Higher dimensional SdS black holes: Greybody factors and power spectra for emission of scalar fields non-minimally coupled to gravity.

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Outline

- The gravitational background and the action for the massless scalar field
- What is a grey-body factor and how to compute it (analytic approach)
- Numerical results for the grey-body factors on the brane and in the bulk
- Power spectra for brane and bulk emission
- Relative and total emissivity bulk-over-brane ratios

The gravitational background for the system

The spherically-symmetric, static and uncharged **Tangherlini de-Sitter** black hole is described by the following metric:

$$dS^2 = -h(r)dt^2 + h(r)^{-1}dr^2 + r^2 d\Omega^2_{n+2} \tag{1}$$

where $d\Omega^2_{n+2}$ is the $(n+2)$ unit-sphere surface element and the metric function is given by:

$$h(r) = 1 - \frac{\mu}{r^{n+1}} - \Lambda r^2 \tag{2}$$

The equation: $h(r) = 1 - \frac{\mu}{r^{(n+1)}} - \Lambda r^2 = 0$ has in general $n+3$ roots corresponding to $n+3$ horizons for this spacetime. We have only 2 real and positive roots if the following condition holds:

$$\mu^2 \Lambda^{(n+1)} < \frac{4(n+1)^{(n+1)}}{(n+3)^{(n+3)}}. \tag{3}$$

Bulk: The Action for the massless scalar field

The action for a massless scalar field $\Phi(x)$ propagating in the aforementioned gravitational background:

$$S = -\frac{1}{2}\int \left[g^{\mu\nu}\partial_\nu\Phi\partial_\mu\Phi + \xi\mathbf{R}\Phi^2\right]\sqrt{-g}\,dx^D \quad , \quad \mathbf{R} = \frac{2D}{D-2}\Lambda. \tag{4}$$

(*where* $D := 4+n$) leads to the following e.o.m:

$$\frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}\,g^{\mu\nu}\partial_\nu\Phi\right) = \xi\mathbf{R}\Phi. \tag{5}$$

We assume the following factorized ansatz for the field:

$$\Phi(t,r,\theta,\phi,\theta_i) = e^{-i\omega t}R(r)Y_{l,m}(\theta,\phi,\theta_i) \quad , \quad \omega > 0 \tag{6}$$

and the angular part of eq.5 can be expressed as

$$\frac{r^2}{\sqrt{-g}}\left[\sum_{j=1}^{n+2}\partial_{\theta_j}\left(\sqrt{-g}\,g^{\theta_j\theta_j}\partial_{\theta_j}Y_{l,m}\right)\right] = -l(l+n+1)Y_{l,m}. \tag{7}$$

Radial part of the e.o.m. in the bulk

The radial part of the e.o.m. in the bulk is then:

$$\frac{1}{r^n}\frac{d}{dr}\left[r^{(n+2)}h(r)\frac{dR(r)}{dr}\right]+\left[\frac{(\omega r)^2}{h(r)}-\lambda(r)\right]R(r)=0 \tag{8}$$

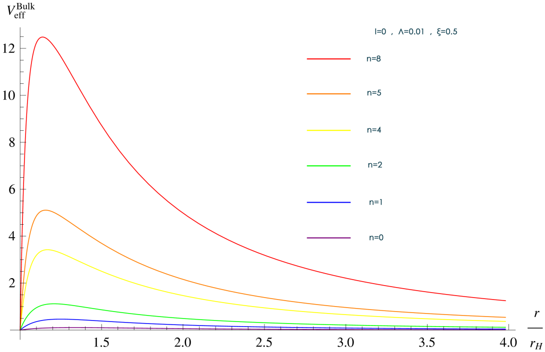
where we have defined: $\lambda(r) := l(l+n+1)+\xi\mathbf{R}r^2$.

By introducing a new "*tortoise*" radial coordinate (r_*) through : $h(r)dr_* = dr$ and redefining the radial function: $u(r) := R(r)r^{(\frac{n}{2}+1)}$ we can recast the e.o.m. to a "*Schrodinger-like form*":

$$\partial^2_{r_*}u(r_*) - V_{eff}(r)u(r_*) = -\omega^2 u(r_*). \tag{9}$$

The effective potential term reads:

$$V_{eff}^{bulk}(r) = h(r)\left[\frac{h'(r)}{r}\left(\frac{n+2}{2}\right) + \frac{h^2(r)}{r^2}\frac{n(n+2)}{4} + \frac{\lambda(r)}{r^2}\right]. \qquad (10)$$



$V_{eff}^{bulk}(r) \to 0$ close to r_H and r_c. We can compute the probability for the emission using asymptotic free-wave solutions in these two regimes.

Black holes are actually grey!

Black holes $\rightarrow$ perfect black body (?) with energy emission rate:

$$\frac{dE}{dt} = \frac{\omega}{(exp[\omega/T_H]-1)} \frac{d^{n+3}k}{(2\pi)^{n+3}} \quad , \quad |k|^2 = \omega^2 - m^2. \tag{11}$$

No! Due to the gravitational barrier we have instead a "grey body":

$$\frac{dE}{dt} = \sum_{l=0}^{\infty} |A_l(\omega)|^2 \frac{\omega}{(exp[\omega/T_H]-1)} \frac{d^{n+3}k}{(2\pi)^{n+3}}. \tag{12}$$

This is Hawking radiation!

- "***Greybody factor***" $|A_l|^2$ is the transmission probability for the l-th mode of particles to overcome the gravitational barrier. ("*Backscattering effect*" reduces the total radiation emitted.)
- Calculation of $|A_l|^2 \rightarrow$ scattering problem.

The solution to the e.o.m. in the bulk for $r \to r_H$

Starting from the e.o.m:

$$\frac{1}{r^n}\frac{d}{dr}\left[r^{(n+2)}h(r)\frac{dR(r)}{dr}\right]+\left[\frac{(\omega r)^2}{h(r)}-\lambda(r)\right]R(r)=0 \qquad (13)$$

and applying the following coordinate transformation: $r \to f(r) := \frac{h(r)}{1-\Lambda r^2}$ we end up with:

$$f(1-f)\partial_f^2 R(f)+(1-B_H f)\partial_f R(f)+\left[\frac{(\omega r_H)^2}{A_H^2 f(1-f)}-\frac{\lambda_H(1-\Lambda r_H^2)}{A_H^2(1-f)}\right]R(f)=0 \qquad (14)$$

where we have defined $\lambda_H := l(l+n+1)+\xi \mathbf{R} r_H^2$ and

$$B_H := 1+4\frac{4\Lambda r_H^2}{A_H^2} \quad , \quad A_H := (n+1)-(n+3)\Lambda r_H^2.$$

Applying the field redefinition: $R(f) = f^{\alpha_H}(1-f)^{\beta_H}F(f)$ we transform the previous eq. to a hypergeometric one:

$$f(1-f)\partial_f^2 F(f) + [c_0 - (a_0 + b_0 + 1)f]\partial_f F(f) - a_0 b_0 F(f) = 0 \qquad (15)$$

with the α_H and β_H parameters that characterize the solution near the black hole assuming the following values:

$$\alpha_H := -\frac{i\omega r_H}{A_H}\ , \ \beta_H = \frac{1}{2}(2 - B_H) - \frac{1}{2}\sqrt{(B_H - 2)^2 + \frac{4\lambda_H(1-\Lambda r_H^2)}{A_H^2}}$$

Similarly the hypergeometric indices (a_0, b_0, c_0) depend on all parameters. That is, on two geometrical (r_H, r_c), two field related (l and ξ) and two spacetime related (n and Λ).

We demand that only ingoing modes exist on the horizon of the black hole and so the general solution reduces to:

$$R_{NH} = A_- f^{\alpha_H}(1-f)^{\beta_H}F(a_0, b_0, c_0; f). \qquad (16)$$

The solution to the e.o.m. in the Bulk for $r \to r_c$

We now turn to the cosmological-horizon regime. The e.o.m.

$$\frac{1}{r^n}\frac{d}{dr}\left[r^{(n+2)}h(r)\frac{dR(r)}{dr}\right]+\left[\frac{(\omega r)^2}{h(r)}-\lambda(r)\right]R(r)=0 \qquad (17)$$

under the change of coordinate: $r \to h(r) \approx 1-\Lambda r^2$ is brought to the form:

$$h(1-h)\partial_h^2 R(h)+\left[1-\frac{(5+n)}{2}h\right]\partial_h R(h)+\left[\frac{(\omega r)^2}{4h(1-h)}-\frac{\lambda_c}{4(1-h)}\right]R(h)=0 \qquad (18)$$

where $\lambda_c := l(l+n+1)+\xi \mathbf{R} r_c^2$. Redefining the field according to $R(h)=h^{\alpha_c}(1-h)^{\beta_c}F(h)$ results once again to a hypergeometric equation.

The parameters characterizing the hypergeometric eq. at $r \to r_c$ are:
$a_c = \frac{i\omega r_c}{2}$, $\beta_c = -\frac{1}{4}\left[(n+1) + \sqrt{(n+1)^2 - (2\omega r_c)^2 + 4\lambda_c}\right]$

while the resulting general solution is:

$$\begin{aligned} P_{FF}(h) = & \, Ch^{\alpha_c}(1-h)^{\beta_c} F(a_1, b_1, c_1; h) \\ & + Dh^{-\alpha_c}(1-h)^{\beta_c} F(1+a_1-c_1, 1+b_1-c_1, 2-c_1; h) \end{aligned} \tag{19}$$

Note that in contrast to the r_H case we have no physical reason to restrict the solution to purely ingoing or outgoing modes on r_c. In terms of the tortoise coordinate (r_*) defined through : $h(r)dr_* = dr$ the solution can be written as:

$$P_{FF}(h) \approx Ce^{-i\omega r_*} + De^{i\omega r_*} \tag{20}$$

and so the grey-body factor for the scattering process is simply

$$|A_l|^2 = 1 - |R_l|^2 = 1 - \left|\frac{D}{C}\right|^2 . \tag{21}$$

Matching of the solutions

Matching of the asymptotic solutions ensures the existence of a complete solution in the area between r_H and r_c. After stretching both solutions towards the intermediate zone we end up with the following asymptotic expressions:

$$P_{NH}(r) \approx r^{\epsilon_+}\Sigma_1 + r^{\epsilon_-}\Sigma_2 \tag{22}$$

$$P_{FF}(r) \approx r^{\epsilon_+}\left[C\Sigma 3 + D\Sigma_4\right] + r^{\epsilon_-}\left[C\Sigma_5 + D\Sigma_6\right] \tag{23}$$

where $\epsilon_\pm := -\frac{1}{2}\left[(n+1) \pm \sqrt{(n+1)^2 + 4l(l+n+1)}\right]$ and the Σ coefficients are given by:

$$\Sigma_1 := \frac{r_H^{-\epsilon_+}\Gamma[c_0]\Gamma[a_0+b_0-c_0]}{\Gamma[a_0]\Gamma[b_0]} \quad , \quad \Sigma_2 := \frac{r_H^{-\epsilon_-}\Gamma[c_0]\Gamma[c_0-a_0-b_0]}{\Gamma[c_0-a_0]\Gamma[c_0-b_0]}$$

$$\Sigma_3 := \frac{\Lambda^{\epsilon_+/2}\Gamma[c_1]\Gamma[c_1-a_1-b_1]}{\Gamma[c_1-a_1]\Gamma[c_1-b_1]} \quad , \quad \Sigma_4 := \frac{\Lambda^{\epsilon_+/2}\Gamma[2-c_1]\Gamma[c_1-a_1-b_1]}{\Gamma[1-a_1]\Gamma[1-b_1]}$$

$$\Sigma_5 := \frac{\Lambda^{\epsilon_-/2}\Gamma[c_1]\Gamma[a_1+b_1-c_1]}{\Gamma[a_1]\Gamma[b_1]} \quad , \quad \Sigma_6 := \frac{\Lambda^{\epsilon_-/2}\Gamma[2-c_1]\Gamma[a_1+b_1-c_1]}{\Gamma[1+a_1-c_1]\Gamma[1+b_1-c_1]}$$

The asymptotic limit of $|A|^2$

In terms of the Σ coefficients we get:

$$|A_l|^2 = 1 - \left|\frac{\Sigma_2\Sigma_3 - \Sigma_1\Sigma_5}{\Sigma_1\Sigma_6 - \Sigma_2\Sigma_4}\right|^2 \tag{24}$$

Our result reproduces the well known asymptotic limit for the greybody factor in the low energy regime. For a free scalar field ($\xi \to 0$) and the dominant mode ($l = 0$) in the limit ($\omega \to 0$) one should get:

$$|A_0|^2(\omega \to 0) = \frac{4(r_c r_H)^{(n+2)}}{(r_c^{n+2} + r_H^{n+2})^2} \tag{25}$$

The second horizon creates a "*finite-size universe*" for the particle to propagate, and particles with $\omega \to 0$ have a non-vanishing probability to be emitted.

We found that if $\xi \neq 0$ the asymptotic limit is lost and $|A_0|^2(\omega \to 0) = \mathcal{O}(\omega^2)$.

The brane emission case.

The induced metric on the Brane is obtained "by projection" from the bulk metric by setting $\theta_i = \frac{\pi}{2} \forall i$:

$$dS^2 = -h(r)dt^2 + h(r)^{-1}dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right) \ , \ h(r) = 1 - \frac{\mu}{r^{(n+1)}} - \Lambda r^2. \tag{26}$$

The corresponding e.o.m. this time is:

$$\frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}g^{\mu\nu}\partial_\nu\Phi\right) = \xi \mathbf{R}\Phi \quad , \quad \mathbf{R} = 12\Lambda + \frac{n(n-1)\mu}{r^{(n+3)}}. \tag{27}$$

The field ansatz is once again of the form:

$$\Phi(t, r, \theta, \phi, \theta_i) = e^{-i\omega t}R(r)Y_{l,m}(\theta, \phi) \quad , \quad \omega > 0 \tag{28}$$

The radial e.o.m. is

$$\frac{d}{dr}\left[r^2 h(r)\frac{dR(r)}{dr}\right] + \left[\frac{(\omega r)^2}{h(r)} - \lambda(r)\right]R(r) = 0 \tag{29}$$

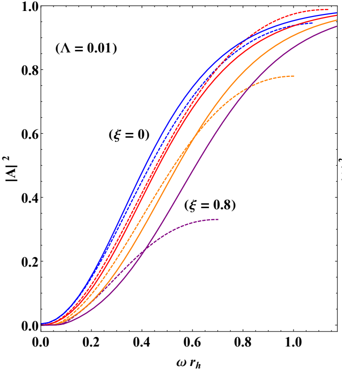
where this time:

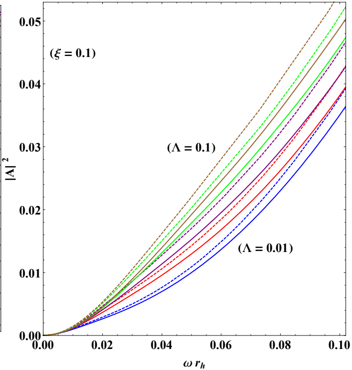
$$\lambda := l(l+1) + \xi \mathbf{R} r^2 \tag{30}$$

Results for the grey-body factors on the brane

We computed the g.f. both in an analytic as well as in a numerical way.

(Left: variable ξ , Right: variable Λ) $[n = 2, l = 0]$ (dashed: analytic , solid: numerical):

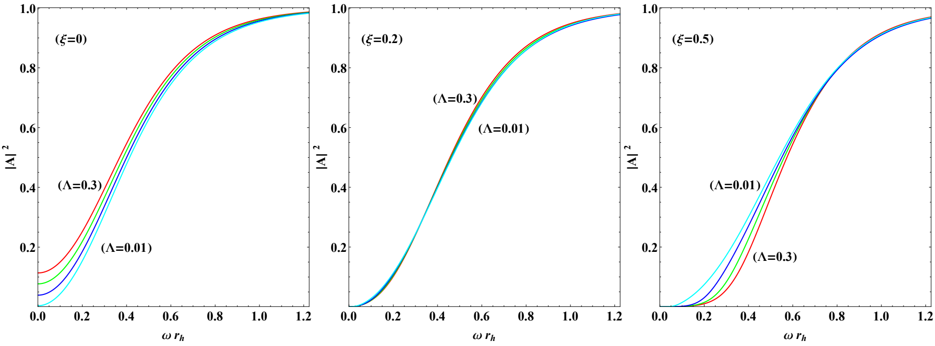




Very good agreement in the low energy regime for small values of ξ and Λ

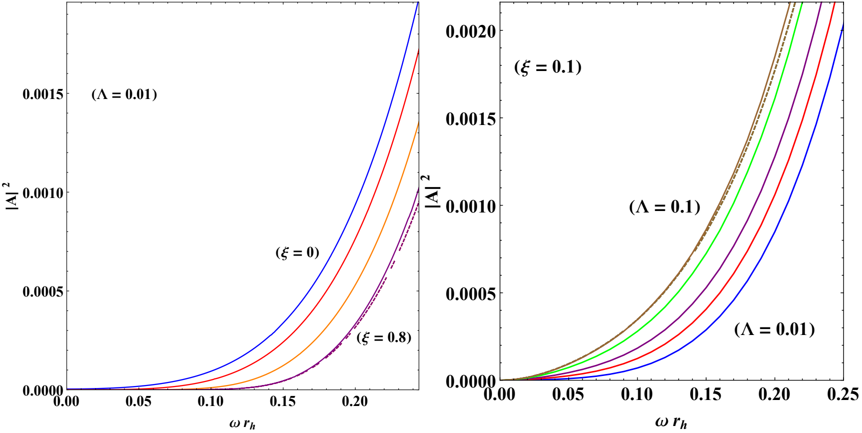
The combined effect of ξ and Λ on $|A|^2$ on the brane

The cosmological constant assumes a "dual role". Enhancing or suppressing the g.f. depending on the value of ξ.

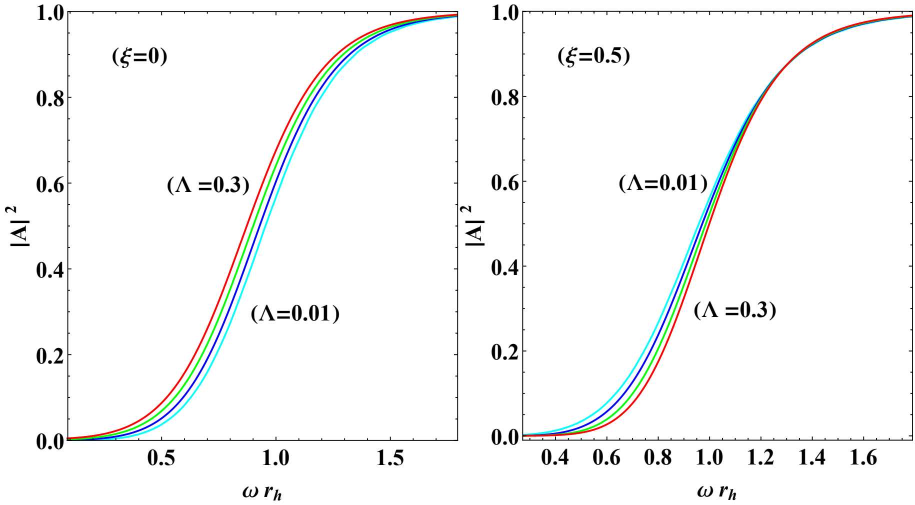


Asymptotic limit vanishes for non-minimally coupled fields.

(Left: variable ξ , Right: variable Λ) $[n = 2, l = 0]$ (dashed: analytic , solid: numerical):



The combined effect of ξ and Λ on $|A|^2$ in the bulk



Power spectra for SdS Black holes

The differential energy emission rate is:

$$\frac{d^2E}{dtd\omega} = \frac{1}{2\pi}\sum_l \frac{N_l\,|A|^2\,\omega}{exp\,(\omega/T_{BH}) - 1} \tag{31}$$

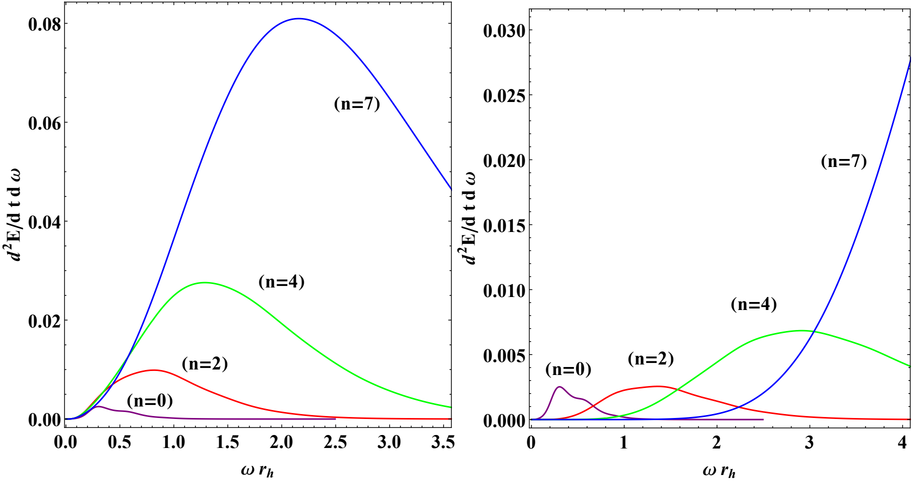
where $N_l = 2l + 1$ on the brane and $N_l = \frac{(2l+n+1)!(l+n)!}{l!(n+1)!}$ in the bulk. While the temperature is given in terms of the surface gravity:

$$T_{BH} = \frac{\kappa_H}{2\pi} = \frac{1}{\sqrt{h(r_0)}}\frac{1}{4\pi r_H}\left[(n+1) - (n+3)\Lambda r_H^2\right] \tag{32}$$

where we have adopted the Bousso-Hawking "normalized" expression for the black hole temperature.

Power spectra. The effect of n

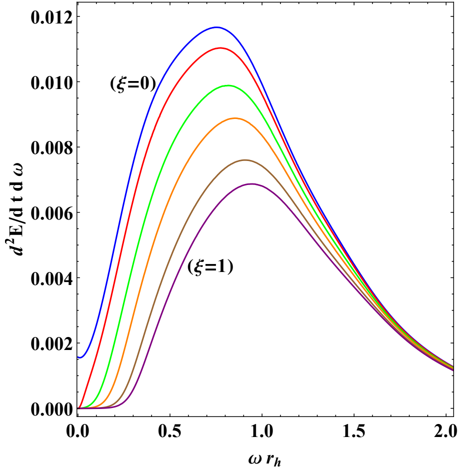
For $\Lambda = 0.1$ and $\xi = 0.3$ [Left: brane , Right: bulk]

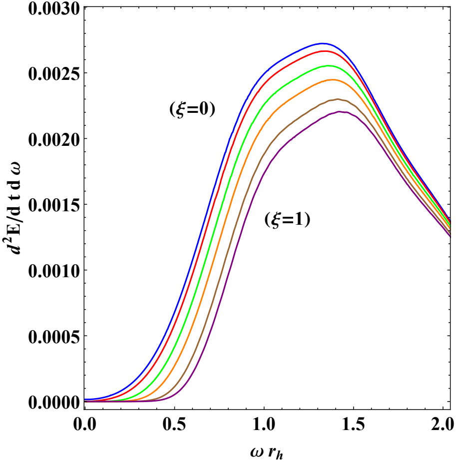


Bulk $\rightarrow$ peaks shift towards high-energy regime (general feature of the power spectra).

Power spectra. The effect of ξ

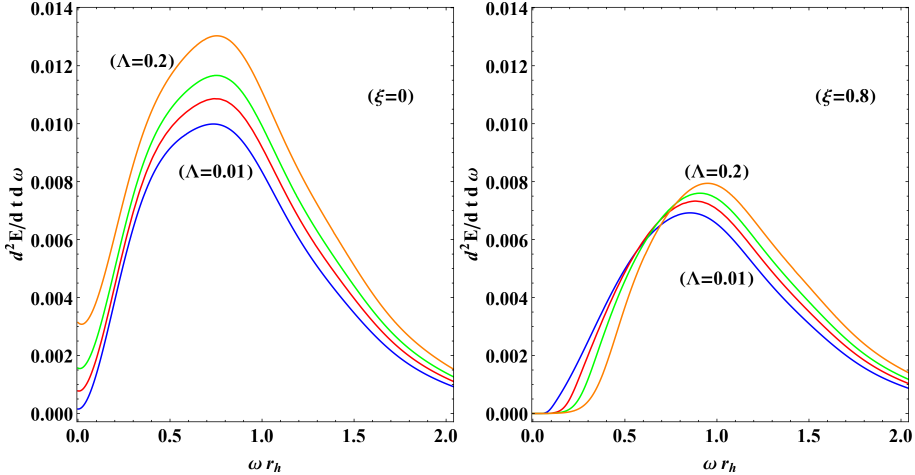
For $\Lambda = 0.1$ and $n = 2$ [L: brane , R: bulk]





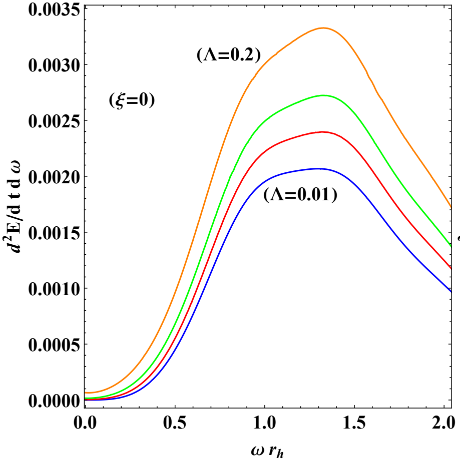
Asymptotic limit suppressed in the bulk.

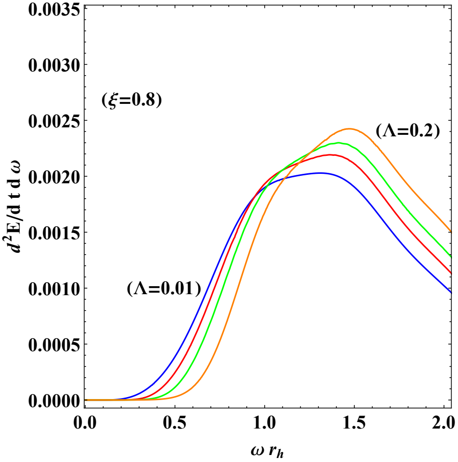
Power spectra for emission on the brane. Λ effect vs ξ effect



For small values of $\xi \rightarrow \Lambda$ results in a global enhancement throughout the energy regime. As ξ increases the Λ role is inversed in the low energy region.

Power spectra for emission in the bulk. Λ effect vs ξ effect

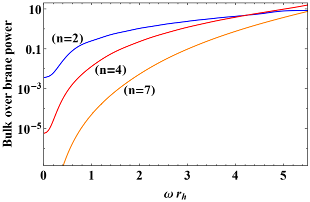


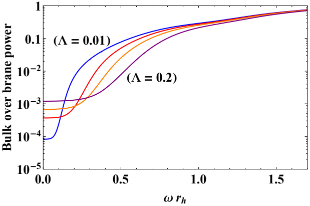


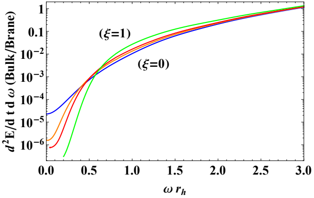
Bulk-over-brane relative emission ratio

We compare the amount of energy emitted on the brane compared to the amount emitted in the bulk. To this end we compute the bulk-over-brane ratio of $d^2E/dtd\omega$

[Left: $\xi = 0.3$, $\Lambda = 0.2$] [Right: $\xi = 0.3$, $n = 2$]







[$\Lambda = 0.1$, $n = 4$]

Bulk-over-brane total emissivity

The amount of energy emitted by the black hole on the brane and in the bulk in the unit of time over the whole frequency range.

Table: Bulk over brane total emissivity for n=2

$\xi \rightarrow$	0.0	0.5	1.0
$\Lambda = 0.01$	0.257506	0.320639	0.393068
0.05	0.27356	0.333195	0.394932
0.2	0.314566	0.357599	0.38618

Table: Bulk over brane total emissivity for n=7

$\xi \rightarrow$	0.0	0.5	1.0
$\Lambda = 0.01$	0.779006	1.60427	3.05751
0.05	0.790883	1.61993	3.0686
0.2	0.824629	1.65866	3.07722

Summary

- We computed both with analytic and numerical methods the grey-body factors and found very good agreement in the low (ξ, Λ, ω) regime.
- The **grey-body factors**: Get suppressed both on brane and in the bulk as ξ or n increase. The Λ on the other hand assumes a "dual" role.
- The **power spectra**: Get enhanced with n due to T_{BH} both on the brane and in the bulk while increase in ξ results in suppression. The dual role of Λ is also reflected here.
- The coupling constant ξ appears to have a **dominant role** in scalar emission in SdS. It destroys the low energy asymptotic value of the energy emission and overthrows the brane-to-bulk ratio.

Thank You!