4}\right)\right\}$$

The one-loop calculation in a nutshell

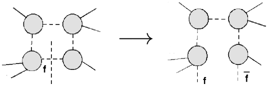
The computation of $pp(p\bar{p}) \to e^+\nu_e\mu^-\bar{\nu}_\mu b\bar{b}$ involves up to six-point functions.
The most generic integrand has therefore the form

$$\mathcal{A}(q) = \sum \underbrace{\frac{N_i^{(6)}(q)}{\bar{D}_{i_0}\bar{D}_{i_1}\cdots\bar{D}_{i_5}}} + \underbrace{\frac{N_i^{(5)}(q)}{\bar{D}_{i_0}\bar{D}_{i_1}\cdots\bar{D}_{i_4}}} + \underbrace{\frac{N_i^{(4)}(q)}{\bar{D}_{i_0}\bar{D}_{i_1}\cdots\bar{D}_{i_3}}} + \underbrace{\frac{N_i^{(3)}(q)}{\bar{D}_{i_0}\bar{D}_{i_1}\bar{D}_{i_2}}} + \cdots$$



In order to apply the OPP reduction, HELAC evaluates numerically the numerators $N_i^6(q), N_i^5(q), ...$ with the values of the loop momentum q provided by CutTools

- generates all inequivalent partitions of 6,5,4,3... blobs attached to the loop, and check all possible flavours (and colours) that can be consistently running inside
- hard-cuts the loop (q is fixed) to get a $n+2$ *tree-like* process



The R_2 contributions (rational terms) are calculated in the same way as the tree-order amplitude, taking into account *extra vertices*

→ MadGraph, RECOLA, OpenLoops

Institute of Nuclear Physics "Demokritos" | Bergische Universitat Wuppertal | Institute of Nuclear Physics PAN | RWTH Aachen University

HELAC-NLO & Associated Tools

Content
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Projects

HELAC-PHEGAS - A generator for all parton level processes in the Standard Model

HELAC-DIPOLES - Dipole formalism for the arbitrary helicity eigenstates of the external partons

HELAC-1LOOP - A program for numerical evaluation of QCD virtual corrections to scattering amplitudes

ONELOOP - A program for the evaluation of one-loop scalar functions

CUTTOOLS - A program implementing the OPP reduction method to compute one-loop amplitudes

PARNI - A program for importance sampling and density estimation

KALEU - A general-purpose parton-level phase space generator

HELAC-ONIA - An automatic matrix element generator for heavy quarkonium physics

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People

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:: top ::

Last modified by Malgorzata Worek
Thursday, January 10th, 2013

Proof of concept: the first NLO public code

$pp \to W^+(l^+\nu)W^-(l^-\bar{\nu})b\bar{b}j$, full final state for $t\bar{t}j$

PRL **116,** 052003 (2016) PHYSICAL REVIEW LETTERS week ending 5 FEBRUARY 2016

Top Quark Pair Production in Association with a Jet with Next-to-Leading-Order QCD Off-Shell Effects at the Large Hadron Collider

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[1]*INFN, Laboratori Nazionali di Frascati, Via E. Fermi 40, I-00044 Frascati, Italy*

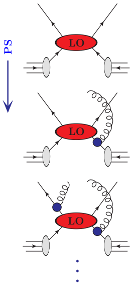
[2]*Institut für Theoretische Teilchenphysik und Kosmologie, RWTH Aachen University, D-52056 Aachen, Germany*

(Received 2 October 2015; revised manuscript received 1 December 2015; published 5 February 2016)

We present a complete description of top quark pair production in association with a jet in the dilepton channel. Our calculation is accurate to next-to-leading order (NLO) in QCD and includes all nonresonant diagrams, interferences, and off-shell effects of the top quark. Moreover, nonresonant and off-shell effects due to the finite W gauge boson width are taken into account. This calculation constitutes the first fully realistic NLO computation for top quark pair production with a final state jet in hadronic collisions. Numerical results for differential distributions as well as total cross sections are presented for the Large Hadron Collider at 8 TeV. With our inclusive cuts, NLO predictions reduce the unphysical scale dependence by more than a factor of 3 and lower the total rate by about 13% compared to leading-order QCD predictions. In addition, the size of the top quark off-shell effects is estimated to be below 2%.

DOI: 10.1103/PhysRevLett.116.052003

Method

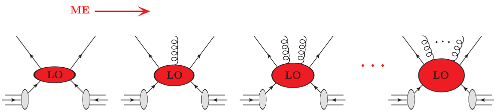


Parton showers

resummation of (soft-)collinear limit
$\rightarrow$ intrajet evolution

- matrix elements (ME) and parton showers (PS) are approximations in different regions of phase space
- MEPS combines multiple LOPS – keeping either accuracy
- NLOPS elevate LOPS to NLO accuracy
- MENLOPS supplements core NLOPS with higher multiplicities LOPS
-

Method



Matrix elements

fixed-order in α_s
$\rightarrow$ hard wide-angle emissions
$\rightarrow$ interference terms

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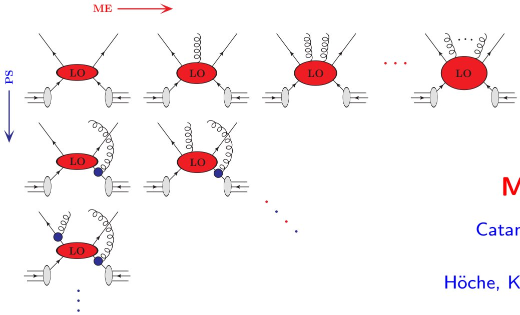

MEPS (CKKW,MLM)

Catani, Krauss, Kuhn, Webber JHEP11(2001)063
Lönnblad JHEP05(2002)046
Höche, Krauss, Schumann, Siegert JHEP05(2009)053
Lönnblad, Prestel JHEP02(2013)094

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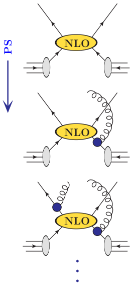

NLOPS (MC@NLO, POWHEG, S-MC@NLO)

Frixione, Webber JHEP06(2002)029
Nason JHEP11(2004)040, Frixione et.al. JHEP11(2007)070
Höche, Krauss, MS, Siegert JHEP09(2012)049

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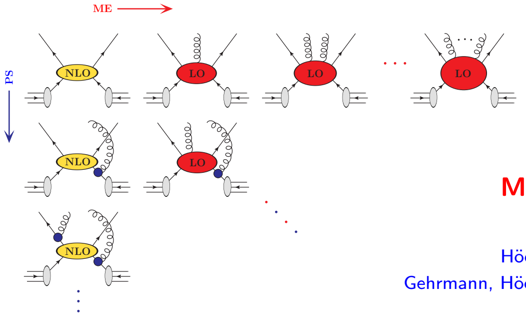

MENLOPS

Hamilton, Nason JHEP06(2010)039
Höche, Krauss, MS, Siegert JHEP08(2011)123
Gehrmann, Höche, Krauss, MS, Siegert JHEP01(2013)144

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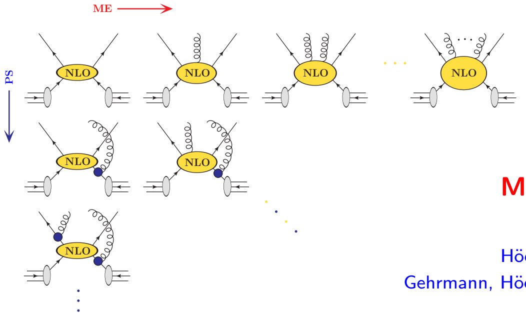

MEPS@NLO

Lavesson, Lönnblad JHEP12(2008)070
Höche, Krauss, MS, Siegert JHEP04(2013)027
Gehrmann, Höche, Krauss, MS, Siegert JHEP01(2013)144
Lönnblad, Prestel JHEP03(2013)166

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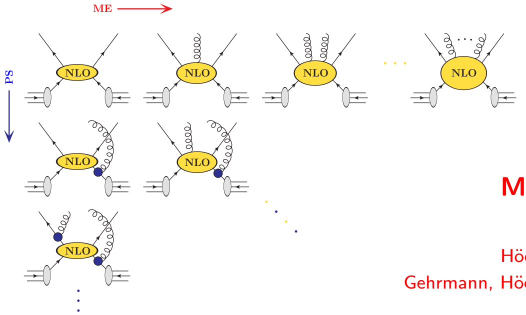
Method



MEPS@NLO

Lavesson, Lönnblad JHEP12(2008)070
Höche, Krauss, MS, Siegert JHEP04(2013)027
Gehrmann, Höche, Krauss, MS, Siegert JHEP01(2013)144
Lönnblad, Prestel JHEP03(2013)166

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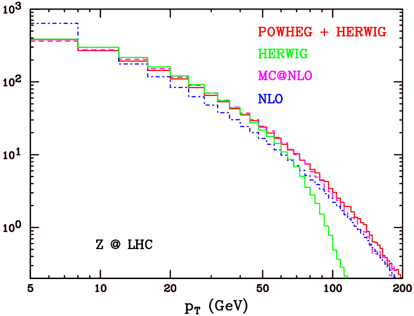
- **MEPS@NLO combines multiple NLOPS – keeping either accuracy**

The POWHEG method

- SMC programs and higher order corrections have been considered complementary approaches for long time. Nowadays it is possible to merge them.
- Double counting of extra emission problem has been adressed and solved first by the MC@NLO approach [Frixione&Webber JHEP 0206:029,2002]
- POWHEG improves over it by being shower independent and by allowing the generation of positive weighted events only [Nason JHEP,2004]
- The resulting events have NLO accuracy and the correct Sudakov suppression

This is achieved by:

1. Generating hardest emission with full tree level matrix element and virtual corrections.
2. The shower generates subsequent emissions, performing (N)LL resummation of collinear/soft logs.
3. Vetoring emissions harder than the first.

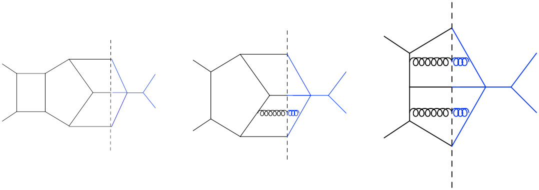


$$d\sigma_{\text{POWHEG}} = \bar{B}(\Phi_n)\, d\Phi_n \Big\{ \Delta_{\text{POWHEG}}(\Phi_n, p_T^{\min}) + \Delta_{\text{POWHEG}}(\Phi_n, k_T) \frac{R(\Phi_n, \Phi_{\text{rad}})}{B(\Phi_n)}\, \theta(k_T - p_T)\, d\Phi_{\text{rad}} \Big\}$$

$$\bar{B} = B(\Phi_n) + V(\Phi_n) + \int [R(\Phi_n, \Phi_{\text{rad}}) - C(\Phi_n, \Phi_{\text{rad}})]\, d\Phi_{\text{rad}}$$

What do we need for an NNLO calculation ?

$$p_1, p_2 \to p_3, \ldots, p_{m+2}$$



What do we need for an NNLO calculation ?

$$p_1, p_2 \to p_3, \dots, p_{m+2}$$

$$\begin{aligned}
\sigma_{NNLO} \to & \int_m d\Phi_m \left(2Re(M_m^{(0)*} M_m^{(2)}) + \left| M_m^{(1)} \right|^2 \right) J_m(\Phi) && VV \\
+ & \int_{m+1} d\Phi_{m+1} \left(2Re \left(M_{m+1}^{(0)*} M_{m+1}^{(1)} \right) \right) J_{m+1}(\Phi) && RV \\
+ & \int_{m+2} d\Phi_{m+2} \left| M_{m+2}^{(0)} \right|^2 J_{m+2}(\Phi) && RR
\end{aligned}$$

$RV + RR \to$ Antenna-S, Colorfull-S, STRIPPER, q_T, N-jetiness

A. Gehrmann-De Ridder, T. Gehrmann and M. Ritzmann, JHEP **1210** (2012) 047
P. Bolzoni, G. Somogyi and Z. Trocsanyi, JHEP **1101** (2011) 059
M. Czakon and D. Heymes, Nucl. Phys. B **890** (2014) 152
S. Catani and M. Grazzini, Phys. Rev. Lett. **98** (2007) 222002
R. Boughezal, C. Focke, X. Liu and F. Petriello, Phys. Rev. Lett. **115** (2015) no.6, 062002

OPP AT TWO LOOPS

coefficients of MI $\oplus$ spurious terms

$$
\begin{aligned}
\frac{N(q)}{\bar{D}_0 \bar{D}_1 \cdots \bar{D}_{m-1}} &= \sum_{i_0<i_1<i_2<i_3}^{m-1} \frac{d(i_0 i_1 i_2 i_3) + \tilde{d}(q; i_0 i_1 i_2 i_3)}{\bar{D}_{i0} \bar{D}_{i1} \bar{D}_{i2} \bar{D}_{i2}} \\
&+ \sum_{i_0<i_1<i_2}^{m-1} \frac{c(i_0 i_1 i_2) + \tilde{c}(q; i_0 i_1 i_2)}{\bar{D}_{i0} \bar{D}_{i1} \bar{D}_{i2}} \\
&+ \sum_{i_0<i_1}^{m-1} \frac{b(i_0 i_1) + \tilde{b}(q; i_0 i_1)}{\bar{D}_{i0} \bar{D}_{i1}} \\
&+ \sum_{i_0}^{m-1} \frac{a(i_0) + \tilde{a}(q; i_0)}{\bar{D}_{i0}} \\
&+ \text{rational terms}
\end{aligned}
$$

G. Ossola, C. G. Papadopoulos and R. Pittau, Nucl. Phys. B **763**, 147 (2007)

OPP at two loops

- Write the "OPP-type" equation at two loops

$$\frac{N(l_1,l_2;\{p_i\})}{D_1 D_2 \ldots D_n} = \sum_{m=1}^{\min(n,8)} \sum_{S_{m;n}} \frac{\Delta_{i_1 i_2 \ldots i_m}(l_1,l_2;\{p_i\})}{D_{i_1} D_{i_2} \ldots D_{i_m}}$$

$$\sum \frac{\Delta_{i_1 i_2 \ldots i_m}(l_1,l_2;\{p_i\})}{D_{i_1} D_{i_2} \ldots D_{i_m}} \rightarrow \text{spurious} \oplus \text{ISP} - \text{irreducible integrals}$$

OPP AT TWO LOOPS

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$$\frac{N(l_1, l_2; \{p_i\})}{D_1 D_2 \ldots D_n} = \sum_{m=1}^{\min(n,8)} \sum_{S_{m;n}} \frac{\Delta_{i_1 i_2 \ldots i_m}(l_1, l_2; \{p_i\})}{D_{i_1} D_{i_2} \ldots D_{i_m}}$$

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ISP-irreducible integrals → use IBPI to Master Integrals

Libraries in the future: `QCD2LOOP`, `TwOLOop`

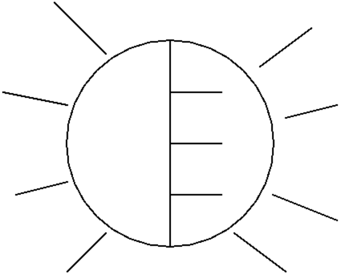
P. Mastrolia, E. Mirabella, G. Ossola and T. Peraro, Phys. Lett. B **718** (2012) 173

J. Gluza, K. Kajda and D. A. Kosower, Phys. Rev. D **83** (2011) 045012

H. Ita, arXiv:1510.05626 [hep-th].

C. G. Papadopoulos, R. H. P. Kleiss and I. Malamos, PoS Corfu **2012** (2013) 019.

IBPI: The current approach



- m independent momenta l loops, $N = l(l+1)/2 + lm$ scalar products
- basis composed by $D_1 \ldots D_N$, allows to express all scalar products $D_i = (\{k,l\} + p_i)^2 - M_i^2$
- $$F[a_1, \ldots, a_N] = \int d^d k d^d l \frac{1}{D_1^{a_1} \ldots D_N^{a_N}}$$
  $$\int d^d k d^d l \; \frac{\partial}{\partial \{k^\mu, l^\mu\}} \left(\frac{\{k^\mu, l^\mu, v^\mu\}}{D_1^{a_1} \ldots D_N^{a_N}} \right) = 0$$
- IBP Laporta: FIRE, AIR, Reduze reduce these to MI

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- Or numerical: `SecDec`, Weinzierl

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F. V. Tkachov, Phys. Lett. B **100** (1981) 65.

K. G. Chetyrkin and F. V. Tkachov, Nucl. Phys. B **192** (1981) 159.

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S. Laporta, Int. J. Mod. Phys. A **15** (2000) 5087

C. Anastasiou and A. Lazopoulos, JHEP **0407** (2004) 046

C. Studerus, Comput. Phys. Commun. **181** (2010) 1293

A. V. Smirnov, Comput. Phys. Commun. **189** (2014) 182

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Z. Bern, L. J. Dixon and D. A. Kosower, Phys. Lett. B **302** (1993) 299.

V. A. Smirnov, Phys. Lett. B **460** (1999) 397

T. Gehrmann and E. Remiddi, Nucl. Phys. B **580** (2000) 485 [hep-ph/9912329].

J. M. Henn, Phys. Rev. Lett. **110** (2013) 25, 251601 [arXiv:1304.1806 [hep-th]].

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S. Borowka, G. Heinrich, S. P. Jones, M. Kerner, J. Schlenk and T. Zirke, Comput. Phys. Commun. **196** (2015) 470

S. Becker, C. Reuschle and S. Weinzierl, JHEP **1012** (2010) 013

IBPI: The current approach

- Find a better IBP algorithm ... Generating function technique, Baikov ?

P. A. Baikov, Nucl. Instrum. Meth. A **389** (1997) 347

V. A. Smirnov and M. Steinhauser, Nucl. Phys. B **672** (2003) 199

K. J. Larsen and Y. Zhang, Phys. Rev. D **93** (2016) no.4, 041701

$$F_{a_1 \ldots a_N} = \sum_{i=\text{masters}} c^{(i)}_{a_1 \ldots a_N} G_i$$

- Baikov polynomial $\leftrightarrow$ LZ construction
- Sector $\leftrightarrow$ cut

$$\delta\left((k+p)^2 - m^2\right) \leftrightarrow \oint_{z=0} dz \frac{1}{z^{n=1}}$$

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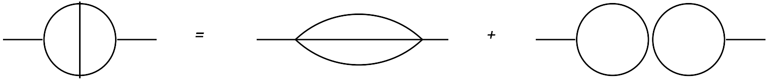

$$F_{11111} = \frac{(3d-10)(3d-8)}{(d-4)^2 (p^2)^2} F_{10011} + \frac{(3d-10)(3d-8)}{(d-4)^2 (p^2)^2} F_{01101} - 2\frac{(d-3)}{(d-4)\, p^2} F_{11110}$$

Differential Equations Approach

The integral is a function of external momenta, so one can set-up differential equations by differentiating and using IBP

$$p_j^\mu \frac{\partial}{\partial p_i^\mu} G[a_1, \ldots, a_n] \to \sum C_{a'_1,\ldots,a'_n} G[a'_1, \ldots, a'_n]$$

- Find the proper parametrization; Bring the system of equations in a form suitable to express the MI in terms of GPs

$$\partial_m f(\varepsilon, \{x_i\}) = \varepsilon A_m(\{x_i\}) f(\varepsilon, \{x_i\})$$
$$\partial_m A_n - \partial_n A_m = 0 \quad [A_m, A_n] = 0$$

J. M. Henn, Phys. Rev. Lett. **110** (2013) 25, 251601 [arXiv:1304.1806 [hep-th]].

- Boundary conditions: expansion by regions or regularity conditions.

B. Jantzen, A. V. Smirnov and V. A. Smirnov, Eur. Phys. J. C **72** (2012) 2139 [arXiv:1206.0546 [hep-ph]].

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Differential Equations Approach

- Iterated Integrals — K. T. Chen, Iterated path integrals, Bull. Amer. Math. Soc. 83 (1977) 831
- Multiple Polylogarithms, Symbol algebra
- Goncharov Polylogarithms

$$\mathcal{G}\left(a_n, \ldots, a_1, x\right) = \int_0^x dt \frac{1}{t - a_n} \mathcal{G}\left(a_{n-1}, \ldots, a_1, t\right)$$

with the special cases, $\mathcal{G}(x) = 1$ and

$$\mathcal{G}\left(\underbrace{0, \ldots 0}_{n}, x\right) = \frac{1}{n!} \log^n\left(x\right)$$

- Shuffle algebra

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- Iterated Integrals
- Multiple Polylogarithms, Symbol algebra

A. B. Goncharov, M. Spradlin, C. Vergu and A. Volovich, Phys. Rev. Lett. **105** (2010) 151605.

C. Duhr, H. Gangl and J. R. Rhodes, JHEP **1210** (2012) 075 [arXiv:1110.0458 [math-ph]].

C. Bogner and F. Brown

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$$\mathcal{G}\left(a_1, a_2; x\right) \mathcal{G}\left(b_1; x\right) = \mathcal{G}\left(a_1, a_2, b_1; x\right) + \mathcal{G}\left(a_1, b_1, a_2; x\right) + \mathcal{G}\left(b_1, a_1, a_2; x\right)$$

The Simplified Differential Equations Approach

C. G. Papadopoulos, JHEP **1407** (2014) 088

Making the whole procedure systematic (algorithmic) and straightforwardly expressible in terms of GPs.

- Introduce one parameter

$$G_{11\ldots1}(x) = \int \frac{d^d k}{i\pi^{d/2}} \frac{1}{(k^2)\,(k + x\,p_1)^2\,(k + p_1 + p_2)^2 \ldots (k + p_1 + p_2 + \ldots + p_n)^2}$$

- Now the integral becomes a function of x, which allows to define a differential equation with respect to x, schematically given by

$$\frac{\partial}{\partial x} G_{11\ldots1}(x) = -\frac{1}{x} G_{11\ldots1}(x) + x p_1^2 G_{12\ldots1} + \frac{1}{x} G_{02\ldots1}$$

- and using IBPI we obtain, for instance for the one-loop 3 off-shell legs

$$\begin{aligned} m_1 x G_{121} + \tfrac{1}{x} G_{021} &= \left(\frac{1}{x-1} + \frac{1}{x - m_3/m_1}\right)\left(\frac{d-4}{2}\right) G_{111} \\ &+ \frac{d-3}{m_1 - m_3}\left(\frac{1}{x-1} - \frac{1}{x - m_3/m_1}\right)\left(\frac{G_{101} - G_{110}}{x}\right) \end{aligned}$$

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The Simplified Differential Equations Approach

- The integrating factor M is given by

$$M = x\,(1-x)^{\frac{4-d}{2}}\,(-m_3 + m_1 x)^{\frac{4-d}{2}}$$

- and the DE takes the form, $d = 4 - 2\varepsilon$,

$$\frac{\partial}{\partial x} M G_{111} = c_\Gamma \frac{1}{\varepsilon}\,(1-x)^{-1+\varepsilon}\,(-m_3 + m_1 x)^{-1+\varepsilon}\left(\left(-m_1 x^2\right)^{-\varepsilon} - (-m_3)^{-\varepsilon}\right)$$

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How far we can go with the Simplified Differential Equations approach ?

TWO-LOOP, FOUR-POINT, TWO OFF-SHELL LEGS

C. G. Papadopoulos, JHEP **1407** (2014) 088

C. G. Papadopoulos, D. Tommasini and C. Wever, JHEP **1501** (2015) 072

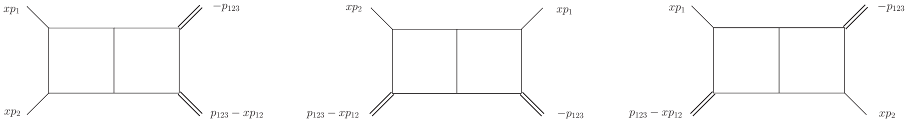

FIGURE : The parametrization of external momenta for the three planar double boxes of the families P_{12} (left), P_{13} (middle) and P_{23} (right) contributing to pair production at the LHC. All external momenta are incoming.

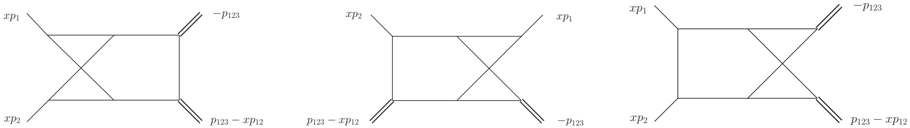

FIGURE : The parametrization of external momenta for the three non-planar double boxes of the families N_{12} (left), N_{13} (middle) and N_{34} (right) contributing

Two-loop, four-point, two off-shell legs

$$p(q_1)p'(q_2) \to V_1(-q_3)V_2(-q_4), \quad q_1^2 = q_2^2 = 0, \quad q_3^2 = M_3^2, \quad q_4^2 = M_4^2.$$

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$$q_1 = xp_1, \quad q_2 = xp_2, \quad q_3 = p_{123} - xp_{12}, \quad q_4 = -p_{123}, \quad p_i^2 = 0,$$
$$s_{12} := p_{12}^2, \qquad s_{23} := p_{23}^2, \qquad q := p_{123}^2,$$

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$$S = (q_1 + q_2)^2 \qquad T = (q_1 + q_3)^2$$

$$S = s_{12}x^2, \quad T = q - (s_{12} + s_{23})x, \quad M_3^2 = (1-x)(q - s_{12}x), \quad M_4^2 = q.$$

$U = (q_1 + q_4)^2 : \; S + T + U = M_3^2 + M_4^2.$

Triangle rule:

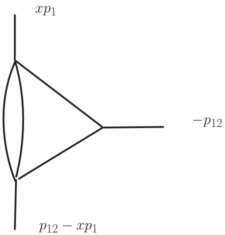

Figure : Required parametrization for off mass-shell triangles after possible pinching of internal line(s).

Planar topologies

$$G^{P_{12}}_{a_1\cdots a_9}(x,s,\epsilon) := e^{2\gamma_E\epsilon}\int \frac{d^dk_1}{i\pi^{d/2}}\frac{d^dk_2}{i\pi^{d/2}}\frac{1}{k_1^{2a_1}(k_1+xp_1)^{2a_2}(k_1+xp_{12})^{2a_3}(k_1+p_{123})^{2a_4}}$$
$$\times\frac{1}{k_2^{2a_5}(k_2-xp_1)^{2a_6}(k_2-xp_{12})^{2a_7}(k_2-p_{123})^{2a_8}(k_1+k_2)^{2a_9}},$$

$$G^{P_{13}}_{a_1\cdots a_9}(x,s,\epsilon) := e^{2\gamma_E\epsilon}\int \frac{d^dk_1}{i\pi^{d/2}}\frac{d^dk_2}{i\pi^{d/2}}\frac{1}{k_1^{2a_1}(k_1+xp_1)^{2a_2}(k_1+xp_{12})^{2a_3}(k_1+p_{123})^{2a_4}}$$
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$$G^{P_{23}}_{a_1\cdots a_9}(x,s,\epsilon) := e^{2\gamma_E\epsilon}\int \frac{d^dk_1}{i\pi^{d/2}}\frac{d^dk_2}{i\pi^{d/2}}\frac{1}{k_1^{2a_1}(k_1+xp_1)^{2a_2}(k_1+p_{123}-xp_2)^{2a_3}(k_1+p_{123})^{2a_4}}$$
$$\times\frac{1}{k_2^{2a_5}(k_2-p_1)^{2a_6}(k_2+xp_2-p_{123})^{2a_7}(k_2-p_{123})^{2a_8}(k_1+k_2)^{2a_9}},$$

Two-loop, four-point, two off-shell legs

Planar topologies

$$P_{12}: \{010000011, 001010001, 001000011, 100000011, 101010010, 101010100, 101000110, 010010101, 101000011, 101000012, 100000111, 100000112, 001010011, 001010012, 010000111, 010010011, 101010110, 111000011, 101000111, 101010011, 011010011, 011010012, 110000111, 110000112, 010010111, 010010112, 111010011, 111000111, 111010111, 111\mathrm{m}10111, 11101\mathrm{m}111\},$$

$$P_{13}: \{000110001, 001000011, 001010001, 001101010, 001110010, 010000011, 010101010, 010110010, 001001011, 001010011, 001010012, 001011011, 001101001, 001101011, 001110001, 001110002, 001110011, 001111001, 001111011, 001211001, 010010011, 010110001, 010110011, 011010011, 011010021, 011110001, 011110011, 011111011, \mathrm{m}11111011\},$$

$$P_{23}: \{001010001, 001010011, 010000011, 010000101, 010010011, 010010101, 010010111, 011000011, 011010001, 011010010, 011010011, 011010012, 011010100, 011010101, 011010111, 011020011, 012010011, 021010011, 100000011, 101000011, 101010010, 101010011, 101010100, 110000111, 111000011, 111010011, 111010111, 111\mathrm{m}10111\}.$$

Non-planar topologies

$$G^{N12}_{a_1\cdots a_9}(x,s,\epsilon) := e^{2\gamma_E\epsilon}\int\frac{d^dk_1}{i\pi^{d/2}}\frac{d^dk_2}{i\pi^{d/2}}\frac{1}{k_1^{2a_1}(k_1+xp_1)^{2a_2}(k_1+xp_{12})^{2a_3}(k_1+p_{123})^{2a_4}}$$
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$$G^{N13}_{a_1\cdots a_9}(x,s,\epsilon) := e^{2\gamma_E\epsilon}\int\frac{d^dk_1}{i\pi^{d/2}}\frac{d^dk_2}{i\pi^{d/2}}\frac{1}{k_1^{2a_1}(k_1+xp_1)^{2a_2}(k_1+xp_{12})^{2a_3}(k_1+p_{123})^{2a_4}}$$
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$$G^{N34}_{a_1\cdots a_9}(x,s,\epsilon) := e^{2\gamma_E\epsilon}\int\frac{d^dk_1}{i\pi^{d/2}}\frac{d^dk_2}{i\pi^{d/2}}\frac{1}{k_1^{2a_1}(k_1+xp_1)^{2a_2}(k_1+xp_{12})^{2a_3}(k_1+p_{123})^{2a_4}}$$
$$\times\frac{1}{k_2^{2a_5}(k_2-xp_1)^{2a_6}(k_2-p_{123})^{2a_7}(k_1+k_2+xp_{12}-p_{123})^{2a_8}(k_1+k_2)^{2a_9}}.$$

Non-planar topologies

N_{12} : {100001010, 000110010, 000110001, 000101010, 000101001, 101010010, 100110010, 100101020, 100101010, 100101001, 001110010, 001110002, 001110001, 001101001, 101110020, 101110010, 101101002, 101101001, 100111020, 100111010, 100102011, 100101011, 001120011, 001111002, 001111001, 001110011, 000111011, 101011011, 100111011, 1m0111011, 001111011, 0m1111011, 101111011, 1m1111011, 1m1111m11},

N_{13} : {010000110, 000110010, 001000101, 001000110, 001010001, 010110100, 001110100, 001010102, 001110002, 000110110, 001010101, 001010110, 001100110, 001110001, 001110010, 010100110, 010110101, 002010111, 001120011, 001210110, 011010102, 001110120, 001010111, 001110210, 001110011, 001110101, 001110110, 002110110, 011000111, 011010101, 011100110, 011110001, 011110110, m11010111, 010110111, m01110111, 0m1110111, 00111m111, 001110111, 011010111, 011110101, 011110111, m11110111},

N_{34} : {001001010, 001010010, 010010010, 100000110, 100010010, 000010111, 010010110, 001010102, 001010101, 010010101, 001020011, 010000111, 001010011, 010010011, 101010020, 101010010, 101010100, 101000011, 110010120, 110010110, 010010112, 010010121, 010010111, 010020111, 020010111, 011010102, 001010111, 011010101, 110000211, 011020011, 110000111, 011010011, 111000101, 111010010, 101010101, 101010011, 111010110, 111010101, 101010111, 11m010111, 110m10111, 11001m111, 110010111, m11010111, 011m10111, 01101m111, 011010111, 111000111, 111010011, 111010111, 111m10111}.

GP-indices

$$I(P_{12}) = \left\{0, 1, \frac{q}{s_{12}}, \frac{s_{12}}{q}, \frac{q}{q - s_{23}}, 1 - \frac{s_{23}}{q}, 1 + \frac{s_{23}}{s_{12}}, \frac{s_{12}}{s_{12} + s_{23}}\right\},$$

$$I(P_{13}) = \left\{0, 1, \frac{q}{s_{12}}, \frac{s_{12} + s_{23}}{s_{12}}, \frac{q}{q - s_{23}}, \xi_-, \xi_+, \frac{q(q - s_{23})}{q^2 - (q + s_{12})s_{23}}\right\},$$

$$I(P_{23}) = \left\{0, 1, \frac{q}{s_{12}}, 1 + \frac{s_{23}}{s_{12}}, \frac{q}{q - s_{23}}, \frac{q}{s_{12} + s_{23}}, \frac{q - s_{23}}{s_{12}}\right\},$$

$$\xi_\pm = \frac{q s_{12} \pm \sqrt{q s_{12} s_{23}(-q + s_{12} + s_{23})}}{q s_{12} - s_{12} s_{23}}.$$

$$I(N_{12}) = I(P_{23}),$$

$$I(N_{34}) = I(P_{12}) \cup I(P_{23}) \cup \left\{\frac{s_{12}}{q - s_{23}}, \frac{s_{12} + s_{23}}{q}, \frac{q^2 - q s_{23} - s_{12} s_{23}}{s_{12}(q - s_{23})}, \frac{s_{12}^2 + q s_{23} + s_{12} s_{23}}{s_{12}(s_{12} + s_{23})}\right\},$$

$$I(N_{13}) = I(P_{23}) \cup \left\{\xi_-, \xi_+, 1 + \frac{q}{s_{12}} + \frac{q}{-q + s_{23}}\right\}.$$

Example

$$G^{P_{13}}_{011111011}(x,s,\epsilon) = \frac{A_3(\epsilon)}{x^2 s_{12}(-q+x(q-s_{23}))^2}\left\{\frac{-1}{2\epsilon^4}+\frac{1}{\epsilon^3}\left(-GP\left(\frac{q}{s_{12}};x\right)+2\,GP\left(\frac{q}{q-s_{23}};x\right)\right.\right.$$

$$\left.+2\ GP(0;x)-GP(1;x)+\log\left(-s_{12}\right)+\frac{9}{4}\right)+\frac{1}{4\epsilon^2}\left(18\ GP\left(\frac{q}{s_{12}};x\right)-36\ GP\left(\frac{q}{q-s_{23}};x\right)\right.$$

$$\left.-8\ GP\left(0,\frac{q}{s_{12}};x\right)+16\ GP\left(0,\frac{q}{q-s_{23}};x\right)+8\ GP\left(\frac{s_{23}}{s_{12}}+1,\frac{q}{q-s_{23}};x\right)+\cdots\right)$$

$$+\frac{1}{\epsilon}\left(9\left(GP\left(0,\frac{q}{s_{12}};x\right)+GP(0,1;x)\right)-4\left(GP\left(0,0,\frac{q}{s_{12}};x\right)+GP(0,0,1;x)\right)+\cdots\right)$$

$$\left.+6\left(GP\left(0,0,1,\xi_-;x\right)+GP\left(0,0,1,\xi_+;x\right)\right)-2\,GP\left(0,0,\frac{q}{q-s_{23}},\frac{q\left(q-s_{23}\right)}{q^2-s_{23}\left(q+s_{12}\right)};x\right)+\cdots\right\}.$$

$$A_3(\epsilon) = -e^{2\gamma_E\epsilon}\frac{\Gamma(1-\epsilon)^3\Gamma(1+2\epsilon)}{\Gamma(3-3\epsilon)}.$$

C. G. Papadopoulos, D. Tommasini and C. Wever, JHEP **1501** (2015) 072

5BOX - ONE LEG OFF-SHELL: ALL FAMILIES

C. G. Papadopoulos, D. Tommasini and C. Wever, arXiv:1511.09404 [hep-ph].

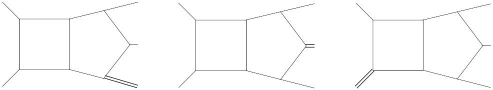

FIGURE : The three planar pentaboxes of the families P_1 (left), P_2 (middle) and P_3 (right) with one external massive leg.

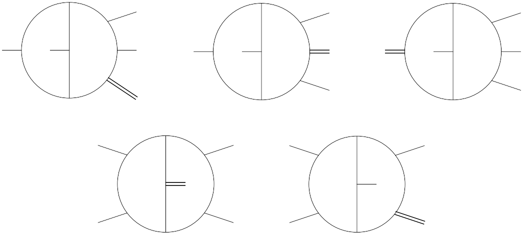

FIGURE : The five non-planar families with one external massive leg.

$$p(q_1)p'(q_2) \to V(q_3)j_1(q_4)j_2(q_5), \quad q_1^2 = q_2^2 = 0, \quad q_3^2 = M_3^2, \quad q_4^2 = q_5^2 = 0.$$

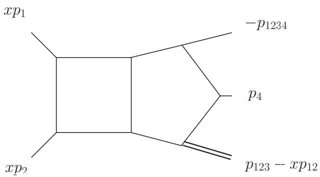

FIGURE : The parametrization of external momenta in terms of x for the planar pentabox of the family P_1. All external momenta are incoming.

$$s_{12} := p_{12}^2, \quad s_{23} := p_{23}^2, \quad s_{34} := p_{34}^2, \quad s_{45} := p_{45}^2 = p_{123}^2, \quad s_{51} := p_{15}^2 = p_{234}^2,$$

$$q_1^2 = q_2^2 = q_4^2 = q_5^2 = 0 \quad q_3^2 = (s_{45} - s_{12}x)(1-x)$$
$$q_{12}^2 = s_{12}x^2 \;\; q_{23}^2 = s_{45}(1-x) + s_{23}x \;\; q_{34}^2 = (s_{34} - s_{12}(1-x))\,x \;\; q_{45}^2 = s_{45} \;\; q_{51}^2 = s_{51}x$$

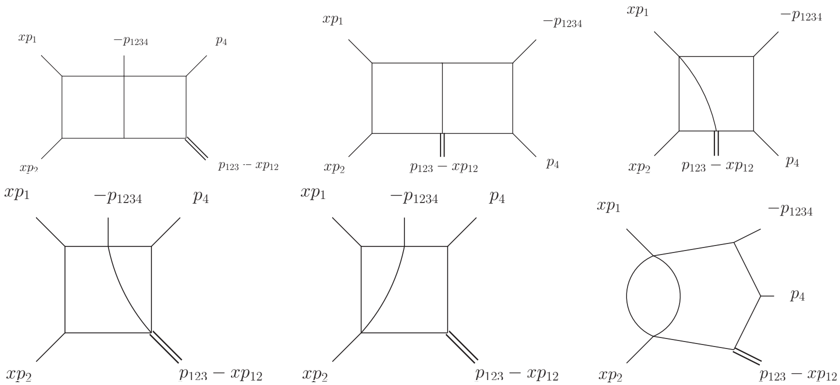

FIGURE : The five-point Feynman diagrams, besides the pentabox itself in Figure 4, that are contained in the family P_1. All external momenta are incoming.

5BOX - ONE LEG OFF-SHELL: P1

$$G^{P_1}_{a_1\cdots a_{11}}(x,s,\epsilon) := e^{2\gamma_E\epsilon}\int \frac{d^dk_1}{i\pi^{d/2}}\frac{d^dk_2}{i\pi^{d/2}}\frac{1}{k_1^{2a_1}(k_1+xp_1)^{2a_2}(k_1+xp_{12})^{2a_3}(k_1+p_{123})^{2a_4}}$$

$$\times\frac{1}{(k_1+p_{1234})^{2a_5}k_2^{2a_6}(k_2-xp_1)^{2a_7}(k_2-xp_{12})^{2a_8}(k_2-p_{123})^{2a_9}(k_2-p_{1234})^{2a_{10}}(k_1+k_2)^{2a_{11}}},$$

P_1 : {10000000101, 01000000101, 00100000101, 10000001001, 01000000011, 00100000011, 10100001100,
10100001010, 10100101000, 01000101001, 10100100100, 10100000102, 10100000101, 10100000011,
10000001102, 10000001101, 10000001011, 01000100101, 01000001101, 01000001011, 00100100102,
00100100101, 11100000101, 11100000011, 11000001102, 11000001101, 11000001012, 11000001011,
11000000111, 10100000112, 10000001111, 01100100102, 01100100101, 01100100011, 01100000111,
01000101102, 01000101101, 01000101011, 01000100111, 01000001111, 00100100111, 10100101100,
10100100101, 10100001101, 10100001011, 10100000111, 111m0000111, 110000m1111, 11000001111,
10100101110, 10100100111, 10100001111, 011001m0111, 01100100111, 010m0101111, 01000101111,
11100100101, 11100001101, 11100001011, 11100000111, 111m0101101, 111001m1101, 11100101101,
1110m101011, 11100101011, 111m0100111, 11100100111, 111000m1111, 111m0001111, 11100001111,
111001m0111, 11100101111, 111001m1111, 111m0101111},

Choosing m= −1 or 2

$$\partial_x \mathbf{G} = \mathbf{M}\left(\{s_{ij}\}, \varepsilon, x\right)\mathbf{G}$$

$$(M_D)_{IJ} = \delta_{IJ} M_{II}(\varepsilon = 0), I, J = 1 \ldots 74$$

$\mathbf{G} \to \mathbf{S}^{-1}\mathbf{G}$, $\mathbf{S} = \exp\left(\int dx\, \mathbf{M}_D\right)$ and $\mathbf{M} \to \mathbf{S}^{-1}(\mathbf{M} - \mathbf{M}_D)\mathbf{S}$.

$$M_{IJ} = N_{IJ}(\varepsilon)\left(\sum_{i=1}^{20}\sum_{j=1}^{2}\sum_{k=0}^{1} \frac{C_{IJ;ijk}\varepsilon^k}{(x - l_i)^j} + \sum_{j=0}^{1}\sum_{k=0}^{1} \tilde{C}_{IJ;jk}\varepsilon^k x^j\right).$$

Letters (20):

$$0,\quad 1,\quad \frac{s_{45}}{s_{45}-s_{23}},\quad \frac{s_{45}}{s_{12}},\quad 1-\frac{s_{34}}{s_{12}},\quad 1+\frac{s_{23}}{s_{12}},$$

$$1-\frac{s_{34}-s_{51}}{s_{12}},\quad \frac{s_{45}-s_{23}}{s_{12}},\quad -\frac{s_{51}}{s_{12}},\quad \frac{s_{45}}{-s_{23}+s_{45}+s_{51}},\quad \frac{s_{45}}{s_{34}+s_{45}},$$

$$\frac{s_{12}s_{23}-2s_{12}s_{45}-s_{12}s_{51}-s_{23}s_{34}+s_{34}s_{45}-s_{45}s_{51}\pm\sqrt{\Delta_1}}{2s_{12}(s_{23}-s_{45}-s_{51})},\quad \frac{s_{12}s_{23}-s_{12}s_{45}-s_{12}s_{51}-s_{23}s_{34}+s_{34}s_{45}-s_{45}s_{51}\pm\sqrt{\Delta_2}}{2s_{12}(s_{23}-s_{45}-s_{51})},$$

$$\frac{s_{12}s_{23}-s_{12}s_{51}-s_{23}s_{34}+s_{34}s_{45}-s_{45}s_{51}\pm\sqrt{\Delta_1}}{2s_{12}(s_{23}+s_{34}-s_{51})},\quad \frac{s_{12}s_{45}\pm\sqrt{\Delta_3}}{s_{12}s_{34}+s_{12}s_{45}},\quad \frac{s_{45}}{s_{12}+s_{23}},$$

$$\Delta_1 = (s_{12}(s_{51}-s_{23}) + s_{23}s_{34} + s_{45}(s_{51}-s_{34}))^2 + 4s_{12}s_{45}s_{51}(s_{23}+s_{34}-s_{51})$$
$$\Delta_2 = (s_{12}(-s_{23}+s_{45}+s_{51}) + s_{23}s_{34} + s_{45}(s_{51}-s_{34}))^2 - 4s_{12}s_{45}s_{51}(-s_{23}+s_{45}+s_{51})$$
$$\Delta_3 = -(s_{12}s_{34}s_{45}(s_{12}-s_{34}-s_{45}))$$

5BOX P1 - DE

$$\partial_x \mathbf{G} = \mathbf{M}\left(\{s_{ij}\}, \varepsilon, x\right)\mathbf{G}$$

$$\left(M_D\right)_{IJ} = \delta_{IJ}\, M_{II}\left(\varepsilon = 0\right), I, J = 1\ldots 74$$

$$\mathbf{G} \to \mathbf{S}^{-1}\mathbf{G},\ \mathbf{S} = \exp\left(\int dx\, \mathbf{M}_D\right) \text{ and } \mathbf{M} \to \mathbf{S}^{-1}\left(\mathbf{M} - \mathbf{M}_D\right)\mathbf{S}.$$

$$M_{IJ} = N_{IJ}\left(\varepsilon\right)\left(\sum_{i=1}^{20}\sum_{j=1}^{2}\sum_{k=0}^{1}\frac{C_{IJ;ijk}\varepsilon^k}{\left(x - l_i\right)^j} + \sum_{j=0}^{1}\sum_{k=0}^{1}\tilde{C}_{IJ;jk}\varepsilon^k x^j\right).$$

Letters (20):

$$0,\quad 1,\quad \frac{s_{45}}{s_{45}-s_{23}},\quad \frac{s_{45}}{s_{12}},\quad 1-\frac{s_{34}}{s_{12}},\quad 1+\frac{s_{23}}{s_{12}},$$

$$1-\frac{s_{34}-s_{51}}{s_{12}},\quad \frac{s_{45}-s_{23}}{s_{12}},\quad -\frac{s_{51}}{s_{12}},\quad \frac{s_{45}}{-s_{23}+s_{45}+s_{51}},\quad \frac{s_{45}}{s_{34}+s_{45}},$$

$$\frac{s_{12}s_{23}-2s_{12}s_{45}-s_{12}s_{51}-s_{23}s_{34}+s_{34}s_{45}-s_{45}s_{51}\pm\sqrt{\Delta_1}}{2s_{12}(s_{23}-s_{45}-s_{51})},\quad \frac{s_{12}s_{23}-s_{12}s_{45}-s_{12}s_{51}-s_{23}s_{34}+s_{34}s_{45}-s_{45}s_{51}\pm\sqrt{\Delta_2}}{2s_{12}(s_{23}-s_{45}-s_{51})},$$

$$\frac{s_{12}s_{23}-s_{12}s_{51}-s_{23}s_{34}+s_{34}s_{45}-s_{45}s_{51}\pm\sqrt{\Delta_1}}{2s_{12}(s_{23}+s_{34}-s_{51})},\quad \frac{s_{12}s_{45}\pm\sqrt{\Delta_3}}{s_{12}s_{34}+s_{12}s_{45}},\quad \frac{s_{45}}{s_{12}+s_{23}},$$

$$\begin{aligned}
\Delta_1 &= (s_{12}(s_{51}-s_{23})+s_{23}s_{34}+s_{45}(s_{51}-s_{34}))^2+4s_{12}s_{45}s_{51}(s_{23}+s_{34}-s_{51})\\
\Delta_2 &= (s_{12}(-s_{23}+s_{45}+s_{51})+s_{23}s_{34}+s_{45}(s_{51}-s_{34}))^2-4s_{12}s_{45}s_{51}(-s_{23}+s_{45}+s_{51})\\
\Delta_3 &= -(s_{12}s_{34}s_{45}(s_{12}-s_{34}-s_{45}))
\end{aligned}$$

5box P1 - DE

$$\partial_x \mathbf{G} = \mathbf{M}\left(\{s_{ij}\}, \varepsilon, x\right)\mathbf{G}$$

$$(M_D)_{IJ} = \delta_{IJ} M_{II}(\varepsilon = 0), I, J = 1 \ldots 74$$

$$\mathbf{G} \to \mathbf{S}^{-1}\mathbf{G},\ \mathbf{S} = \exp\left(\int dx\, \mathbf{M}_D\right) \text{ and } \mathbf{M} \to \mathbf{S}^{-1}\left(\mathbf{M} - \mathbf{M}_D\right)\mathbf{S}.$$

$$M_{IJ} = N_{IJ}(\varepsilon)\left(\sum_{i=1}^{20}\sum_{j=1}^{2}\sum_{k=0}^{1}\frac{C_{IJ;ijk}\varepsilon^k}{(x-l_i)^j} + \sum_{j=0}^{1}\sum_{k=0}^{1}\tilde{C}_{IJ;jk}\varepsilon^k x^j\right).$$

Letters (20):

$$0,\quad 1,\quad \frac{s_{45}}{s_{45}-s_{23}},\quad \frac{s_{45}}{s_{12}},\quad 1-\frac{s_{34}}{s_{12}},\quad 1+\frac{s_{23}}{s_{12}},$$

$$1-\frac{s_{34}-s_{51}}{s_{12}},\quad \frac{s_{45}-s_{23}}{s_{12}},\quad -\frac{s_{51}}{s_{12}},\quad \frac{s_{45}}{-s_{23}+s_{45}+s_{51}},\quad \frac{s_{45}}{s_{34}+s_{45}},$$

$$\frac{s_{12}s_{23}-2s_{12}s_{45}-s_{12}s_{51}-s_{23}s_{34}+s_{34}s_{45}-s_{45}s_{51}\pm\sqrt{\Delta_1}}{2s_{12}(s_{23}-s_{45}-s_{51})},\quad \frac{s_{12}s_{23}-s_{12}s_{45}-s_{12}s_{51}-s_{23}s_{34}+s_{34}s_{45}-s_{45}s_{51}\pm\sqrt{\Delta_2}}{2s_{12}(s_{23}-s_{45}-s_{51})},$$

$$\frac{s_{12}s_{23}-s_{12}s_{51}-s_{23}s_{34}+s_{34}s_{45}-s_{45}s_{51}\pm\sqrt{\Delta_1}}{2s_{12}(s_{23}+s_{34}-s_{51})},\quad \frac{s_{12}s_{45}\pm\sqrt{\Delta_3}}{s_{12}s_{34}+s_{12}s_{45}},\quad \frac{s_{45}}{s_{12}+s_{23}},$$

$$\Delta_1 = (s_{12}(s_{51}-s_{23})+s_{23}s_{34}+s_{45}(s_{51}-s_{34}))^2+4s_{12}s_{45}s_{51}(s_{23}+s_{34}-s_{51})$$
$$\Delta_2 = (s_{12}(-s_{23}+s_{45}+s_{51})+s_{23}s_{34}+s_{45}(s_{51}-s_{34}))^2-4s_{12}s_{45}s_{51}(-s_{23}+s_{45}+s_{51})$$
$$\Delta_3 = -(s_{12}s_{34}s_{45}(s_{12}-s_{34}-s_{45}))$$

5box P1 - DE

$$\partial_x \mathbf{G} = \mathbf{M}\left(\left\{s_{ij}\right\}, \varepsilon, x\right) \mathbf{G}$$

$$\left(M_D\right)_{IJ} = \delta_{IJ}\, M_{II}\left(\varepsilon = 0\right), I, J = 1 \ldots 74$$

$$\mathbf{G} \rightarrow \mathbf{S}^{-1}\mathbf{G},\ \mathbf{S} = \exp\left(\int dx\, \mathbf{M}_D\right) \text{ and } \mathbf{M} \rightarrow \mathbf{S}^{-1}\left(\mathbf{M} - \mathbf{M}_D\right)\mathbf{S}.$$

$$M_{IJ} = N_{IJ}\left(\varepsilon\right)\left(\sum_{i=1}^{20}\sum_{j=1}^{2}\sum_{k=0}^{1}\frac{C_{IJ;ijk}\varepsilon^k}{(x-l_i)^j} + \sum_{j=0}^{1}\sum_{k=0}^{1}\tilde{C}_{IJ;jk}\varepsilon^k x^j\right).$$

Letters (20):

$$0,\quad 1,\quad \frac{s_{45}}{s_{45}-s_{23}},\quad \frac{s_{45}}{s_{12}},\quad 1-\frac{s_{34}}{s_{12}},\quad 1+\frac{s_{23}}{s_{12}},$$

$$1-\frac{s_{34}-s_{51}}{s_{12}},\quad \frac{s_{45}-s_{23}}{s_{12}},\quad -\frac{s_{51}}{s_{12}},\quad \frac{s_{45}}{-s_{23}+s_{45}+s_{51}},\quad \frac{s_{45}}{s_{34}+s_{45}},$$

$$\frac{s_{12}s_{23}-2s_{12}s_{45}-s_{12}s_{51}-s_{23}s_{34}+s_{34}s_{45}-s_{45}s_{51}\pm\sqrt{\Delta_1}}{2s_{12}(s_{23}-s_{45}-s_{51})},\quad \frac{s_{12}s_{23}-s_{12}s_{45}-s_{12}s_{51}-s_{23}s_{34}+s_{34}s_{45}-s_{45}s_{51}\pm\sqrt{\Delta_2}}{2s_{12}(s_{23}-s_{45}-s_{51})},$$

$$\frac{s_{12}s_{23}-s_{12}s_{51}-s_{23}s_{34}+s_{34}s_{45}-s_{45}s_{51}\pm\sqrt{\Delta_1}}{2s_{12}(s_{23}+s_{34}-s_{51})},\quad \frac{s_{12}s_{45}\pm\sqrt{\Delta_3}}{s_{12}s_{34}+s_{12}s_{45}},\quad \frac{s_{45}}{s_{12}+s_{23}},$$

$$\Delta_1 = \left(s_{12}(s_{51}-s_{23})+s_{23}s_{34}+s_{45}(s_{51}-s_{34})\right)^2+4s_{12}s_{45}s_{51}(s_{23}+s_{34}-s_{51})$$
$$\Delta_2 = \left(s_{12}(-s_{23}+s_{45}+s_{51})+s_{23}s_{34}+s_{45}(s_{51}-s_{34})\right)^2-4s_{12}s_{45}s_{51}(-s_{23}+s_{45}+s_{51})$$
$$\Delta_3 = -\left(s_{12}s_{34}s_{45}(s_{12}-s_{34}-s_{45})\right)$$

5BOX P1 - DE

$$M_{IJ} = N_{IJ}(\varepsilon)\left(\sum_{i=1}^{20}\sum_{j=1}^{2}\sum_{k=0}^{1}\frac{C_{IJ;ijk}\varepsilon^k}{(x-l_i)^j} + \sum_{j=0}^{1}\sum_{k=0}^{1}\tilde{C}_{IJ;jk}\varepsilon^k x^j\right).$$

$$\int_0^x dt \frac{1}{(t-a_n)^2}\mathcal{G}(a_{n-1},\dots,a_1,t) \qquad \int_0^x dt\ t^m\ \mathcal{G}(a_{n-1},\dots,a_1,t)$$

Fuchsian

$N_{IJ}(\varepsilon) = n_J(\varepsilon)/n_I(\varepsilon),\ G_I \to n_I(\varepsilon) G_I$

$$M_{IJ} = \left(\sum_{i=1}^{20}\sum_{j=1}^{2}\sum_{k=0}^{1}\frac{C_{IJ;ijk}\varepsilon^k}{(x-l_i)^j} + \sum_{j=0}^{1}\sum_{k=0}^{1}\tilde{C}_{IJ;jk}\varepsilon^k x^j\right).$$

$$\mathbf{G} \to (\mathbf{I}-\mathbf{K}_i)\,\mathbf{G}, \quad \mathbf{M} \to (\mathbf{M} - \partial_x\mathbf{K}_i - \mathbf{K}_i\mathbf{M})(\mathbf{I}-\mathbf{K}_i)^{-1} \quad i = 1,2,3$$

$$\partial_x \mathbf{G} = \left(\varepsilon\sum_{a=1}^{19}\frac{\mathbf{M}_a}{(x-l_a)}\right)\mathbf{G}$$

5BOX P1 - DE

Fuchsian

$N_{IJ}(\varepsilon) = n_J(\varepsilon)/n_I(\varepsilon),\ G_I \to n_I(\varepsilon)\, G_I$

$$M_{IJ} = \left(\sum_{i=1}^{20} \sum_{j=1}^{2} \sum_{k=0}^{1} \frac{C_{IJ;ijk}\varepsilon^k}{(x-l_i)^j} + \sum_{j=0}^{1} \sum_{k=0}^{1} \tilde{C}_{IJ;jk}\varepsilon^k x^j \right).$$

$$\mathbf{G} \to (\mathbf{I} - \mathbf{K}_i)\,\mathbf{G}, \quad \mathbf{M} \to (\mathbf{M} - \partial_x \mathbf{K}_i - \mathbf{K}_i \mathbf{M})(\mathbf{I} - \mathbf{K}_i)^{-1} \quad i = 1,2,3$$

$$\partial_x \mathbf{G} = \left(\varepsilon \sum_{a=1}^{19} \frac{\mathbf{M}_a}{(x - l_a)} \right) \mathbf{G}$$

5BOX P1 - DE

Fuchsian

$N_{IJ}(\varepsilon) = n_J(\varepsilon)/n_I(\varepsilon)$, $G_I \to n_I(\varepsilon)\, G_I$

$$M_{IJ} = \left(\sum_{i=1}^{20} \sum_{j=1}^{2} \sum_{k=0}^{1} \frac{C_{IJ;ijk}\varepsilon^k}{(x-l_i)^j} + \sum_{j=0}^{1} \sum_{k=0}^{1} \tilde{C}_{IJ;jk}\varepsilon^k x^j \right).$$

$$\mathbf{G} \to (\mathbf{I}-\mathbf{K}_i)\,\mathbf{G}, \quad \mathbf{M} \to (\mathbf{M} - \partial_x \mathbf{K}_i - \mathbf{K}_i \mathbf{M})\,(\mathbf{I}-\mathbf{K}_i)^{-1} \quad i = 1,2,3$$

$\mathbf{M}(\varepsilon = 0)$ contains $(x-l_i)^{-2}$ and x^0

$$(\mathbf{K}_1)_{IJ} = \begin{cases} \int dx (\mathbf{M}(\varepsilon=0))_{IJ} & I,J \neq 69,74 \\ 0 & I,J = 69,74 \end{cases}$$

$$(\mathbf{K}_2)_{IJ} = \begin{cases} \int dx (\mathbf{M}(\varepsilon=0))_{IJ} & I,J \neq 74 \\ 0 & I,J = 74 \end{cases}$$

$$(\mathbf{K}_3)_{IJ} = \int dx (\mathbf{M}(\varepsilon=0))_{IJ}$$

M.A. Barkatou and E.Pflügel, Journal of Symbolic Computation, **44** (2009),1017

$$\partial_x \mathbf{G} = \left(\varepsilon \sum_{a=1}^{19} \frac{\mathbf{M}_a}{(x-l_a)} \right) \mathbf{G}$$

Fuchsian

$N_{IJ}(\varepsilon) = n_J(\varepsilon)/n_I(\varepsilon), \; G_I \to n_I(\varepsilon)\, G_I$

$$M_{IJ} = \left(\sum_{i=1}^{20} \sum_{j=1}^{2} \sum_{k=0}^{1} \frac{C_{IJ;ijk}\varepsilon^k}{(x-l_i)^j} + \sum_{j=0}^{1} \sum_{k=0}^{1} \tilde{C}_{IJ;jk}\varepsilon^k x^j \right).$$

$$\mathbf{G} \to (\mathbf{I} - \mathbf{K}_i)\,\mathbf{G}, \quad \mathbf{M} \to (\mathbf{M} - \partial_x \mathbf{K}_i - \mathbf{K}_i \mathbf{M})(\mathbf{I} - \mathbf{K}_i)^{-1} \quad i = 1,2,3$$

$$\partial_x \mathbf{G} = \left(\varepsilon \sum_{a=1}^{19} \frac{\mathbf{M}_a}{(x-l_a)} \right) \mathbf{G}$$

5box P1 - solution

- Solution:

$$
\begin{aligned}
\mathbf{G} \;=\;& \varepsilon^{-2}\mathbf{b}_0^{(-2)} + \varepsilon^{-1}\left(\sum \mathcal{G}_a \mathbf{M}_a \mathbf{b}_0^{(-2)} + \mathbf{b}_0^{(-1)}\right) \\
+\;& \varepsilon^{0}\left(\sum \mathcal{G}_{ab} \mathbf{M}_a \mathbf{M}_b \mathbf{b}_0^{(-2)} + \sum \mathcal{G}_a \mathbf{M}_a \mathbf{b}_0^{(-1)} + \mathbf{b}_0^{(0)}\right) \\
+\;& \varepsilon\left(\sum \mathcal{G}_{abc} \mathbf{M}_a \mathbf{M}_b \mathbf{M}_c \mathbf{b}_0^{(-2)} + \sum \mathcal{G}_{ab} \mathbf{M}_a \mathbf{M}_b \mathbf{b}_0^{(-1)} + \sum \mathcal{G}_a \mathbf{M}_a \mathbf{b}_0^{(0)} + \mathbf{b}_0^{(1)}\right) \\
+\;& \varepsilon^{2}\left(\sum \mathcal{G}_{abcd} \mathbf{M}_a \mathbf{M}_b \mathbf{M}_c \mathbf{M}_d \mathbf{b}_0^{(-2)} + \sum \mathcal{G}_{abc} \mathbf{M}_a \mathbf{M}_b \mathbf{M}_c \mathbf{b}_0^{(-1)}\right. \\
+\;& \left.\sum \mathcal{G}_{ab} \mathbf{M}_a \mathbf{M}_b \mathbf{b}_0^{(0)} + \sum \mathcal{G}_a \mathbf{M}_a \mathbf{b}_0^{(1)} + \mathbf{b}_0^{(2)}\right)
\end{aligned}
$$

$\mathbf{b}_0^{(k)}$, $k = -2, \ldots, 2$ representing the x-independent boundary terms in the limit $x = 0$ at order ε^k

$$
\mathbf{G} \underset{x \to 0}{\sim} \sum_{k=-2}^{2} \varepsilon^k \sum_{n=0}^{k+2} \mathbf{b}_n^{(k)} \log^n(x) + \text{subleading terms.}
$$

$\mathcal{G}_{a,b,\ldots} = \mathcal{G}(l_a, l_b, \ldots ; x)$ with $a, b, c, d = 1, \ldots, 19$.

- Uniform transcendental: UT multi- vs one-parameter DE

$\mathbf{M}_a$ depend on kinematics, but eigenvalues not: $(x - l_a)^{-n_a \varepsilon}$, n_a positive integers, $x \to l_a$.

5BOX P1 - SOLUTION

- Solution:

$$\begin{aligned}\mathbf{G} &= \varepsilon^{-2}\mathbf{b}_0^{(-2)} + \varepsilon^{-1}\left(\sum \mathcal{G}_a \mathbf{M}_a \mathbf{b}_0^{(-2)} + \mathbf{b}_0^{(-1)}\right) \\ &+ \varepsilon^0 \left(\sum \mathcal{G}_{ab}\mathbf{M}_a\mathbf{M}_b\mathbf{b}_0^{(-2)} + \sum \mathcal{G}_a\mathbf{M}_a\mathbf{b}_0^{(-1)} + \mathbf{b}_0^{(0)}\right) \\ &+ \varepsilon \left(\sum \mathcal{G}_{abc}\mathbf{M}_a\mathbf{M}_b\mathbf{M}_c\mathbf{b}_0^{(-2)} + \sum \mathcal{G}_{ab}\mathbf{M}_a\mathbf{M}_b\mathbf{b}_0^{(-1)} + \sum \mathcal{G}_a\mathbf{M}_a\mathbf{b}_0^{(0)} + \mathbf{b}_0^{(1)}\right) \\ &+ \varepsilon^2 \left(\sum \mathcal{G}_{abcd}\mathbf{M}_a\mathbf{M}_b\mathbf{M}_c\mathbf{M}_d\mathbf{b}_0^{(-2)} + \sum \mathcal{G}_{abc}\mathbf{M}_a\mathbf{M}_b\mathbf{M}_c\mathbf{b}_0^{(-1)}\right. \\ &+ \left.\sum \mathcal{G}_{ab}\mathbf{M}_a\mathbf{M}_b\mathbf{b}_0^{(0)} + \sum \mathcal{G}_a\mathbf{M}_a\mathbf{b}_0^{(1)} + \mathbf{b}_0^{(2)}\right)\end{aligned}$$

$\mathbf{b}_0^{(k)}$, $k = -2, \ldots, 2$ representing the x-independent boundary terms in the limit $x = 0$ at order ε^k

$$\mathbf{G} \underset{x\to 0}{\sim} \sum_{k=-2}^{2} \varepsilon^k \sum_{n=0}^{k+2} \mathbf{b}_n^{(k)} \log^n(x) + \text{subleading terms.}$$

$\mathcal{G}_{a,b,\ldots} = \mathcal{G}(l_a, l_b, \ldots; x)$ with $a, b, c, d = 1, \ldots, 19$.

- Uniform transcendental: UT multi- vs one-parameter DE

  $\mathbf{M}_a$ depend on kinematics, but eigenvalues not: $(x - l_a)^{-n_a\varepsilon}$, n_a positive integers, $x \to l_a$.

5BOX - BOUNDARY TERMS

- Resumed

$$G_{res} = \lim_{x \to 0} G = \sum_j c_j x^{i_0 + j\epsilon} + d_j x^{i_0+1+j\epsilon} + \mathcal{O}(x^{i_0+2}),$$

- DE: using the above and equating terms $x^{i+j\epsilon}$, linear equations for c_i and d_i
- *bottom-up*: MI with homogeneous DE treated exactly
- MI needing special treatment (20)
  - Expansion by regions (11)
  - Shifted boundary point (6)
  - Extraction from known integrals (3)

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$$\{(10100000101), (10100000102), (11000001012), (11000001011), (01000101011), (10100100111),$$
$$(10100001111), (111m0100111), (111000m1111), (11100001111), (111001m0111)\}.$$

  - Shifted boundary point (6)

$$\begin{aligned} \infty : &\quad \{(10100000011), (10000001011), (11100000011), (01100100011), (10100100111)\} \\ (s_{12} - s_{34} + s_{51})/s_{12} : &\quad \{(01000001011)\} \end{aligned}$$

  - Extraction from known integrals (3)

$$\begin{aligned} G_{11100001011}(x, s_{12}, s_{34}, s_{51}) &= G_{11100100101}(x' = 1, s'_{12}, s'_{23}, s'_{45}), \\ G_{11100101011}(x, s_{12}, s_{34}, s_{51}) &= G_{11100101101}(x' = 1, s'_{12}, s'_{23}, s'_{45}), \\ G_{111m0101011}(x, s_{12}, s_{34}, s_{51}) &= G_{111m0101101}(x' = 1, s'_{12}, s'_{23}, s'_{45}), \\ s'_{12} = x^2 s_{12}, \qquad s'_{23} = x s_{51}, &\qquad s'_{45} = -x s_{12} + x s_{34} + x^2 s_{12}. \end{aligned} \tag{1}$$

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Systematic approach: combining information from the expansion by regions technique (asy2) and the DE itself

Mellin-Barnes, XSummer

5BOX - ON-SHELL

All planar one-shell 5box by taking the limit $x \to 1$.

- $x = 1$ corresponds to l_2

$$\mathbf{G} = \sum_{n \geq -2} \varepsilon^n \sum_{i=0}^{n+2} \frac{1}{i!} \mathbf{c}_i^{(n)} \log^i (1-x)$$

- with $\mathbf{M}_2$ the residue matrix at $x = 1$ and
- $\mathbf{G}_{trunc} \equiv \mathbf{G}_{reg}(x = 1)$

$$\mathbf{G}_{x=1} = \left(\mathbf{I} + \frac{3}{2}\mathbf{M}_2 + \frac{1}{2}\mathbf{M}_2^2 \right) \mathbf{G}_{trunc}$$

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$$\mathbf{c}_i^{(n)} = \mathbf{M}_2 \mathbf{c}_{i-1}^{(n-1)} \quad i \geq 1$$

$$\mathbf{G}_{reg} = \sum_{n \geq -2} \varepsilon^n \mathbf{c}_0^{(n)}.$$

characteristic polynomial: $x^{61}(1+x)^9(2+x)^4$

$$\mathbf{G} = \mathbf{G}_{reg} + \frac{\left((1-x)^{-2\varepsilon} - 1\right)}{(-2\varepsilon)}\mathbf{X} + \frac{\left((1-x)^{-\varepsilon} - 1\right)}{(-\varepsilon)}\mathbf{Y}$$

$$\mathbf{X} = \sum_{n \geq -1} \varepsilon^n \mathbf{X}^{(n)} \quad \mathbf{Y} = \sum_{n \geq -1} \varepsilon^n \mathbf{Y}^{(n)}.$$

$$(-1)^n {\mathbf{M}_2}^n = {\mathbf{M}_2}^2 \left(2^{n-1} - 1\right) + \mathbf{M}_2 \left(2^{n-1} - 2\right), \quad n \geq 1.$$

minimal polynomial: $x(x+1)(x+2)$

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5BOX - NUMERICAL CHECKS

- $\mathcal{O}(3,000)$ GPs for all 74 MI
- Directly computed by using **GiNaC**
- All invariants negative Euclidean: perfect agreement with `SecDec`
- $\mathcal{O}(10)$ secs.
  `HyperInt` analytic extraction of imaginary parts before numerics: increasing efficiency by $\mathcal{O}(100)$
- Physical region awaiting tests for 5boxes. Direct timing $\mathcal{O}(1000)$ secs.

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J. Vollinga and S. Weinzierl, Comput. Phys. Commun. **167** (2005) 177

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E. Panzer, Comput. Phys. Commun. **188** (2014) 148

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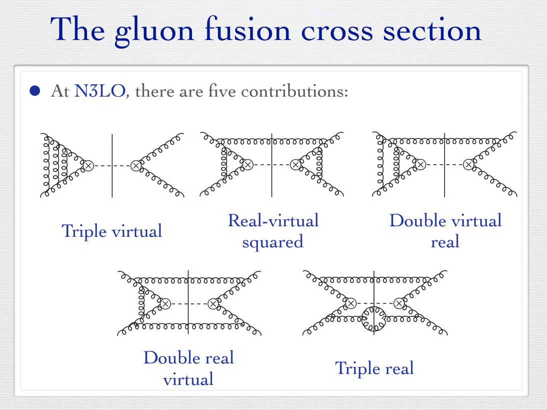
Recent calculations beyond NLO

Processes currently known through NNLO

H+0jet	fully inclusive N^3LO	Higgs couplings	1503.06056
H+1jet	exclusive	Higgs couplings	1604.04085,1408.5325,1504.07922, 1505.03893
WBF	exclusive VBF cuts	Higgs couplings	1506.02660
H->bb	exclusive, massless	Higgs couplings boosted	1110.2368,1501.07226
W+0jet	fully exclusive, decays	PDFs	0903.2120,1208.5967
Z/gamma+0jet	fully exclusive, decays	PDFs	0903.2120,1208.5967
W+j	fully exclusive, decays	PDFs	1504.02131
Z+j	decay, off-shell effects	PDFs	1601.04569,1507.20850, 1507.02850
ZH	decays to bb at NLO	Higgs couplings	1407.4747,1601.00658
WH	fully exclusive	Higgs couplings	1312.1669, 1601.00658
ZZ	fully exclusive, off-shell	trilinear gauge couplings,BSM	1405.2219, 1507.06257,1509.06734
WW	fully inclusive	trilinear gauge couplings,BSM	1408.5243,1511.08617
$W\gamma$,$Z\gamma$	fully exclusive	trilinear gauge couplings,BSM	1601.06751
$\gamma\gamma$	fully differential	Background studies	1110.2375,1603.02663
tt pair	fully exclusive, stable tops	top cross section ,mass pt, FB asymmetry,PDFs BSM	1601.05375, 1506.04037
single top	fully exclusive, stable tops, t-channel	Vtb,width, PDfs	1404.7116
top decay	exclusive	Top couplings	1210.2808, 1301.7133
dijets	gluon-gluon	PDFs,strong couplings,BSM	1407.5558

Adapted from K. Melnikov, Aspen Winter Conference 2016

QCD N3LO



C. Anastasiou, C. Duhr, F. Dulat, E. Furlan, T. Gehrmann, F. Herzog, A. Lazopoulos and B. Mistlberger, arXiv:1602.00695

Higgs at N3LO

- In the limit $m_t \to \infty$, the Higgs boson couples directly to gluons:

$$\mathcal{L} = \mathcal{L}_{QCD,5} - \frac{1}{4v} C_1 \, H \, G^a_{\mu\nu} \, G^{\mu\nu}_a$$

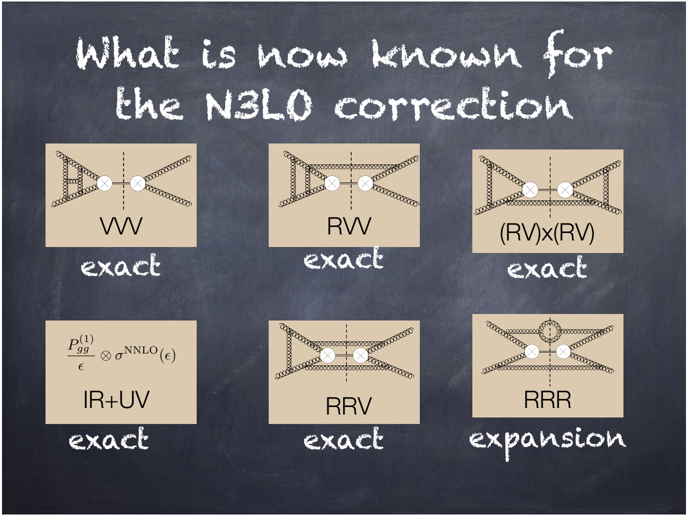
- In this limit, the cross section is known
  - ➡ at NLO. [Dawson; Djouadi, Spira, Zerwas]
  - ➡ at NNLO. [Anastasiou, Melnikov; Harlander, Kilgore; Ravindran, Smith, van Neerven]
  - ➡ at N3LO. [Anastasiou, Dulat, CD, Furlan, Gehrmann, Herzog, Lazopoulos, Mistlberger]

- The N3LO cross section is only known as an expansion around threshold:

$$\sigma = \tau \sum_{ij} \int_\tau^1 \frac{dz}{z} \, \mathcal{L}_{ij}(\tau/z) \, \frac{\hat{\sigma}_{ij}(z)}{z} \qquad z = \frac{m_H^2}{\hat{s}}$$

$$\hat{\sigma}(z) = \sigma_{-1} + \sigma_0 + (1-z)\,\sigma_1 + \mathcal{O}(1-z)^2 \qquad \tau = \frac{m_H^2}{S} \simeq 10^{-4}$$

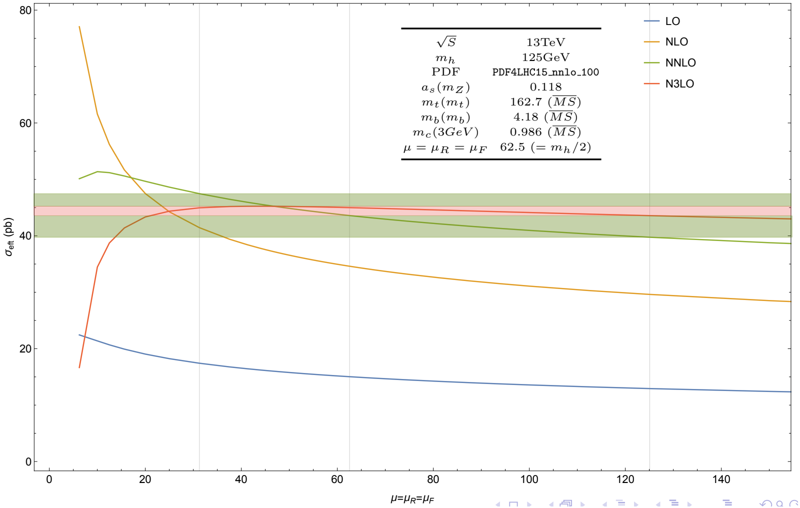
Anastasiou, Loops&Legs, April 2016



How tough of a problem?

- Two orders of magnitude more Feynman diagrams than NNLO
- 1028 N3LO master integrals (27 at NNLO)
- 72 boundary conditions for the N3LO master integrals (5 at NNLO)

Higgs at N3LO



Higgs at N3LO

$$\sigma = 48.48 \pm 1.55\,^{+2.07}_{-3.09}\,\text{pb} = 48.48\,\text{pb} \pm 3.19\%\,^{+4.27\%}_{-6.37\%}$$

- Most precise prediction of the Higgs cross section to date!
- Perturbative stability of the cross section under control.
  - ➡ Scale variation gives a reliable estimate of higher-order QCD corrections.
- Places where we can improve:
  - ➡ top-bottom interference at NNLO in QCD.
  - ➡ N3LO PDFs.
  - ➡ Exact mixed QCD-EW corrections.
  - ➡ NNLO corrections including exact top-mass dependence.

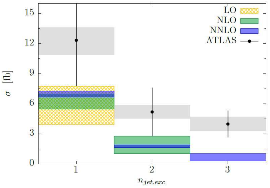
NNLO H + jet production, large mass limit

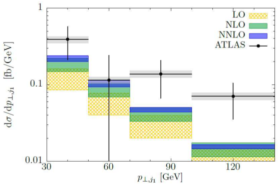
Boughezal, Caola, Melnikov, Petriello, Schulze (13,15), Chen, Gehrmann, Jaquier, NG (14), Boughezal, Focke, Giele, Liu, Petriello (15), Caola, Melnikov, Schulze (15)

- ✓ large K-factor

$$\sigma_{NLO}/\sigma_{LO} \sim 1.6$$
$$\sigma_{NNLO}/\sigma_{NLO} \sim 1.3$$

- ✓ significantly reduced scale dependence $\mathcal{O}(4\%)$
- ✓ Three independent computations:
  - ✚ STRIPPER
  - ✚ N-jettiness
  - ✚ Antenna (gluons only)
- ✓ Fully differential and allows for arbitrary cuts on the final state

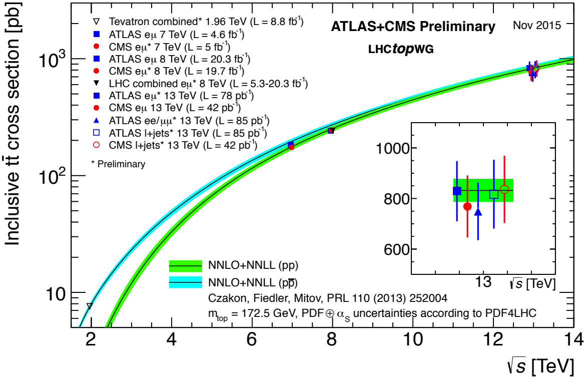




$t\bar{t}$ at NNLO

Czakon, Heymes, Mitov

Total Cross Section



General Remarks

- High precision should be associated with fixed order perturbation theory:
  - Clear advantage: not many ambiguities
  - But: beware of range of applicability
  - Currently at next-to-next-to-leading order for on-shell production

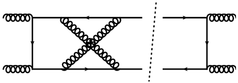
    MC, Bärnreuther, Fiedler, Heymes, Mitov `12 - `15

  - Partial independent results by:

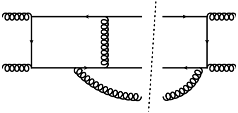
    Abelof, Gehrmann-De Ridder, Maierhofer, Pozzorini `14
    Catani, Grazzini, Torre `14 - `15

Contributions

- 2-loop virtual corrections (V-V)
  MC `07, Bärnreuther, MC, Fiedler `13
  complete numerical results partial analytical results:
  Bonciani, Ferroglia, Gehrmann, Maitre, von Manteuffel, Studerus `08–`13
  divergences of two-loop amplitudes:
  Ferroglia, Neubert, Pecjak, Yang `09

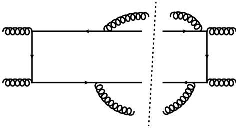


- 1-loop virtual with one extra parton (R-V)
  from next-to-leading order corrections to tt+jet
  code by Stefan Dittmaier

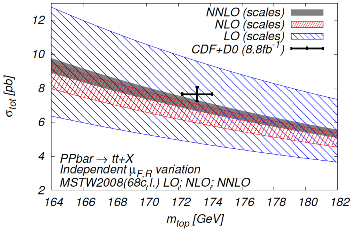


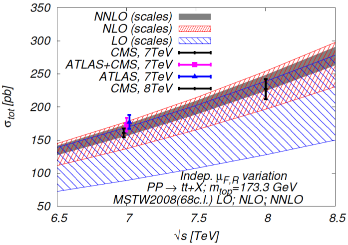
- 2 extra emitted partons at tree level (R-R)
  MC `10 `11 *new subtraction scheme STRIPPER*
  MC, Heymes `14 *4-d formulation of STRIPPER*

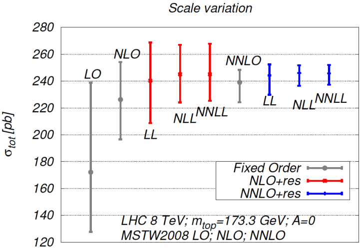
- One-loop squared amplitudes
  original results not used:
  Körner, Merebashvili, Rogal `07, Anastasiou, Aybat `08



Perturbation Theory Convergence







Concurrent uncertainties:

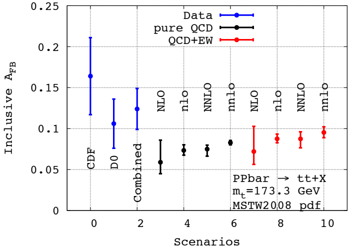
Scales	~ **3%**
pdf (at 68%cl)	~ **2-3%**
α_S (parametric)	~ **1.5%**
m_{top} (parametric)	~ **3%**

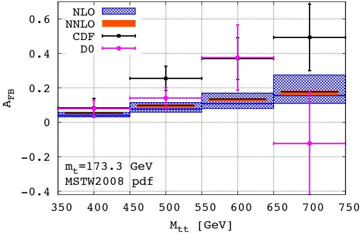
Soft gluon resummation makes a difference: **5%** → **3%**

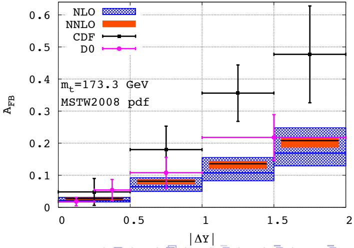
MC, Fiedler, Mitov `13

Data vs Precision QCD

MC, Fiedler, Mitov `14



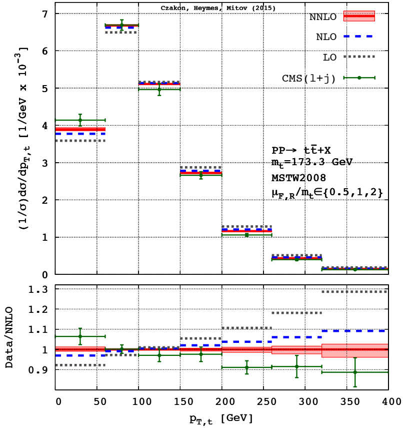




Differential Distributions @ LHC

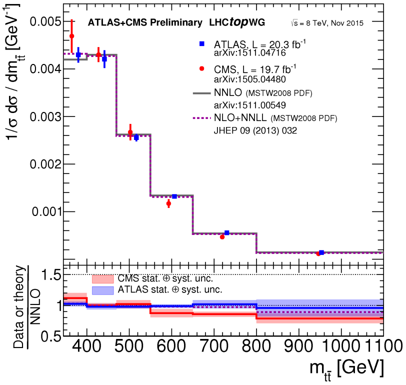
- Even with fixed scale the agreement with data quite good
- Apparently convergence poor in normalized distributions

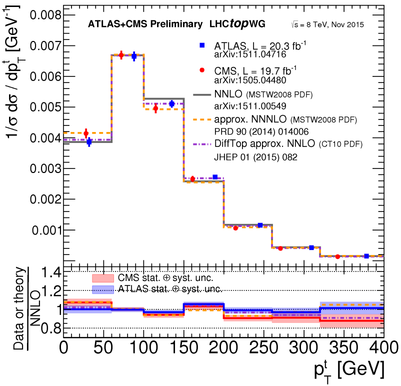
MC, Heymes, Mitov `15



Differential Distributions @ LHC

- Much better agreement with ATLAS data
- Lesson for the theorist: "spot-on agreement"may be dangerous



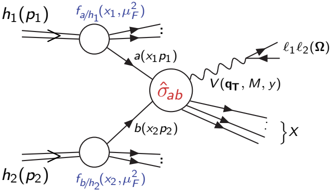


Drell-Yan

Drell–Yan $\mathbf{q_T}$ distribution

$$\mathbf{h_1(p_1) + h_2(p_2) \rightarrow V + X \rightarrow \ell_1 + \ell_2 + X}$$

where $V = Z^0/\gamma^*, W^\pm$



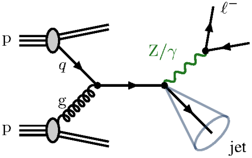
QCD factorization formula:

$$\frac{d\sigma}{d^2\mathbf{q_T}\, dM^2\, dy\, d\mathbf{\Omega}} = \sum_{a,b} \int_0^1 dx_1 \int_0^1 dx_2\, f_{a/h_1}(x_1, \mu_F^2)\, f_{b/h_2}(x_2, \mu_F^2)\, \frac{d\hat{\sigma}_{ab}}{d^2\mathbf{q_T} dM^2 d\hat{y} d\mathbf{\Omega}}(\hat{s};\alpha_S,\mu_R^2,\mu_F^2).$$

R. Gavin, Y. Li, F. Petriello and S. Quackenbush, Comput. Phys. Commun. **182** (2011) 2388

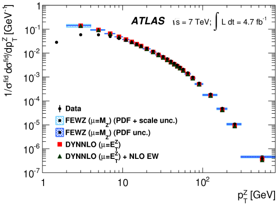
S. Catani, L. Cieri, G. Ferrera, D. de Florian and M. Grazzini, Phys. Rev. Lett. **103** (2009) 082001

Example: Inclusive p_T spectrum of Z



$pp \to Z/\gamma^* \to \ell^+\ell^- + X$

- large cross section
- clean leptonic signature



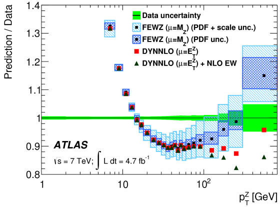
- fully inclusive wrt QCD radiation
- only reconstruct ℓ^+, ℓ^- so clean and precise measurement
- potential to constrain gluon PDFs

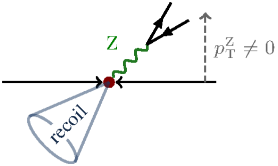
NNLO QCD Z+Jet

Gehrmann-De Ridder, Gehrmann, NG, Huss, Morgan (15)
Boughezal, Campbell, Ellis, Focke, Giele, Liu, Petriello (15)
Boughezal, Liu, Petriello (16)

p_T OF Z

Example: Inclusive p_T spectrum of Z





✚ low $p_T^Z \leq 10$ GeV, resummation required

✚ $p_T^Z \geq 20$ GeV, fixed order prediction about 10% below data

✘ *Very precise measurement of Z p_T poses problems to theory*,
D. Froidevaux, HiggsTools School

FEWZ/DYNNLO are Z + 0 jet @ NNLO

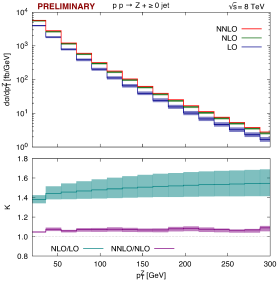
✘ Only NLO accurate in this distribution

✓ Requiring recoil means Z + 1 jet @ NNLO required

$V + 1$JET

$V + 1$jet

Example: Inclusive p_T spectrum of Z

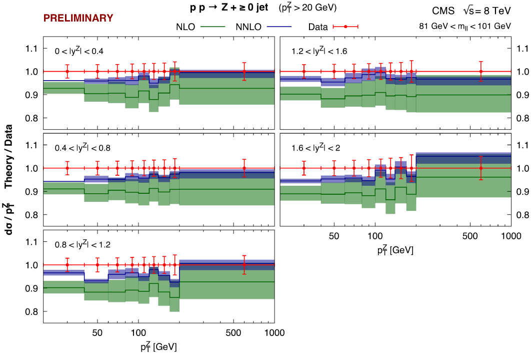


- ✓ NLO corrections $\sim 40 - 60\%$
- ✓ significant reduction of scale uncertainties NLO $\rightarrow$ NNLO
- ✓ NNLO corrections relatively flat $\sim 4 - 8\%$

Can the NNLO corrections resolve the discrepancy in theory v data?

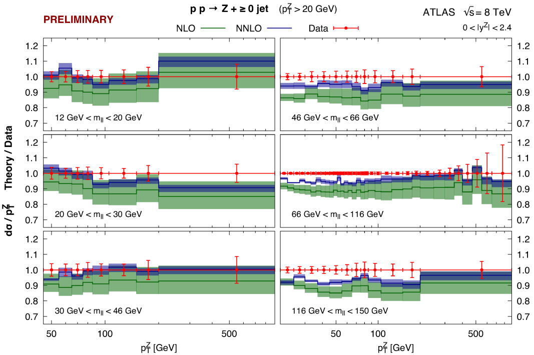
– p. 13

Double-differential: $d\sigma/dp_T^Z$ binned in y^Z - CMS



- improvement of theory vs. data comparison
- significant reduction of scale uncertainties

Double-differential: $d\sigma/dp_T^Z$ binned in $m_{\ell\ell}$ - ATLAS



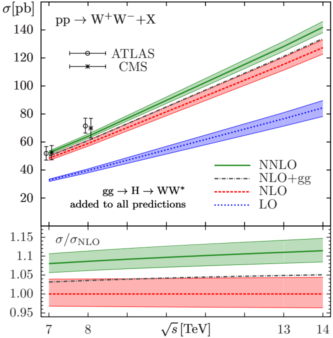
- improvement of theory vs. data comparison
- significant reduction of scale uncertainties

– p. 18

Diboson

$$V + V'$$

pp $\rightarrow$ WW at NNLO



Gehrmann, Grazzini, Kallweit, Maierhofer, von Manteuffel, Pozzorini, Rathlev, Tancredi (14)

- ✓ Provides a handle on the determination of triple gauge couplings, and possible new physics
- ✓ Severe contamination of the W^+W cross section due to top-quark resonances

$\frac{\sqrt{s}}{\text{TeV}}$	σ_{LO}	σ_{NLO}	σ_{NNLO}	$\sigma_{gg\to H\to WW^*}$
7	$29.52^{+1.6\%}_{-2.5\%}$	$45.16^{+3.7\%}_{-2.9\%}$	$49.04^{+2.1\%}_{-1.8\%}$	$3.25^{+7.1\%}_{-7.8\%}$
8	$35.50^{+2.4\%}_{-3.5\%}$	$54.77^{+3.7\%}_{-2.9\%}$	$59.84^{+2.2\%}_{-1.9\%}$	$4.14^{+7.2\%}_{-7.8\%}$
13	$67.16^{+5.5\%}_{-6.7\%}$	$106.0^{+4.1\%}_{-3.2\%}$	$118.7^{+2.5\%}_{-2.2\%}$	$9.44^{+7.4\%}_{-7.9\%}$
14	$73.74^{+5.9\%}_{-7.2\%}$	$116.7^{+4.1\%}_{-3.3\%}$	$131.3^{+2.6\%}_{-2.2\%}$	$10.64^{+7.5\%}_{-8.0\%}$

- ✓ The NNLO QCD corrections increase the NLO result by an amount varying from 9% to 12% as $\sqrt{s}$ increases from 7 to 14 TeV.

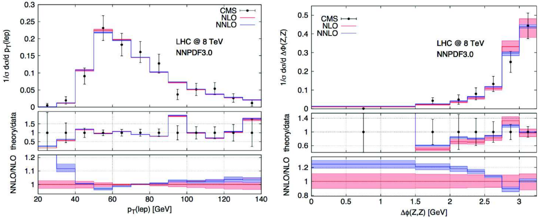
– p. 36

Z boson pair production with decays

Grazzini, Kallweit, Rathlev (15)

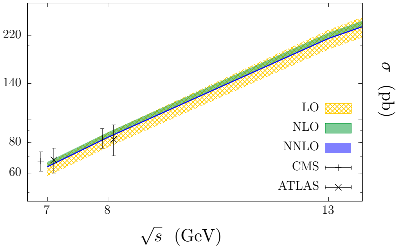
✓ The NNLO corrections increase the NLO result by an amount varying from 11% to 17% as $\sqrt{s}$ increases from 7 to 14 TeV.

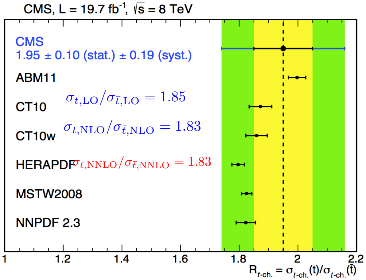
✓ The loop-induced gluon fusion contribution provides about 60% of the total NNLO effect.



✓ NNLO effects improve agreement with data for the $\Delta\phi$ distribution.

Single top production (t-channel)



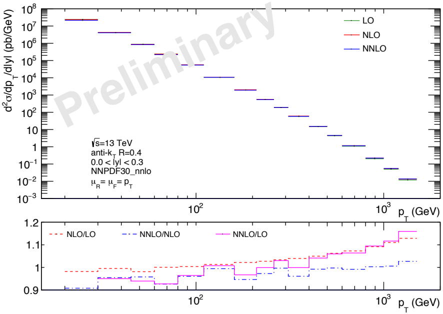


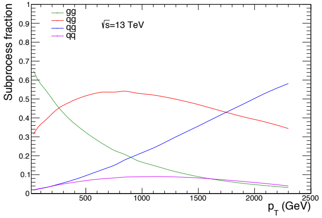
Burcherseifer, Caola, K.M.

$p_\perp$	σ_{LO}, pb	σ_{NLO}, pb	δ_{NLO}	σ_{NNLO}, pb	δ_{NNLO}
0 GeV	$53.8^{+3.0}_{-4.3}$	$55.1^{+1.6}_{-0.9}$	$+2.4\%$	$54.2^{+0.5}_{-0.2}$	-1.6%
20 GeV	$46.6^{+2.5}_{-3.7}$	$48.9^{+1.2}_{-0.5}$	$+4.9\%$	$48.3^{+0.3}_{-0.02}$	-1.2%
40 GeV	$33.4^{+1.7}_{-2.5}$	$36.5^{+0.6}_{-0.03}$	$+9.3\%$	$36.5^{+0.1}_{+0.1}$	-0.1%
60 GeV	$22.0^{+1.0}_{-1.5}$	$25.0^{+0.2}_{+0.3}$	$+13.6\%$	$25.4^{-0.1}_{+0.2}$	$+1.6\%$

The precision on the inclusive cross section is about one percent. Ratio of top and anti-top cross sections is sensitive to parton distribution functions at relatively large values of x and should be used as one of the standard candles for PDF determinations.

Di-jet production





Results are for gluon-gluon and quark-gluon (preliminary) initial states. Not all color factors included for quark-gluon channel. Flat NNLO/NLO K-factors; small corrections (may change if other channels included). Results for various orders obtained with NNLO PDFs.

Curie, Gehrmann-de Ridder, Gehrmann, Glover, Pires

Summary - Where are we now?

- ✓ Witnessed a revolution that has established NLO as the new standard
  - previously impossible calculations now achieved
  - very high level of automation for numerical code
  - standardisation of interfaces - linkage of one-loop and real radiation providers
  - take up by experimental community
- ✓ Substantial progress in NNLO in past couple of years
  - several different approaches for isolating IR singularities
  - several new calculations available

Summary - Where are we going?

✓ NNLO automation?

- as we gain analytical and numerical experience with NNLO calculations, can we benefit from (some of) the developments at NLO, and the improved understanding of amplitudes
- automation of two-loop contributions?
- automation of infrared subtraction terms?
- standardisation of interfaces - linkage to one-loop and real radiation providers?
- interface with experimental community

Next few years:

✓ Les Houches wishlist to focus theory attention

✓ New high precision calculations such as, e.g. N3LO σ_H, could reduce Missing Higher Order uncertainty by a factor of two

✓ NNLO is emerging as standard for benchmark processes such as V+jet or dijet production leading to improved pdfs etc. could reduce theory uncertainty due to inputs by a factor of two

Accuracy and Precision (A. David)