Joint University Accelerator School

Solutions of Examination of Transverse Beam Dynamics

JUAS 2014

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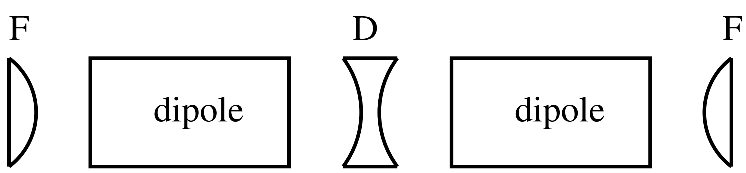
1 Exercise: [7 points]

TLEP is a high luminosity circular e^+e^- collider to study the Higgs boson and physics at the electroweak scale. The TLEP project is conceived as a ring with a circumference 80 km long. Let us assume that the optics of this machine is made of identical FODO cells, each cell with the following structure:



Assuming for the calculations the thin lens approximation, and considering the following parameters:

Parameter	Value
Beam energy	120 GeV
Total FODO cell length	50 m
Quadrupole gradient G	10 T/m
Quadrupole length	1.5 m
Dipole length	21.3 m

answer the following questions:

1. Calculate: (1) for each dipole the bending angle and the necessary magnetic field; (2) the value of the focal length of each quadrupole [0.5+0.5 pt]

(1) Solution. Knowing the length of each cell L=50 m and the total circumference of the machine (80 km), we can calculate the number of bending magnets (we know that there are two per cell):

$$N_{\text{cell}} = 80000/50 = 1600$$

cells and

$$N_{\text{bending}} = 2 \times 1600 = 3200$$

bending magnets. Therefore, the necessary bending angle for each dipole is

$$\theta = 2\pi/N_{bending} = 1.96 \times 10^{-3} \text{ [rad]}.$$

In order to calculate the magnetic field, we calculate first the rigidity

$$B\rho = p/e = 3.3356 \cdot p[GeV/c] = 400.272 \text{ T m}.$$

Combining this expression with $\theta = l_B/\rho$ (where l_B is the length of the bending magnet), we obtain

$$B = 400.272 \cdot \frac{\theta}{l_B} = 0.037 \text{ T}.$$

(2) Solution. We can use the beam rigidity (or the particle momentum) to calculate the normalized quadrupole strength:

$$k = \frac{G}{B\rho} = \frac{G}{p/e} = 0.299792 \cdot \frac{G}{p[\text{GeV/c}]} = 0.299792 \cdot \frac{10\text{T/m}}{120\text{GeV/c}} = 0.025 \text{ m}^{-2}$$

an the focal length (using the quadrupole length $l_q = 1.5$ m):

$$f = \frac{1}{k \cdot l_q} = 26.67 \text{ m}$$

1. Is this lattice stable? [1 pt]

Solution. To answer this question let us use the transfer matrix for this FODO cell. We can calculate it multiplying the corresponding matrices for each element in thin lens approximations. However, we know from the exercises bulletin 5 (student will have access to all kind of material) that the 3×3 transfer matrix for this kind of cell is given by

$$\begin{aligned} M_{cell} &= \begin{pmatrix} 1 - \frac{L^2}{8f^2} & L\left(1 + \frac{L}{4f}\right) & L\left(1 + \frac{L}{8f}\right)\theta \\ -\frac{L}{4f^2}\left(1 - \frac{L}{4f}\right) & 1 - \frac{L^2}{8f^2} & 2\left(1 - \frac{L}{8f} - \frac{L^2}{32f^2}\right)\theta \\ 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} \cos\mu & \beta\sin\mu & L\left(1 + \frac{L}{8f}\right)\theta \\ -\frac{\sin\mu}{\beta} & \cos\mu & 2\left(1 - \frac{L}{8f} - \frac{L^2}{32f^2}\right)\theta \\ 0 & 0 & 1 \end{pmatrix} \end{aligned}$$

Then we have
$\cos\mu = 1 - \frac{L^2}{8f^2} = 0.56 < 1$
and therefore it fullfils the stability condition.

1. What are the vertical and horizontal tunes for this machine? [1 pt]

Solution. The phase advance per cell is $\mu = 0.976$ rad

$Q_{x,y} = \frac{N_{cell}\mu_{x,y}}{2\pi} = 248.54$

1. Compute the maximum and the minimum value for the betatron functions, and the maximum and minimum value of the dispersion [1+1 pts]

The maximum beta function for this lattice is:
$\beta_{max} = \frac{L}{\sin\mu}\left(1 + \frac{L}{4f}\right) = 88.66$ m
and the minimum can be calculated subtituting ($f \to -f$):
$\beta_{min} = \frac{L}{\sin\mu}\left(1 - \frac{L}{4f}\right) = 32.07$ m. We usually assume no dispersion in the vertical plane so $D_y = 0$. The maximum horizontal dispersion is at the focusing quadrupoles of this lattice and taking the corresponding matrix element:
$D_{x,max} = m_{13} = L\left(1 + \frac{L}{8f}\right)\theta = 0.12$ m
and the corresponding minimum dispersion will be at the defocusing quadrupoles ($f \to -f$):
$D_{x,min} = L\left(1 - \frac{L}{8f}\right)\theta = 0.075$ m

1. Calculate the natural chromaticities for this machine [2 pts]

Solution. From the definition of natural chromaticity:

$$\begin{aligned}\xi_N &= -\frac{1}{4\pi}\oint \beta(s)k(s)ds \\ &= -\frac{1}{4\pi} \times N_{cell}\int_{cell} \beta(s)k(s)ds \\ &= -\frac{N_{cell}}{4\pi}\sum_{i\in\{quads\}} \beta_i(kl_q)_i\end{aligned}$$

Here we have used the following approximation valid for thin lens:

$$\int_{cell} \beta(s)k(s)ds \simeq \sum_{i\in\{quads\}} \beta_i(kl_q)_i$$

where we sum over each quadrupole i in the cell. In the case of the FODO cell we have two half focusing quadrupoles and one defocusing quadrupole. Taking into account that $(kl_q)_i = 1/f_i$, we have:

$$\begin{aligned}\xi_N &\simeq -\frac{N_{cell}}{4\pi}\sum_{i\in\{quads\}} \beta_i(kl_q)_i \\ &= -\frac{N_{cell}}{4\pi}\left[\beta_{max}\left(\frac{1}{2f}\right) + \beta_{min}\left(-\frac{1}{f}\right) + \beta_{max}\left(\frac{1}{2f}\right)\right] \\ &= -\frac{N_{cell}}{4\pi}\left[\beta_{max}\left(\frac{1}{f}\right) + \beta_{min}\left(-\frac{1}{f}\right)\right] \\ &= -\frac{N_{cell}}{4\pi\sin\mu}\left[\left(L+\frac{L^2}{4f}\right)\frac{1}{f} - \left(L-\frac{L^2}{4f}\right)\frac{1}{f}\right] \\ &= -\frac{N_{cell}}{8\pi\sin\mu}\frac{L^2}{f^2} = -270.15\end{aligned}$$

Here we have used the expression for β_{max} and β_{min}.

2 Exercise: [4 points]

Let us now assume that in TLEP there is an Interaction Point (IP) with $\beta_x^* = 0.5$ m and $\beta_y^* = 0.1$ cm. The peak luminosity available by a e^+e^- collider can be written as:

$$L = \frac{N_\mathrm{b} N_{e^-} N_{e^+} f_\mathrm{rev}}{4\pi\sigma_x^*\sigma_y^*}\ [\mathrm{cm}^{-2}\mathrm{s}^{-1}]$$

where $N_\mathrm{b} = 80$ is the number of bunches per beam (we assume the same number of bunches for both the e^- and the e^+ beams), $N_{e^-} = N_{e^+} = 5\times 10^{11}$ is the number of particles per bunch (we assume the same number for both e^- and e^+ bunches), and f_rev is the revolution frequency. The horizontal and vertical normalised beam emittances are respectively: $\epsilon_{x,N} = 2.2$ mm and $\epsilon_{y,N} = 4.7$ μm.

1. Compute the revolution frequency f_rev, knowing that the circumference is 80 km and that the beam moves nearly at the speed of light [1 pt]

Solution. The revolution period is given by T_{rev} = circumference$/c$ = 80km/c, and therefore the revolution frequency is:

$$f_{rev} = 1/T_{rev} = c/80\ \mathrm{km} \simeq 3.75\ \mathrm{kHz}$$

1. Calculate the beam transverse beam sizes σ_x^* and σ_y^* at the IP, and the luminosity L for two different beam energies: 45 GeV and 120 GeV [1+0.5+0.5 pts]

Solution. For 45 GeV beam energy: in this case the Lorentz factor is $\gamma = 88062.622$, and $\sigma_x^* = \sqrt{\beta_x^* \epsilon_{x,N}/\gamma} \simeq 111.76\ \mu$m and $\sigma_y^* = \sqrt{\beta_y^* \epsilon_{y,N}/\gamma} \simeq 0.23\ \mu$m, and the luminosity is $L \simeq 2.32 \times 10^{34}$ cm^{-2}s^{-1}

For 120 GeV beam energy: in this case the Lorentz factor is $\gamma = 234833.66$, and $\sigma_x^* = \sqrt{\beta_x^* \epsilon_{x,N}/\gamma} \simeq 68.56\ \mu$m and $\sigma_y^* = \sqrt{\beta_y^* \epsilon_{y,N}/\gamma} \simeq 0.14\ \mu$m, and the luminosity is $L \simeq 6.22 \times 10^{34}$ cm^{-2}s^{-1}

1. What is the value of the betatron function at position $s = 0.5$ m from the IP? [1 pt]

Solution. We know that the betatron function in the drift space of a low beta region (where we have the interaction point) depends on the longitudinal coordinate as follows:

$$\beta(s) = \beta^* + \frac{s^2}{\beta^*}$$

Therefore, $\beta_x(0.5\text{m}) = 1$ m, and $\beta_y(0.5\text{m}) = 250$ m

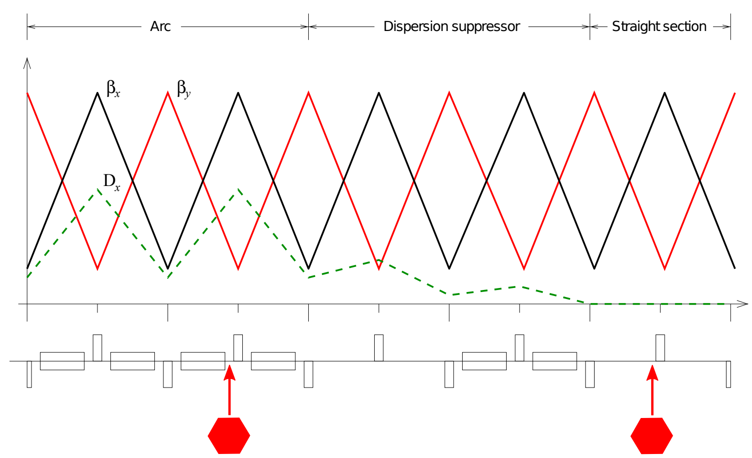
3 Exercise: [5 points]

Given a dispersion suppressor based on the missing-magnet scheme, followed by a straight section with the same focusing properties, reduce the horizontal chromaticity of the system using two sextupoles.

1. Sketch the lattice with the horizontal beta functions and the dispersion. [1 pt]
2. Where do the two sextupoles have to be located? Why? [2 pts]
3. What are the requested sextupoles strength? (work in thin-lens approximation) [2 pts]

Solution

1. The lattice is the following:



2. To correct the chromaticity we need to apply different focusing to particles with different energies. In dispersive regions particles get an energy-based displacement. The first sextupole should then be placed where the dispersion is maxiumum, eg: close to a focusing quadrupole. The second sextupole is meant to correct the geometric aberrations introduced by the first one. In order not to interfere with the correction of the chromaticity, it should be placed in a non dispersive region. A phase advance of $(2n + 1)\pi$ is required between the two sextupoles. The sextupoles positioning shown in the picture satisfies all these requirements, since the phase advance per cell to adopt with the missing dipole scheme is 60°.

3. The required sextupole strength must satisfy:

$$\xi = -\frac{1}{4\pi} \oint k(s)\beta(s)ds + \frac{1}{4\pi} \oint k_2(s)D\beta(s)ds = 0$$

Assuming thin-lens approximation we can rewrite the two integrals:

$$k_2 D\beta L_{\text{sext}} = 3k_F\beta_{\max}L_{\text{quad}} - 3k_D\beta_{\min}L_{\text{quad}}$$

This allows to compute the strenghts (equal for the two sextupoles):

$$k_2 = \frac{3k_F\beta_{\max}L_{\text{quad}} - 3k_D\beta_{\min}L_{\text{quad}}}{D\beta L_{\text{sext}}}$$

4 Exercise: [4 points]

Orbit control: given two kickers located at the two ends of a FODO cell with phase advance 45 degrees (the two kickers are located at L_{cell} distance from each other), compute the strengths of such kickers (in radians) in order to give the beam, initially at $(x_i,\, x_i') = (0,\, 0)$, an arbitrary offset at the end of the cell while preserving its angle, $\left(x_f,\, x_f'\right) = (x_{\text{arbitrary}},\, 0)$.

Solution

The transfer matrix of a periodic cell is:

$$M = \begin{pmatrix} \cos\varphi + \alpha\sin\psi & \beta\sin\varphi \\ -\gamma\sin\varphi & \cos\varphi - \alpha\sin\varphi \end{pmatrix}$$

Substituting the value for the phase advance we get the matrix to apply to the beam right after the first kick k_1:

$$\begin{pmatrix} x_f \\ x_f' \end{pmatrix} = \frac{\sqrt{2}}{2}\begin{pmatrix} 1+\alpha & \beta \\ -\gamma & 1-\alpha \end{pmatrix}\begin{pmatrix} 0 \\ k_1 \end{pmatrix} = \frac{\sqrt{2}}{2}\begin{pmatrix} \beta k_1 \\ (1-\alpha)k_1 \end{pmatrix}$$

From this we see that to achieve an arbitrary x_f we need:

$$k_1 = \frac{\sqrt{2}x_f}{\beta}$$

The second kick, k_2, has only to remove the final tilt:

$$k_2 = -x_f' = -\frac{(1-\alpha)}{\sqrt{2}}k_1$$

Notice that one can reduce the strenght of the kickers by placing them close to a focusing quadrupoles, where β is maximum.