Joint University Accelerator School

Examination of Transverse Beam Dynamics
(solutions)

JUAS 2015

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1 Exercise:

Consider a quadrupole magnet (in thin-lens approximation) focusing electrons with $E = 1$ GeV. Its aperture (diameter) is 20 mm, and its length is 40 mm. The magnetic field at the radius is 0.1 T.

(i) What is the field gradient? (1 pt)

$$G = \frac{B}{d/2} = 10 \text{ T/m}$$

(ii) Calculate the normalised quadrupole field strength, k. What are the units of k? (2 pt)

$$k = \frac{G}{p/q} = 3 \text{ m}^{-2}$$

(iii) What is the maximum angular deviation that this quadrupole can impart to a particle? (3 pt)

$$\Delta x' = L\,k\,x = 0.04 \cdot 3 \cdot 0.01 = 1.2 \text{ mrad}$$

2 Exercise:

The position x of particles can easily be measured, for example using phosphor screens. It is more difficult to measure the particle's angular divergence x'.

(i) Assume a drift section with length L and two phosphor screens located at its ends. Using the transfer matrix formalism, show how the initial angle x'_0 can be derived from the measurements of the displacements x_0 and x at the two screens. (3pt)

$$x = x_0 + x'_0 \cdot L \qquad \Rightarrow \qquad x'_0 = \frac{x - x_0}{L}$$

(ii) Now suppose that there is no initial screen, but instead a focusing quadrupole with focal length f. (Therefore the initial position x_0 cannot be measured anymore). How can the quadrupole be set to measure initial angles x'_0 without knowing x_0? (4 pt)

$$M = \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ \frac{1}{-f} & 1 \end{pmatrix} = \begin{pmatrix} \frac{L}{-f} + 1 & L \\ \frac{1}{-f} & 1 \end{pmatrix}$$

we need to have $M_{11} = 0$ so that

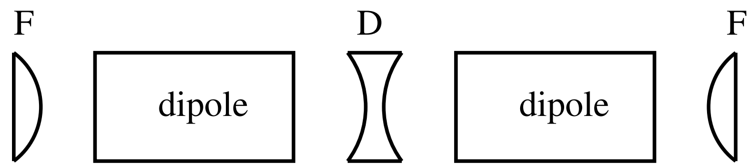
$$x = L \cdot x'_0$$

therefore $f = L$ and

$$x'_0 = \frac{x}{L}.$$

3 Exercise:

A Muon Collider is one of the options being considered for a forthcoming Higgs factory. The Muon Collider design is conceived as a ring with a circumference 6 km long where muons and anti-muons circulate for about 1000 turns before they decay. Let us assume that the optics of this machine is made of identical FODO cells, each cell with the following structure:



Assuming for the calculations the thin lens approximation, and considering the following parameters:

Parameter	Value
Beam energy	1.5 TeV
Total FODO cell length	80 m
Quadrupole gradient G	20 T/m
Quadrupole length	4 m
Dipole length	21.3 m

answer the following questions:

(i) Calculate: (1) for each dipole the bending angle and the necessary magnetic field; (2pt) (2) the value of the focal length of each quadrupole (2pt)

(1) Solution. Knowing the length of each cell L=80 m and the total circumference of the machine (6 km), we can calculate the number of bending magnets (we know that there are two per cell):

$$N_{\text{cell}} = 6000/80 = 75$$

cells and

$$N_{\text{bending}} = 2 \times 75 = 150$$

bending magnets. Therefore, the necessary bending angle for each dipole is

$$\theta = 2\pi/N_{bending} = 0.042 \text{ [rad]} = 2.4 \text{ [degrees]}.$$

In order to calculate the magnetic field, we calculate first the rigidity

$$B\rho = p/e = 3.3356 \cdot p[GeV/c] = 5003.4 \text{ T m}.$$

Combining this expression with $\theta = l_B/\rho$ (where l_B is the length of the bending magnet), we obtain

$$B = 5003.4 \cdot \frac{\theta}{l_B} = 9.9 \text{ T}.$$

(2) Solution. We can use the beam rigidity (or the particle momentum) to calculate the normalised quadrupole strength:

$$k = \frac{G}{B\rho} = \frac{G}{p/e} = 0.299792 \cdot \frac{G}{p[\text{GeV/c}]} = 0.299792 \cdot \frac{20 \text{ T/m}}{1500\text{GeV/c}} = 0.004 \text{ m}^{-2}$$

an the focal length (using the quadrupole length $l_q = 4$ m):

$$f = \frac{1}{k \cdot l_q} = 62.5 \text{ m}$$

(ii) Is this lattice stable? (2pt)

Solution. To answer this question let us use the transfer matrix for this FODO cell. We can calculate it multiplying the corresponding matrices for each element in thin lens approximations. However, we know from the exercises bulletin 5 (student will have access to all kind of material) that the 3×3 transfer matrix for this kind of cell is given by

$$
\begin{aligned}
M_{cell} &= \begin{pmatrix} 1 - \frac{L^2}{8f^2} & L\left(1 + \frac{L}{4f}\right) & L\left(1 + \frac{L}{8f}\right)\theta \\ -\frac{L}{4f^2}\left(1 - \frac{L}{4f}\right) & 1 - \frac{L^2}{8f^2} & 2\left(1 - \frac{L}{8f} - \frac{L^2}{32f^2}\right)\theta \\ 0 & 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} \cos\mu & \beta\sin\mu & L\left(1 + \frac{L}{8f}\right)\theta \\ -\frac{\sin\mu}{\beta} & \cos\mu & 2\left(1 - \frac{L}{8f} - \frac{L^2}{32f^2}\right)\theta \\ 0 & 0 & 1 \end{pmatrix}
\end{aligned}
$$

Then we have

$$\cos\mu = 1 - \frac{L^2}{8f^2} = 0.795 < 1$$

and therefore it is stable.

(iii) What are the vertical and horizontal tunes for this machine? (2pt)

Solution. The phase advance per cell is $\mu \simeq 0.65$ rad

$$Q_{x,y} = \frac{N_{cell}\mu_{x,y}}{2\pi} = 7.76$$

(iv) Compute the maximum and the minimum value for the betatron functions, and the maximum and minimum value of the dispersion (1pt+1pt+1pt+1pt)

Solution.

The maximum beta function for this lattice is:

$\beta_{max} = \frac{L}{\sin\mu}\left(1 + \frac{L}{4f}\right) = 174.49$ m

and the minimum can be calculated substituting $(f \to -f)$:

$\beta_{min} = \frac{L}{\sin\mu}\left(1 - \frac{L}{4f}\right) = 89.89$ m. We usually assume no dispersion in the vertical plane so $D_y = 0$. The maximum horizontal dispersion is at the focusing quadrupoles of this lattice and taking the corresponding matrix element:

$D_{x,max} = \frac{L\theta\left(1 + \frac{1}{2}\sin\frac{\mu}{2}\right)}{4\sin^2\frac{\mu}{2}} = 9.53$ m

and the corresponding minimum dispersion will be at the defocusing quadrupoles $(f \to -f)$:

$D_{x,min} = \frac{L\theta\left(1 - \frac{1}{2}\sin\frac{\mu}{2}\right)}{4\sin^2\frac{\mu}{2}} = 6.90$ m

(v) Calculate the natural chromaticities for this machine (3pt)

Solution. From the definition of natural chromaticity:

$$
\begin{aligned}
\xi_N &= -\frac{1}{4\pi}\oint \beta(s)k(s)ds \\
&= -\frac{1}{4\pi} \times N_{cell}\int_{cell} \beta(s)k(s)ds \\
&= -\frac{N_{cell}}{4\pi}\sum_{i\in\{quads\}} \beta_i(kl_q)_i
\end{aligned}
$$

Here we have used the following approximation valid for thin lens:

$$\int_{cell} \beta(s)k(s)ds \simeq \sum_{i\in\{quads\}} \beta_i (kl_q)_i$$

where we sum over each quadrupole i in the cell. In the case of the FODO cell we have two half focusing quadrupoles and one defocusing quadrupole. Taking into account that $(kl_q)_i = 1/f_i$, we have:

$$\begin{aligned}
\xi_N &\simeq -\frac{N_{cell}}{4\pi} \sum_{i\in\{quads\}} \beta_i (kl_q)_i \\
&= -\frac{N_{cell}}{4\pi}\left[\beta_{max}\left(\frac{1}{2f}\right) + \beta_{min}\left(-\frac{1}{f}\right) + \beta_{max}\left(\frac{1}{2f}\right)\right] \\
&= -\frac{N_{cell}}{4\pi}\left[\beta_{max}\left(\frac{1}{f}\right) + \beta_{min}\left(-\frac{1}{f}\right)\right] \\
&= -\frac{N_{cell}}{4\pi \sin\mu}\left[\left(L + \frac{L^2}{4f}\right)\frac{1}{f} - \left(L - \frac{L^2}{4f}\right)\frac{1}{f}\right] \\
&= -\frac{N_{cell}}{8\pi \sin\mu}\frac{L^2}{f^2} = -8.079
\end{aligned}$$

Here we have used the expression for β_{max} and β_{min}.

(vi) Let's imagine that all focusing quadrupoles of our ring are connected to faulty power supplies that coherently provide $+10\%$ more current than required. What it the tune shift induced? (neglect beta beat effects) (3pt)

Solution. We know that the tune shift is defined as

$$\Delta Q = \oint_{\text{quads}} \frac{\Delta k\,(s)\,\beta\,(s)\,\mathrm{d}s}{4\pi}$$

if the error is only on the focusing quadrupoles, we can write:

$$\Delta Q = \sum_{\text{focusing quads}} \frac{\Delta k\, l_q\, \beta_{max}}{4\pi}$$

since $\Delta k = 0.1 \cdot 0.004\ \text{m}^{-2} = 0.0004\ \text{m}^{-2}$, $l_q = 4$ m, and $\beta_{max} = 174.49$ m:

$$\Delta Q = N_{cell} \frac{\Delta k\, l_q\, \beta_{max}}{4\pi} = 1.66$$

(vii) If instead the errors are randomly distributed, that is, each quadrupole is powered with a random current, Gaussian distributed within 10% RMS from the nominal value, what is the tune shift induced? (neglect beta beat effects) (1pt)

Solution. As all errors add coherently and each belong to a random variable with average 0, the tune shift will be the average of all errors, that is 0:

$$\Delta Q = \sum_{\text{focusing quads}} \frac{\Delta k_{\text{random}}\, l_q\, \beta_{max}}{4\pi} = N_{cell} \frac{\langle \Delta k_{\text{random}} \rangle\, l_q\, \beta_{max}}{4\pi} = 0$$

4 Exercise:

Let's now assume that in such a Muon Collider there is an Interaction Point (IP). Preliminary designs foresee an IP with $\beta_x^* = \beta_y^* = 0.3$ cm. The peak luminosity available at a $\mu^+\mu^-$ collider can be written as:

$$L = \frac{N_{\rm b} N_{\mu^-} N_{\mu^+} f_{\rm rev}}{4\pi\sigma_x^*\sigma_y^*} \ [\mathrm{cm}^{-2}\mathrm{s}^{-1}]$$

where $N_{\rm b} = 4$ is the number of bunches per beam (we assume the same number of bunches for both the μ^- and the μ^+ beams), $N_{\mu^-} = N_{\mu^+} = 2 \times 10^{12}$ is the number of particles per bunch (we assume the same number for both μ^- and μ^+ bunches), and $f_{\rm rev}$ is the revolution frequency. The horizontal and vertical normalised emittances are: $\epsilon_{x,N} = \epsilon_{y,N} = 50$ μm. [Remember that the mass of the muon is $m_{\mu^\pm} \simeq 105.6$ MeV]

(i) Compute the revolution frequency $f_{\rm rev}$, knowing that the circumference is 6 km and that the beam moves nearly at the speed of light (1pt)

Solution. The revolution period is given by T_{rev} = circumference$/c$ = 6km/c, and therefore the revolution frequency is:

$$f_{rev} = 1/T_{rev} = c/6\mathrm{km} \simeq 50 \text{ kHz}$$

(ii) Calculate the beam transverse beam sizes σ_x^* and σ_y^* at the IP, and the luminosity L, at beam energy 1.5 TeV (3pt)

Solution. For 1.5 TeV beam energy: in this case the Lorentz factor is $\gamma = 1.42 \times 10^4$, and $\sigma_x^* = \sqrt{\beta_x^* \epsilon_{x,N}/\gamma} \simeq 3.25$ μm and $\sigma_y^* = \sqrt{\beta_y^* \epsilon_{y,N}/\gamma} \simeq 3.25$ μm, and the luminosity is $L \simeq 6.03 \times 10^{35}$ cm^{-2}s^{-1}

(iii) What is the value of the betatron function at position $s = 0.5$ m from the IP? (2pt)

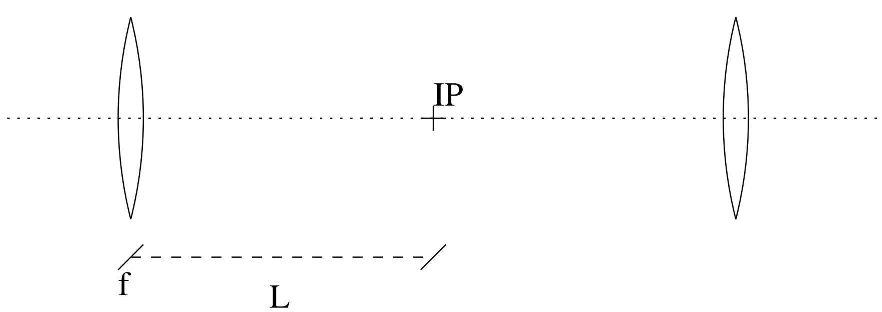
Solution. We know that the betatron function in the drift space of a low beta region (where we have the interaction point) depends on the longitudinal coordinate as follows:

$$\beta(s) = \beta^* + \frac{s^2}{\beta^*}$$

Therefore, $\beta(0.5\mathrm{m}) = 83.336$ m

5 Exercise:

Consider the following low-beta insertion around an interaction point (IP). The quadrupoles are placed with mirror-symmetry with respect to the IP:



The beam enters the quadrupole with twiss parameters $\beta_0 = 20$ m and $\alpha_0 = 0$. The drift space has length $L = 10$ m.

(i) Determine the focal length of the quadrupole in order to locate the waist at the IP. (4 pt)

(ii) What is the value of $\beta^\star$? (2pt)

(iii) What is the phase advance between the quadrupole and the IP? (4 pt)

Solution.

$$M = \begin{pmatrix} \frac{L}{-f}+1 & L \\ \frac{1}{-f} & 1 \end{pmatrix}$$

$$\begin{pmatrix} \beta & -\alpha \\ -\alpha & \gamma \end{pmatrix}_{\mathrm{IP}} = M \cdot \begin{pmatrix} \beta & -\alpha \\ -\alpha & \gamma \end{pmatrix}_0 \cdot M^T$$

$$\begin{pmatrix} \beta_{\mathrm{IP}} & 0 \\ 0 & 1/\beta_{\mathrm{IP}} \end{pmatrix} = M \cdot \begin{pmatrix} \beta_0 & 0 \\ 0 & 1/\beta_0 \end{pmatrix} \cdot M^T$$

From which one can find:

$$f = 20 \text{ m}.$$

The phase advance can be computed remembering that

$$M_{0\to s} = \begin{pmatrix} \sqrt{\frac{\beta_s}{\beta_0}}\left(\cos\psi_s + \alpha_0 \sin\psi_s\right) & \sqrt{\beta_s\beta_0}\sin\psi_s \\ \frac{(\alpha_0-\alpha_s)\cos\psi_s-(1+\alpha_0\alpha_s)\sin\psi_s}{\sqrt{\beta_s\beta_0}} & \sqrt{\frac{\beta_0}{\beta_s}}\left(\cos\psi_s - \alpha_s\sin\psi_s\right) \end{pmatrix}$$

In this case, $\alpha_0 = \alpha_{\mathrm{IP}} = 0$,

$$\mathrm{Trace}\,(M) = \frac{3}{2} = \left(\sqrt{\frac{\beta^\star}{\beta_0}} + \sqrt{\frac{\beta^0}{\beta^\star}}\right)\cos\Delta\mu$$

$$\Delta\mu = \arccos\left(\frac{3}{2}\cdot\frac{1}{\sqrt{\frac{\beta^\star}{\beta_0}}+\sqrt{\frac{\beta^0}{\beta^\star}}}\right) = \arccos\left(\frac{3}{2}\cdot\frac{1}{2.1213}\right) = 45 \text{ degrees}$$

Alternatively, given that the system:

$$M = Q\cdot D\cdot D\cdot Q$$

is indeed periodic, one can say:

$$M = \begin{pmatrix} 1-\frac{2L}{f} & 2L \\ \frac{2L}{f^2}-\frac{2}{f} & 1-\frac{2L}{f} \end{pmatrix}$$

$$\cos\Delta\mu_{\mathrm{twice}} = \frac{1}{2}\mathrm{Trace}\,(M) = \frac{1}{2}\mathrm{Trace}\left(2-\frac{4L}{f}\right) = 0$$

$$\Delta\mu_{\mathrm{twice}} = 90 \text{ degrees} \qquad \Rightarrow \Delta\mu = 45 \text{ degrees}$$

Table of points

Exercise	question	pt
1 (6pt)	(i)	1
	(ii)	2
	(iii)	3
2 (7pt)	(i)	3
	(ii)	4
3 (19)	(i)	2+2
	(ii)	2
	(iii)	2
	(iv)	1+1+1+1
	(v)	3
	(vi)	3
	(vii)	1

Exercise	question	pt
4 (6pt)	(i)	1
	(ii)	3
	(iii)	2
5 (10pt)	(i)	4
	(ii)	2
	(iii)	4
TOTAL		48