Non-linear effects

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Bibliography

- Books on non-linear dynamical systems
  - M. Tabor, Chaos and Integrability in Nonlinear Dynamics, An Introduction, Willey, 1989.
  - A.J Lichtenberg and M.A. Lieberman, Regular and Chaotic Dynamics, 2nd edition, Springer 1992.
- Books on beam dynamics
  - E. Forest, Beam Dynamics - A New Attitude and Framework, Harwood Academic Publishers, 1998.
  - H. Wiedemann, Particle accelerator physics, 3rd edition, Springer 2007.
- Lectures on non-linear beam dynamics
  - A. Chao, Advanced topics in Accelerator Physics, USPAS, 2000.
  - A. Wolski, Lectures on Non-linear dynamics in accelerators, Cockroft Institute 2008.
  - W. Herr, Lectures on Mathematical and Numerical Methods for Non-linear Beam Dynamics in Rings, CAS 2013.
  - L. Nadolski, Lectures on Non-linear beam dynamics, Master NPAC, LAL, Orsay 2013.

Contents of the 1st lecture

- Accelerator performance parameters and non-linear effects
- Linear and non-linear oscillators
  - Integral and frequency of motion
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Accelerator performance parameters

$$L = \frac{N_b^2 k_b \gamma}{4\pi \epsilon_n \beta^*}$$

$$\bar{P} = \bar{I}E = f_N N e E$$

$$B = \frac{N_p}{4\pi^2 \epsilon_x \epsilon_y}$$

- Colliders
  - Luminosity (i.e. rate of particle production)
    - N_b bunch population
    - k_b number of bunches
    - γ relativistic reduced energy
    - ε_n normalized emittance
    - β^* "betatron" amplitude function at collision point
- High intensity accelerators
  - Average beam power
    - $\bar{I}$ mean current intensity
    - E energy
    - f_N repetition rate
    - N number of particles/pulse
- Synchrotron light sources (low emittance rings)
  - Brightness (photon density in phase space)
    - N_p number of photons
    - $\varepsilon_{x,,y}$ transverse emittances
- Non-linear effects limit performance of particle accelerators but impact also design cost

Non-linear effects in colliders

$$L = \frac{N_b^2 k_b \gamma}{4\pi \epsilon_n \beta^*}$$

- At injection
  - Non-linear magnets (sextupoles, octupoles)
  - Magnet imperfections and misalignments
  - Power supply ripple
  - Ground motion (for e+/e-)
  - Electron (Ion) cloud
- At collision
  - Insertion Quadrupoles
  - Magnets in experimental areas (solenoids, dipoles)
  - Beam-beam effect (head on and long range)
- Limitations affecting (integrated) luminosity
  - Particle losses causing
    - Reduced lifetime
    - Radio-activation (super-conducting magnet quench)
    - Reduced machine availability
  - Emittance blow-up
  - Reduced number of bunches (either due to electron cloud or long-range beam-beam)
  - Increased crossing angle
  - Reduced intensity
- Cost issues
  - Number of magnet correctors and families (power convertors)
  - Magnetic field and alignment tolerances

$$\bar{P} = \bar{I}E = f_N NeE$$

- Non-linear magnets (sextupoles, octupoles)
- Magnet imperfections and misalignments
- Injection chicane
- Magnet fringe fields
- Space-charge effect

- Limitations affecting beam power
  - Particle losses causing
    - Reduced intensity
    - Radio-activation (hands-on maintenance)
    - Reduced machine availability
  - Emittance blow-up which can lead to particle loss
- Cost issues
  - Number of magnet correctors and families (power convertors)
  - Magnetic field and alignment tolerances
  - Design of the collimation system

$$B = \frac{N_p}{4\pi^2 \epsilon_x \epsilon_y}$$

- Chromaticity sextupoles
- Magnet imperfections and misalignments
- Insertion devices (wigglers, undulators)
- Injection elements
- Ground motion
- Magnet fringe fields
- Space-charge effect (in the vertical plane for damping rings)
- Electron cloud (Ion) effects

- Limitations affecting beam brightness
  - Reduced injection efficiency
  - Particle losses causing
    - Reduced lifetime
    - Reduced machine availability
  - Emittance blow-up which can lead to particle loss
- Cost issues
  - Number of magnet correctors and families (power convertors)
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Reminder: Harmonic oscillator

- Described by the differential equation:

$$\frac{d^2u(t)}{dt^2} + \omega_0^2 u(t) = 0$$

- The solution obtained by the substitution $u(t) = e^{\lambda t}$

  and the solutions of the characteristic polynomial are

  $\lambda_\pm = \pm i\omega_0$ which yields the general solution

$$u(t) = ce^{i\omega_0 t} + c^* e^{-i\omega_0 t} = C_1 \cos(\omega_0 t) + C_2 \sin(\omega_0 t) = A \sin(\omega_0 t + \phi)$$

- The amplitude and phase depend on the initial conditions

$$A = \frac{\left(\frac{du}{dt}(0)^2 + \omega_0^2 u(0)^2\right)^{1/2}}{\omega_0} \ , \ \tan(\phi) = \frac{\frac{du}{dt}(0)}{\omega_0 u(0)}$$

- Note that a negative sign in the differential equation provides a solution described by an hyperbolic sine function
- Note also that for no restoring force $\omega_0 = 0$, the motion is unbounded

Integral of motion

- Rewrite the differential equation of the harmonic oscillator as a pair of coupled 1st order equations

$$\frac{du(t)}{dt} = p_u(t)$$
$$\frac{dp_u(t)}{dt} = -\omega_0^2 u(t)$$

which can be combined to provide

$$\frac{dp_u}{dt}p_u + \omega_0^2 u\frac{du}{dt} = \frac{1}{2}\frac{d}{dt}\left(p_u^2 + \omega_0^2 u^2\right) = 0$$ or

$\frac{1}{2}\left(p_u^2 + \omega_0^2 u^2\right) = I_1$ with I_1 an **integral of motion** identified as the mechanical energy of the system

- Solving the previous equation for p_u, the system can be reduced to a first order equation

$$\frac{du}{dt} = \sqrt{2I_1 - \omega_0^2 u^2}$$

Integration by quadrature

- The last equation can be be solved as an explicit integral or "**quadrature**"

$$\int dt = \int \frac{du}{\sqrt{2I_1 - \omega_0^2 u^2}}$$, yielding $$t + I_2 = \frac{1}{\omega_0} \arcsin\left(\frac{u\omega_0}{\sqrt{2I_1}}\right)$$

or the well-known solution $u(t) = \frac{\sqrt{2I_1}}{\omega_0} \sin(\omega_0 t + \omega_0 I_2)$

- **Note**: Although the previous route may seem complicated, it becomes more natural when non-linear terms appear, where a substitution of the type $u(t) = e^{\lambda t}$ is not applicable
- The ability to integrate a differential equation is not just a nice mathematical feature, but deeply characterizes the dynamical behavior of the system described by the equation

Frequency of motion

- The period of the harmonic oscillator is calculated through the previous integral after integration between two extrema (when the velocity $\frac{du}{dt} = \sqrt{2I_1 - \omega_0^2 u^2}$ vanishes), i.e. $x_{\rm ext} = \pm \frac{\sqrt{2I_1}}{\omega_0}$:

$$T = 2\int_{-\frac{\sqrt{2I_1}}{\omega_0}}^{\frac{\sqrt{2I_1}}{\omega_0}} \frac{du}{\sqrt{2I_1 - \omega_0^2 u^2}} = \frac{2\pi}{\omega_0}$$

- The frequency (or the period) of linear systems is independent of the integral of motion (energy)
- Note that this is not true for non-linear systems, e.g. for an oscillator with a non-linear restoring force $\frac{d^2u}{dt^2} + k\ u(t)^3 = 0$
- The integral of motion is $I_1 = \frac{1}{2}p_u^2 + \frac{1}{4}k\ u^4$ and the integration yields

$$T = 2\int_{-(4I_1/k)^{1/4}}^{(4I_1/k)^{1/4}} \frac{du}{\sqrt{2I_1 - \frac{1}{2}k\ u^4}} = \sqrt{\frac{1}{2\pi}\Gamma^2(\frac{1}{4})}\,(I_1\ k)^{-1/4}$$

- This means that the period (frequency) depends on the integral of motion (energy) i.e. the maximum "amplitude"

The pendulum

- An important non-linear equation which can be integrated is the one of the pendulum, for a string of length L and gravitational constant g

$$\frac{d^2\phi}{dt^2} + \frac{g}{L}\sin\phi = 0$$

- For small displacements it reduces to an harmonic oscillator with frequency $\omega_0 = \sqrt{\frac{g}{L}}$
- The integral of motion (scaled energy) is

$$\frac{1}{2}\left(\frac{d\phi}{dt}\right)^2 - \frac{g}{L}\cos\phi = I_1 = E'$$

and the quadrature is written as $t = \int \frac{d\phi}{\sqrt{2(I_1 + \frac{g}{L}\cos\phi)}}$
assuming that for $t = 0\ ,\quad \phi = 0$

Solution for the pendulum

- Using the substitutions $\cos\phi = 1 - 2k^2\sin^2\theta$ with $k = \sqrt{1/2(1 + I_1 L/g)}$, the integral is

$$t = \sqrt{\frac{L}{g}}\int_0^\theta \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}}$$ and can be solved using

Jacobi elliptic functions: $\phi(t) = 2\arcsin\left[k\,\mathrm{sn}\left(t\sqrt{\frac{g}{L}},k\right)\right]$

- For recovering the period, the integration is performed between the two extrema, i.e. $\phi = 0$ and $\phi = \arccos(-I_1 L/g)$, corresponding to $\theta = 0$ and $\theta = \pi/2$, for which

$$T = 4\sqrt{\frac{L}{g}}\int_0^{\pi/2} \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}} = 4\sqrt{\frac{L}{g}}\mathcal{F}(\pi/2,k)$$

i.e. the complete elliptic integral multiplied by four times the period of the harmonic oscillator

- Damped harmonic oscillator:

$$\frac{d^2u(t)}{dt^2} + \frac{\omega_0}{Q}\frac{du(t)}{dt} + \omega_0^2 u(t) = 0$$

  - $Q = \frac{1}{2\zeta}$ is the ratio between the stored and lost energy per cycle with ζ the damping ratio
  - ω_0 is the eigen-frequency of the harmonic oscillator
- General solution can be found by the same ansatz

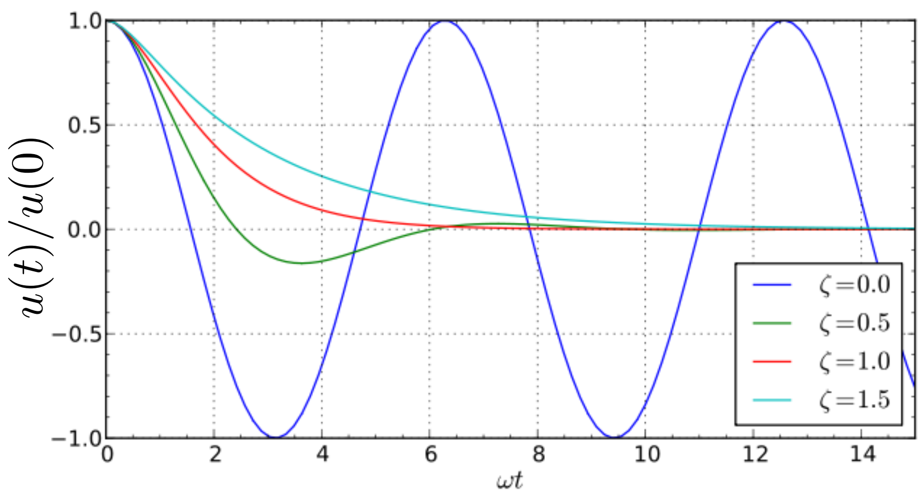
$$u(t) = e^{\lambda t}$$

leading to an auxiliary 2nd order equation

$\lambda^2 + \frac{\omega_0}{Q}\lambda + \omega_0^2 = 0$ with solutions

$$\lambda_\pm = -\frac{\omega_0}{2Q}(-1 \pm \sqrt{1-4Q^2}) = -\omega_0\zeta(-1 \pm \sqrt{1-\frac{1}{\zeta^2}})$$

- Three cases can be distinguished
  - *Overdamping* (λ real, i.e. $\zeta > 1$ or $Q < 1/2$): The system exponentially decays to equilibrium (slower for larger damping ratio values)
  - *Critical damping* ($\zeta = 1$): The system returns to equilibrium as quickly as possible without oscillating.
  - *Underdamping* (λ complex, i.e. $\zeta < 1$ or $Q > 1/2$): The system oscillates with the amplitude gradually decreasing to zero, with a slightly different frequency than the *harmonic* one: $\omega_d = \omega_0 \sqrt{1-\zeta^2}$



- Note that there is no integral of motion, in that case, as the energy is not conserved (dissipative system)

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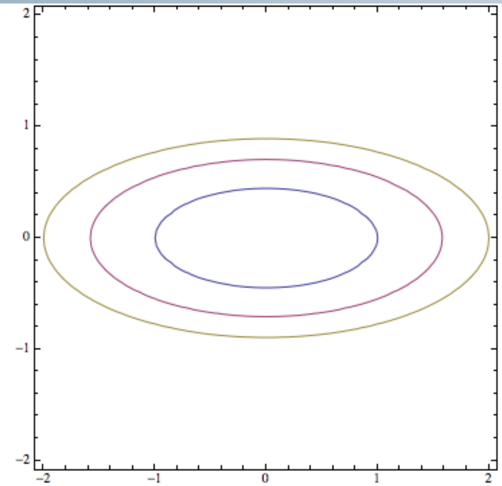
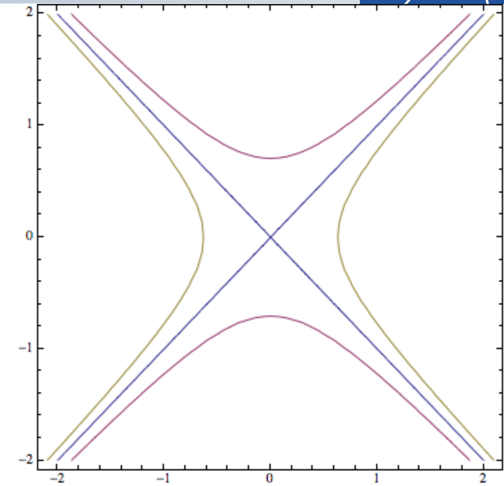
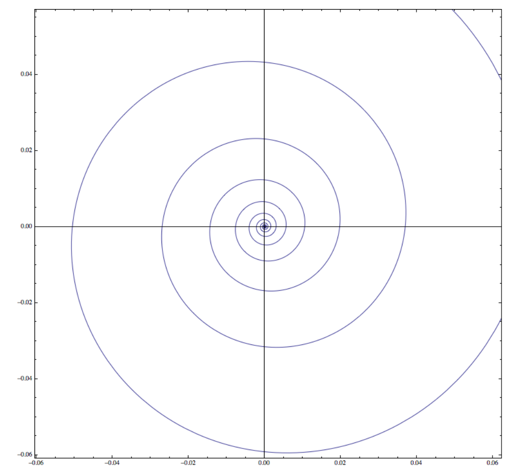
Phase space dynamics

- Valuable description when examining trajectories in phase space $(u, \frac{du}{dt})$
- Existence of integral of motion imposes geometrical constraints on phase flow
- For the harmonic oscillator (left), phase space curves are ellipses around the equilibrium point parameterized by the integral of motion (energy)
- By simply changing the sign of the potential in the harmonic oscillator (right), the phase trajectories become hyperbolas, symmetric around the equilibrium point where two straight lines cross, moving towards and away from it

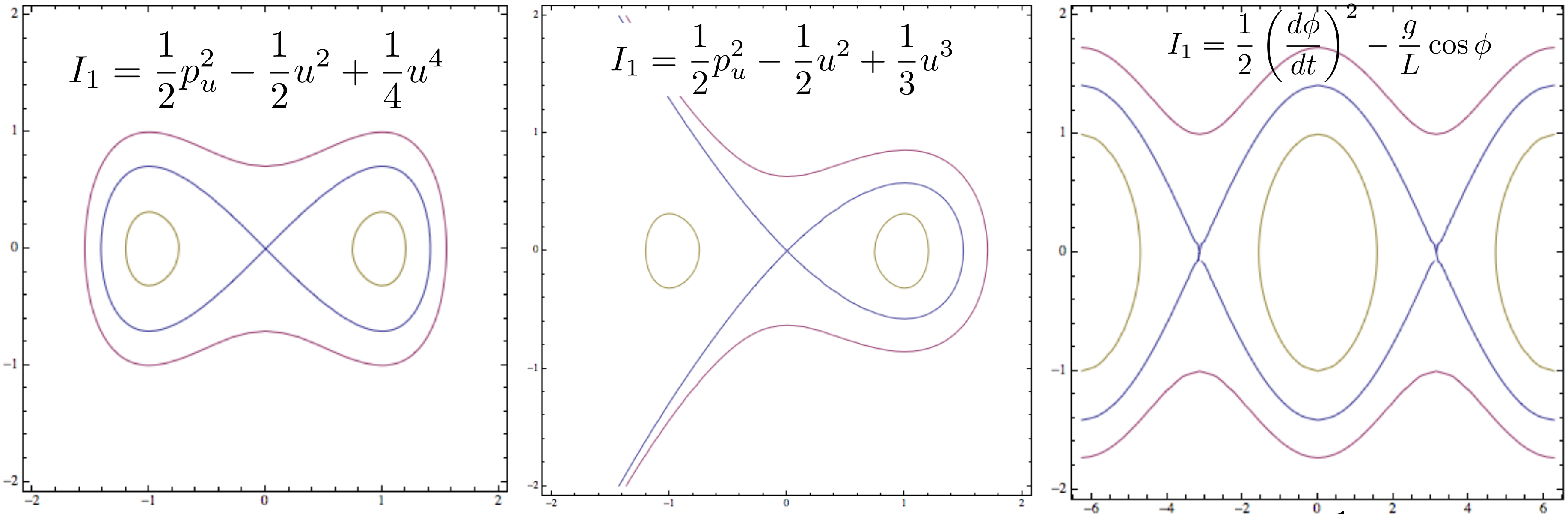






- For the damped harmonic oscillator (above), the phase space trajectories are spiraling towards equilibrium with a rate depending on the damping coefficient

Non-linear oscillators



- Conservative non-linear oscillators have quadrature $I_1 = E = \frac{1}{2}p_u^2 + V(u)$ with potential being a general (polynomial) function of positions
- The equations of motion are $\frac{d^2u}{dt^2} + \frac{dV(u)}{du} = 0$
- Equilibrium points are associated with extrema of the potential
- Considering three non-linear oscillators
  - Quartic potential (left): two minima and one maximum
  - Cubic potential (center): one minimum and one maximum
  - Pendulum (right): periodic minima and maxima

Fixed point analysis

- Consider a general second order system
$$\begin{aligned}\frac{du}{dt} &= f_1(u,p_u)\\ \frac{dp_u}{dt} &= f_2(u,p_u)\end{aligned}$$
- Equilibrium or "**fixed**" points $f_1(u_0,p_{u0}) = f_2(u_0,p_{u0}) = 0$ are determinant for topology of trajectories at their vicinity
- The linearized equations of motion at their vicinity are

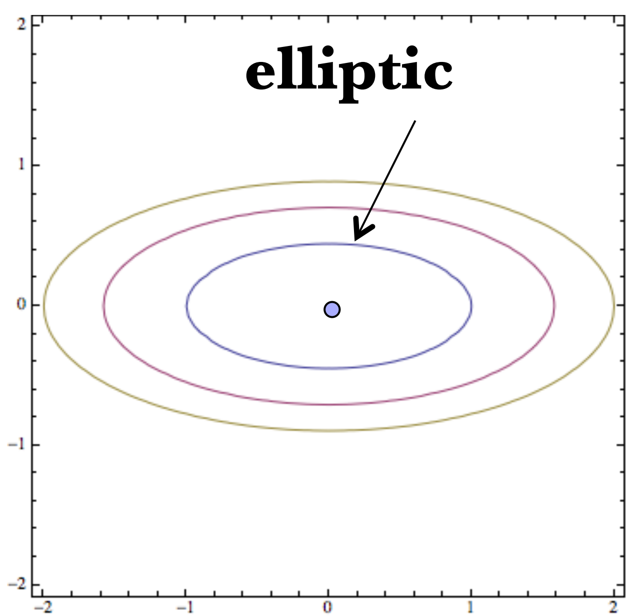
$$\frac{d}{dt}\begin{bmatrix}\delta u\\ \delta p_u\end{bmatrix} = \mathcal{M}_J\begin{bmatrix}\delta u\\ \delta p_u\end{bmatrix} = \underbrace{\begin{bmatrix}\frac{\partial f_1(u_0,p_{u0})}{\partial u} & \frac{\partial f_1(u_0,p_{u0})}{\partial p_u}\\ \frac{\partial f_2(u_0,p_{u0})}{\partial u} & \frac{\partial f_2(u_0,p_{u0})}{\partial p_u}\end{bmatrix}}_{\textbf{Jacobian matrix}}\begin{bmatrix}\delta u\\ \delta p_u\end{bmatrix}$$

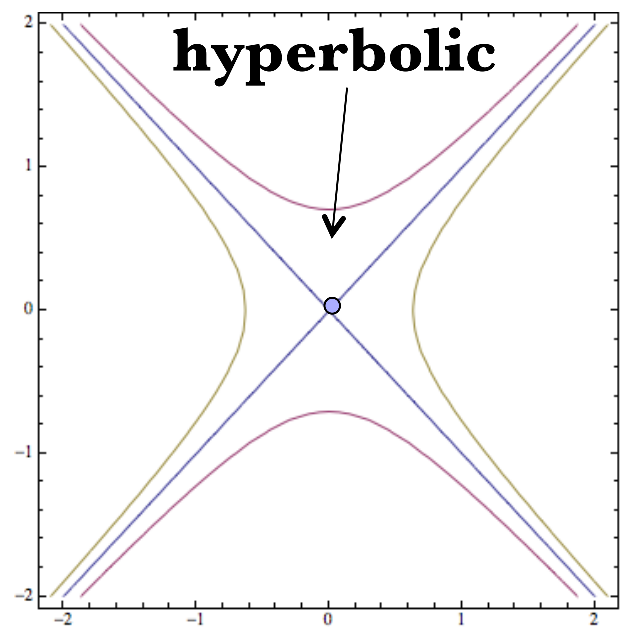
Jacobian matrix

- Fixed point nature is revealed by eigenvalues of $\mathcal{M}_J$, i.e. solutions of the characteristic polynomial $\det|\mathcal{M}_J - \lambda\mathbf{I}| = 0$

Fixed point for conservative systems

- For conservative systems of two dimensions the second order characteristic polynomial has too solutions:
    - Two complex eigenvalues with opposite sign, corresponding to **elliptic** fixed points. Phase space flow is described by ellipses, with particles evolving clockwise or anti-clockwise
    - Two real eigenvalues with opposite sign, corresponding to **hyperbolic** (or saddle) fixed points. Flow described by two lines (or manifolds), incoming (stable) and outcoming (unstable)



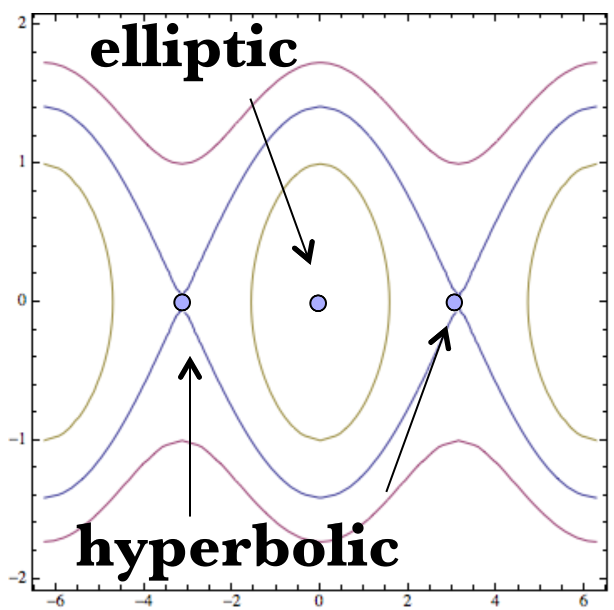


Pendulum fixed point analysis

- The "fixed" points for a pendulum can be found at

$$(\phi_n, \frac{d\phi_n}{dt}) = (\pm n\pi, 0)\ ,\ n = 0, 1, 2\ldots$$

- The Jacobian matrix is $\begin{bmatrix} 0 & 1 \\ -\frac{g}{L}\cos\phi_n & 0 \end{bmatrix}$
- The eigenvalues are $\lambda_{1,2} = \pm i\sqrt{\frac{g}{L}\cos\phi_n}$
- Two cases can be distinguished:
  - $\phi_n = 2n\pi$, for which $\lambda_{1,2} = \pm i\sqrt{\frac{g}{L}}$ corresponding to elliptic fixed points
  - $\phi_n = (2n+1)\pi$, for which $\lambda_{1,2} = \pm\sqrt{\frac{g}{L}}$ corresponding to hyperbolic fixed points
  - The **separatrix** are the stable and unstable manifolds passing through the hyperbolic points, separating bounded **librations** and unbounded **rotations**



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Non-autonomous systems

- Consider a linear system with explicit dependence in time
$$\frac{d^2u}{dt^2} + \omega_0^2 u = F(t)$$
- **Time** now is an **independent** variable and can be considered as an **extra dimension** leading to a completely new type of behavior
- Consider two independent solutions of the homogeneous equation $u_1(t)$ and $u_2(t)$
- The general solution is a sum of the homogeneous solutions $u_h(t) = c_1 u_1(t) + c_2 u_2(t)$ and a particular solution, $u_p(t) = c_3 u_1(t) + c_4 u_2(t)$, where the coefficients are computed as $c_3 = \int \frac{u_2(t)F(t)}{W(t)} dt$, $c_4 = \int \frac{u_1(t)F(t)}{W(t)} dt$

with the **Wronskian** of the system
$$W(t) = u_1(t)\frac{du_2(t)}{dt} - u_2(t)\frac{du_1(t)}{dt}$$

Driven harmonic oscillator

- Consider periodic force pumping energy into the system

$$\frac{d^2u(t)}{dt^2} + \omega_0^2 u(t) = \frac{F}{m}\cos(\omega t)$$

- General solution is a combination of the homogeneous and a particular solution found as

$$u(t) = u_0(t)\sin(\omega_0 t + \phi_0) + \frac{F}{m(\omega_0^2 - \omega^2)}\cos(\omega t)$$

- Obviously a **resonance** condition appears when driving frequency hits the oscillator eigen-frequency. In the limit of $\omega \longrightarrow \omega_0$ the solution becomes

$$u(t) = u_0'(t)\sin(\omega_0 t + \phi_0') + \frac{F}{2m\omega_0} t\sin(\omega)$$

- The 2nd secular term implies unbounded growth of amplitude at resonance

Damped oscillator with periodic driving

- Consider periodic force pumping energy into the system

$$\frac{d^2u(t)}{dt^2} + \frac{\omega_0}{Q}\frac{du(t)}{dt} + \omega_0^2 u(t) = \frac{F}{m}\cos(\omega t)$$

- The solution of the homogeneous system is

$$u_h(t) = u_0(t)e^{-\omega_0\zeta t}\sin(\omega_0\sqrt{1-\zeta^2}\,t + \phi_0)$$

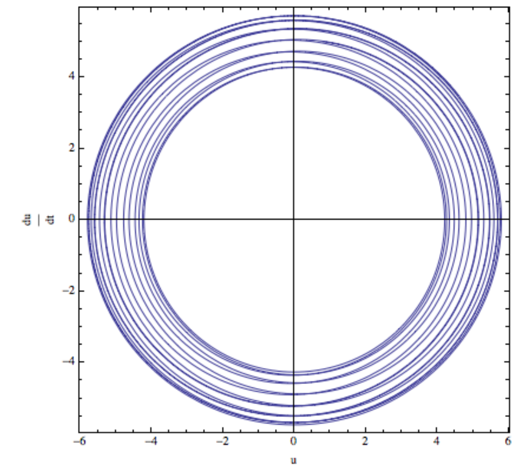
- The particular solution is

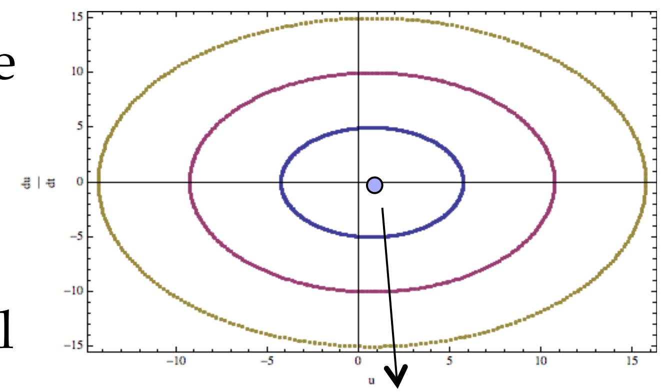
$$u_p(t) = \frac{F\cos(\omega t + \phi_0')}{m\,\omega_0^2\sqrt{(1-\frac{\omega^2}{\omega_0^2})^2 + 4\zeta^2\frac{\omega^2}{\omega_0^2}}}$$

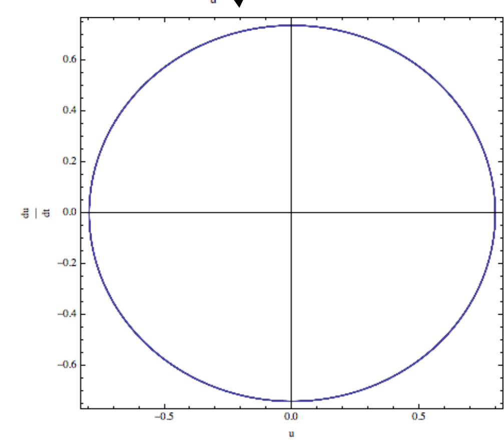
- The homogeneous solution vanishes for $t \to \infty$, leaving only the particular one, for which there is an amplitude maximum for $\omega_0 = \omega$ but no divergence
- In that case the energy pumped into the system compensate the friction, a steady state representing a limit cycle

Phase space for non-autonomous systems

- Plotting the evolution of the **driven oscillator** in $(u, \frac{du}{dt})$ provides trajectories that intersect each other (top)
- The phase space has time as an extra 3rd dimension
- By rescaling the time to become $\tau = \omega t$ and considering every integer interval of the new time variable, the phase space looks like the one of the harmonic oscillator (middle)
- This is the simplest version of a (Poincaré) **surface of section**, which is useful for studying geometrically phase space of multi-dimensional systems
- The fixed point in the surface of section is now a periodic orbit (bottom) defined by $u(t) = \frac{F\cos(\omega t)}{m(\omega_0^2 - \omega^2)}$
- In that case, one can show the existence of two integrals of motion, but when a non-linearity is introduced, the system becomes non-integrable







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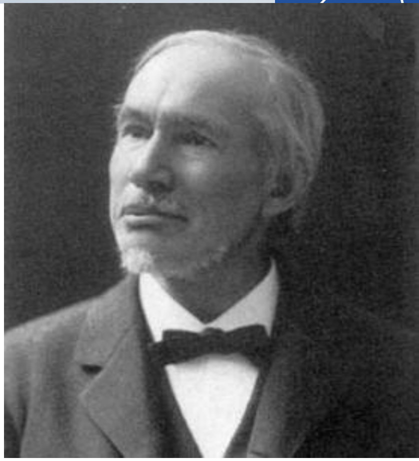


George Hill

- A very important class of equations especially for beam dynamics (but also solid state physics) are linear equations with periodic coefficients

$$\frac{d^2u}{dt^2} + K(t)u = 0$$

with $K(t) = K(t+T)$ a periodic function of time

- These are called **Hill's equations** and can be thought as equations of harmonic oscillator with time dependent (periodic) frequency
- There are two solutions that can be written as $u(t) = \Re\left\{w(t)e^{i\psi(t)}\right\}$ with $w(t) = w(t+T)$ periodic but also $e^{i\psi(t+T)-\psi(t)} = e^{i\sigma}$ with σ a constant which implies that $\frac{d\psi}{dt}(t+T) = \frac{d\psi}{dt}(t)$ is periodic
- The solutions are derived based on **Floquet** theory

Amplitude, phase and invariant

- Differentiating the solutions twice and substituting to Hill's equation, the following two equations are obtained

$$\begin{aligned} \frac{d^2w}{dt^2} - w(\frac{d\psi}{dt})^2 + K(t)w &= 0 \\ 2\frac{dw}{dt}\frac{d\psi}{dt} + w\frac{d^2\psi}{dt^2} &= 0 \end{aligned}$$

- The 2nd one can be integrated to give $\frac{d\psi}{dt} = \frac{1}{w^2}$, i.e. the relation between the "phase" and the amplitude
- Substituting this to the 1st equation, the amplitude equation is derived (or the beta function in accelerator jargon)

$$\frac{d^2w}{dt^2} + K(t)w - \frac{1}{w^3} = 0$$

- By evaluating the quadratic sum of the solution and its derivative an invariant can be constructed, with the form

$$I(u, \frac{du}{dt}, t) = \left[\frac{u^2}{w^2} + \left(w\frac{du}{dt} - \frac{dw}{dt}u\right)^2\right]$$

Normalized coordinates

- Recall the Floquet solutions for betatron motion

$$u(s) = \sqrt{\epsilon\beta(s)}\cos(\psi(s)+\psi_0)$$

$$u'(s) = -\sqrt{\frac{\epsilon}{\beta(s)}}\left(\sin(\psi(s)+\psi_0)+\alpha(s)\cos(\psi(s)+\psi_0)\right)$$

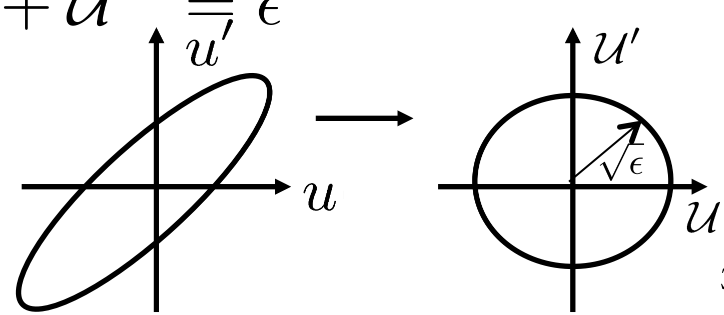
- Introduce new variables

$$\mathcal{U} = \frac{u}{\sqrt{\beta}}\ ,\quad \mathcal{U}' = \frac{d\mathcal{U}}{d\psi} = \frac{\alpha}{\sqrt{\beta}}u + \sqrt{\beta}u'\ ,\quad \phi = \frac{\psi}{\nu} = \frac{1}{\nu}\int\frac{ds}{\beta(s)}$$

- In matrix form $\begin{pmatrix}\mathcal{U}\\ \mathcal{U}'\end{pmatrix} = \begin{pmatrix}\frac{1}{\sqrt{\beta}} & 0\\ \frac{\alpha}{\sqrt{\beta}} & \sqrt{\beta}\end{pmatrix}\begin{pmatrix}u\\ u'\end{pmatrix}$
- Hill' s equation becomes $\frac{1}{\nu^2\beta^{3/2}}\left(\frac{d^2\mathcal{U}}{d\phi^2}+\nu^2\mathcal{U}\right)=0$
- System becomes harmonic oscillator with frequency

$$\begin{pmatrix}\mathcal{U}\\ \mathcal{U}'\end{pmatrix} = \sqrt{\epsilon}\begin{pmatrix}\cos(\nu\phi)\\ -\sin(\nu\phi)\end{pmatrix}\quad \text{or}\quad \mathcal{U}^2+\mathcal{U}'^2 = \epsilon$$

- **Floquet transformation** transforms phase space in circles



Perturbation of Hill's equations

- Hill's equations in normalized coordinates with harmonic perturbation, using $\mathcal{U} = \mathcal{U}_x$ or $\mathcal{U}_y$ and $\phi = \phi_x$ or ϕ_y

$$\frac{d^2\mathcal{U}}{d\phi^2} + \nu^2\mathcal{U} = \nu^2\beta^{3/2}F(\mathcal{U}_x(\phi_x),\mathcal{U}_y(\phi_y))$$

where the *F* is the Lorentz force from perturbing fields

  - **Linear magnet imperfections**: deviation from the design dipole and quadrupole fields due to powering and alignment errors
  - **Time varying fields**: feedback systems (damper) and wake fields due to collective effects (wall currents)
  - **Non-linear magnets**: sextupole magnets for chromaticity correction and octupole magnets for Landau damping
  - **Beam-beam interactions**: strongly non-linear field
  - **Space charge effects**: very important for high intensity beams
  - **non-linear magnetic field imperfections**: particularly difficult to control for super conducting magnets where the field quality is entirely determined by the coil winding accuracy

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  - The pendulum
  - Damped harmonic oscillator
- Phase space dynamics
  - Fixed point analysis
- Non-autonomous systems
  - Driven (damped) harmonic oscillator, resonance conditions
- Linear equations with periodic coefficients – Hill's equations
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- Perturbation theory
  - Non-linear oscillator
  - Perturbation by periodic function – single dipole perturbation
  - Application to single multipole – resonance conditions
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Perturbation theory

- Completely integrable systems are exceptional
- For understanding dynamics of general non-linear systems composed of a part whose solution $u_0(t)$ is known and a part parameterized by a small constant ϵ , perturbation theory is employed
- The general idea is to expand the solution in a power series $u(t) = u_0(t) + \epsilon u_1(t) + \epsilon^2 u_2(t) + \ldots$ and compute recursively the corrections $u_1(t), u_2(t), \ldots$ hoping that a few terms will be sufficient to find an accurate representation of the general solution
- This may not be true for all times, i.e. facing series convergence problems
- In addition, any series expansion breaks in the vicinity of a resonance

Perturbation of non-linear oscillator

- Consider a non-linear harmonic oscillator, $\frac{d^2u}{dt^2} + \omega_0^2 u - \frac{1}{6}\epsilon\omega_0^2 u^3 = 0$
- This is just the pendulum expanded to 3rd order in u
- Note that ϵ is a dimensionless measure of smallness, which may represent a scaling factor of u (e.g. $\epsilon = 1$ without loss of generality)
- Expanding $u(t) = u_0(t) + \epsilon u_1(t) + \epsilon^2 u_2(t) + \ldots$ and separating the equations with equal power in ϵ:
  - Order 0: $\frac{d^2u_0}{dt^2} + \omega_0^2 u_0 = 0 \Rightarrow u_0(t) = A\cos(\omega_0 t)$
  - Order 1: $\frac{d^2u_1}{dt^2} + \omega_0^2 u_1 = \frac{\omega_0^2 u_0^3}{6} = \frac{\omega_0^2 A^3}{6}\cos^3(\omega_0 t) = \frac{\omega_0^2 A^3}{24}(\cos(3\omega_0 t) + 3\cos(\omega_0 t))$
- The 2nd equation has a particular solution with two terms. A well behaved one $u_{1a}(t) = -\frac{A^3}{192}\cos(3\omega t)$ and $u_{1b}(t) = \frac{A^3}{64}(\omega_0 t\cos(\omega t) + 2\cos(\omega_0 t))$ the first part of which grows linearly with time (**secular** term)
- But this cannot be true, the pendulum does not present such behavior. **What did it go wrong?**

Perturbation of non-linear oscillator

- It was already shown that the pendulum has an amplitude dependent frequency, so the frequency has to be developed as well (**Poincaré-Linstead** method):

$$\omega = \omega_0 + \epsilon\omega_1 + \epsilon^2\omega_2 + \dots$$

- Assume that the solution is a periodic function of $\tau = \omega t$ which becomes the new independent variable. The equation at zero order gives the solution $u_0(\tau) = A\cos(\tau)$ and at leading perturbation order becomes

$$\omega_0 \frac{d^2 u_1}{d\tau^2} + \omega_0 u_1 = -2\omega_1 \frac{d^2 u_0}{d\tau^2} + \frac{\omega_0}{6} u_0^3 = \frac{\omega_0 A^3}{24}\cos(3\tau) + \left(\frac{\omega_0 A^3}{8} + 2A\omega_1\right)\cos(\tau)$$

- The last term has to be zero, if not it gives secular terms, thus $\omega_1 = -\frac{A^2\omega_0}{16}$ which reveals the decrease of the frequency with the oscillation amplitude
- Finally, the solution $u_1(t) = \frac{A^3}{192}\left(\cos(\omega_0 t) - \cos(3\omega_0 t)\right)$ is the leading order correction due to the non-linear term

Perturbation by periodic function

- In beam dynamics, perturbing fields are periodic functions
- The problem to solve is a generalization of the driven harmonic oscillator, $\frac{d^2u}{dt^2} + \omega_0^2 u(t) = g(t)$ with a general periodic function $g(t)$, with frequency ω
- The right side can be Fourier analyzed: $g(t) = \sum_{m=-\infty}^{m=+\infty} a_m e^{im\omega t}$
- The homogeneous solution is $u_h(t) = u_0(t)\sin(\omega_0 t + \phi_0)$
- The particular solution can be found by considering that $u(t)$ has the same form as $g(t)$: $u_p(t) = \sum_{m=-\infty}^{m=+\infty} u_{pm} e^{im\omega t}$
- By substituting we find the following relation for the Fourier coefficients of the particular solution $u_{pm} = \frac{a_m}{\omega_0^2 - m^2\omega^2}$
- There is a **resonance condition** for infinite number of frequencies satisfying $\omega_0^2 = m^2\omega^2$

Perturbation by single dipole

- Hill's equations in normalized coordinates with single dipole perturbation:

$$\frac{d^2\mathcal{U}}{d\phi^2} + \nu_0^2\mathcal{U} = \nu_0^2\beta^{3/2}b_1(\phi) = \overline{b_1}(\phi)$$

- The dipole perturbation is periodic, so it can be expanded in a Fourier series

$$\overline{b_1}(\phi) = \sum_{m=-\infty}^{\infty} \overline{b_{1m}}e^{im\phi}$$

- Note, as before that a periodic kick introduces infinite number of integer driving frequencies
- The resonance condition occurs when $\nu_0 = m$ i.e. **integer tunes** should be avoided (remember orbit distortion due to single dipole kick)

- Consider single quadrupole kick in one normalized plane:

$$\frac{d^2\mathcal{U}}{d\phi^2} + \nu_0^2\mathcal{U} = \nu_0^2\beta^2 b_2(\phi)\mathcal{U} = \overline{b_2}(\phi)\mathcal{U}$$

- The quadrupole perturbation can be expanded in a Fourier series

$$\overline{b_2}(\phi) = \sum_{m=-\infty}^{\infty} \overline{b_{2m}} e^{im\phi}$$

- Following the perturbation approach, the 1st order equation becomes $\frac{d^2\mathcal{U}_1}{d\phi^2} + \nu_0^2\mathcal{U}_1 = \sum_{q=-1}^{1}\sum_{m=-\infty}^{\infty} \overline{W}_q\overline{b_{2m}} e^{i(m+q\nu_0)\phi}$ with $\overline{W}_0 = 0$

- For $q = -1$, the resonance conditions are $m - \nu_0 = \nu_0 \to \nu_0 = \frac{m}{2}$
i.e. **integer** and **half-integer tunes** should be avoided

- For $q = 1$, the condition $m + \nu_0 = \nu_0 \to m = 0$ corresponds to a non-vanishing average value $\overline{b_{20}}$, which can be absorbed in the left-hand side providing a **tune-shift**:

$$\nu^2 = \nu_0^2 - \bar{b}_{20} \text{ or } \delta\nu \approx -\frac{\bar{b}_{20}}{2\nu_0} = -\frac{\nu_0\beta^2 b_{20}}{2}$$

Perturbation by single multi-pole

- For a generalized multi-pole perturbation, Hill's equation is:

$$\frac{d^2\mathcal{U}}{d\phi^2} + \nu_0^2\mathcal{U} = \nu_0^2\beta^{\frac{n}{2}+1}b_n(\phi)\mathcal{U}^{n-1} = \overline{b_n}(\phi)\mathcal{U}^{n-1}$$

- As before, the multipole coefficient can be expanded in Fourier series $\overline{b_n}(\phi) = \sum_{m=-\infty}^{\infty} \overline{b_{nm}} e^{im\phi}$
- Following the perturbation steps, the zero-order solution is given by the homogeneous equation $\mathcal{U}_0 = W_1 e^{i\nu_0\phi} + W_{-1}e^{-i\nu_0\phi}$
- Then the position can be expressed as

$$\mathcal{U}_0^{n-1} = \sum_{k=0}^{n-1} \overbrace{\binom{n-1}{k} W_1^{n-1-k} W_{-1}^{k}}^{\overline{W}_q} e^{i\overbrace{(n-1-2k)}^{q = -n+1,\, -n+3,\, \ldots,\, n-1}\nu_0\phi} = \sum_{q=-n+1}^{n-1} \overline{W}_q e^{iq\nu_0\phi}$$

with $\overline{W}_{n-2} = \overline{W}_{n-4} = \overline{W}_{n-6} = \cdots = \overline{W}_{-n+2} = 0$

- The first order solution is written as

$$\frac{d^2\mathcal{U}_1}{d\phi^2} + \nu_0^2\mathcal{U}_1 = \overline{b_n}(\phi)\mathcal{U}_0^{n-1} = \sum_{q=-n+1}^{n-1} \sum_{m=-\infty}^{m=\infty} \bar{b}_{nm}\overline{W}_q e^{i(m+q\nu_0)\phi}$$

Resonances for single multi-pole

- Following the discussion on the periodic perturbation, the solution can be found by setting the leading order solution to be periodic with the same frequency as the right hand side

$$\mathcal{U}_1 = \sum_{q=-n+1}^{n-1} \sum_{m=-\infty}^{m=\infty} \mathcal{U}_{1mq} e^{i(m+q\nu_0)\phi}$$

- Equating terms of equal exponential powers, the Fourier amplitudes are found to satisfy the relationship

$$\mathcal{U}_{1mq} = \frac{\bar{b}_{nm}\overline{W}_q}{\nu_0^2 - (m+q\nu_0)^2}$$

- This provides the **resonance condition** $m \pm |q|\nu_0 = \nu_0$ or $\nu_0 = \dfrac{m}{1 \pm |q|}$ which means that there are resonant frequencies for an "infinite" number of rationals

Tune-shift for single multi-pole

- Note that for even multi-poles and $q = 1$ or $m = 0$, there is a Fourier coefficient $\bar{b}_{n0}$, which is independent of ϕ and represents the average value of the periodic perturbation
- The perturbing term in the r.h.s. is

$$\bar{b}_{n0}\overline{W}_1 e^{i\nu_0\phi} = \nu_0^2 \beta^{\frac{n}{2}+1} b_{n0} \binom{n-1}{\frac{n}{2}-1} W_1^{n-1} W_{-1}^{\frac{n}{2}-1} e^{i\nu_0\phi}$$

which can be obtained for $k = \frac{n}{2} - 1$ (it is indeed an integer only for even multi-poles)

- Following the approach of the perturbed non-linear harmonic oscillator, this term will be secular unless a perturbation in the frequency is considered, thereby resulting to a tune-shift equal to

$$\delta\nu = -\frac{\nu_0 \beta^{\frac{n}{2}+1} b_{n0}}{2} \binom{n-1}{\frac{n}{2}-1} \widetilde{W}^{n-2} \quad \text{with} \quad \widetilde{W}^2 = W_1 W_{-1}$$

- This tune-shift is amplitude dependent for $n > 2$

Single Sextupole Perturbation

- Consider a localized sextupole perturbation in the horizontal plane

$$\frac{d^2\mathcal{U}}{d\phi^2} + \nu_0^2\mathcal{U} = \nu_0^2\beta^{\frac{5}{2}}b_3(\phi)\mathcal{U}^2 = \overline{b_3}(\phi)\mathcal{U}^2$$

- After replacing the perturbation by its Fourier transform and inserting the unperturbed solution to the right hand side

$$\frac{d^2\mathcal{U}_1}{d\phi^2} + \nu_0^2\mathcal{U}_1 = \sum_{q=-2}^{2}\sum_{m=-\infty}^{\infty} \overline{W}_q\overline{b_{3m}}e^{i(m+q\nu_0)\phi} \quad \text{with } \overline{W}_{-1} = \overline{W}_1 = 0$$

- Resonance conditions:

3rd integer → $3\nu_0 = m$ for $q = -2$

integer → $\nu_0 = m$ for $q = 0, 2$

- Note that there is not a tune-spread associated. This is only true for small perturbations (first order perturbation treatment)
- Although perturbation treatment can provide approximations for evolution of motion, there is no exact solution

General multi-pole perturbation

- Equations of motion including any multi-pole error term, in both planes

$$\frac{d^2\mathcal{U}_x}{d\phi_x^2} + \nu_{0x}^2\mathcal{U}_x = \overline{b_{n,r}}(\phi_x)\mathcal{U}_x^{n-1}\mathcal{U}_y^{r-1}$$

- Expanding perturbation coefficient in Fourier series and inserting the solution of the unperturbed system on the rhs gives the following series:

$$\overline{b_{nr}}(\phi_x) = \sum_{m=-\infty}^{\infty} \overline{b_{nrm}} e^{im\phi_x}$$

$$\mathcal{U}_x^{n-1} \approx \mathcal{U}_{0x}^{n-1} = \sum_{q_x=-n+1}^{n-1} \overline{W}_{q_x} e^{iq_x\nu_0\phi_x}$$

$$\mathcal{U}_y^{r-1} \approx \mathcal{U}_{0y}^{r-1} = \sum_{q_y=-r+1}^{r-1} \overline{W}_{q_y} e^{iq_y\nu_{0y}\phi_x}$$

- The equation of motion becomes

$$\frac{d^2\mathcal{U}_x}{d\phi_x^2} + \nu_{0x}^2\mathcal{U}_x = \sum_{m,q_x,q_y} \overline{b_{nrm}} W_{q_x}^x W_{q_y}^y e^{i(m+q_x\nu_{0x}+q_y\nu_{0y})\phi_x}$$

- In principle, same perturbation steps can be followed for getting an approximate solution in both planes

Example: Linear Coupling

- For a localized skew quadrupole we have

$$\frac{d^2\mathcal{U}_x}{d\phi_x^2} + \nu_{0x}^2\mathcal{U}_x = \overline{b_{1,2}}(\phi_x)\mathcal{U}_y$$

- Expanding perturbation coefficient in Fourier series and inserting the solution of the unperturbed system gives the following equation:

$$\frac{d^2\mathcal{U}_x}{d\phi_x^2} + \nu_{0x}^2\mathcal{U}_x = \sum_{m=-\infty}^{\infty}\sum_{q_y=-1}^{q_y=1}\overline{b_{12m}}W_{q_y}^y e^{i(m+q_y\nu_{0y})\phi_x}$$ with $W_0^y = 0$

- The coupling resonance are found for $q_y = \pm 1$

Linear sum resonance

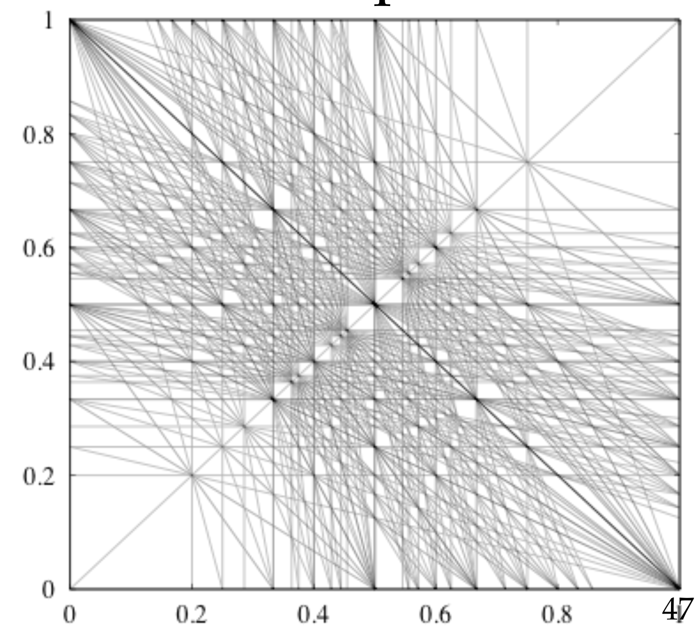
$$m = \nu_{0x} + \nu_{0y}$$

Linear difference resonance

$$m = \nu_{0x} - \nu_{0y}$$

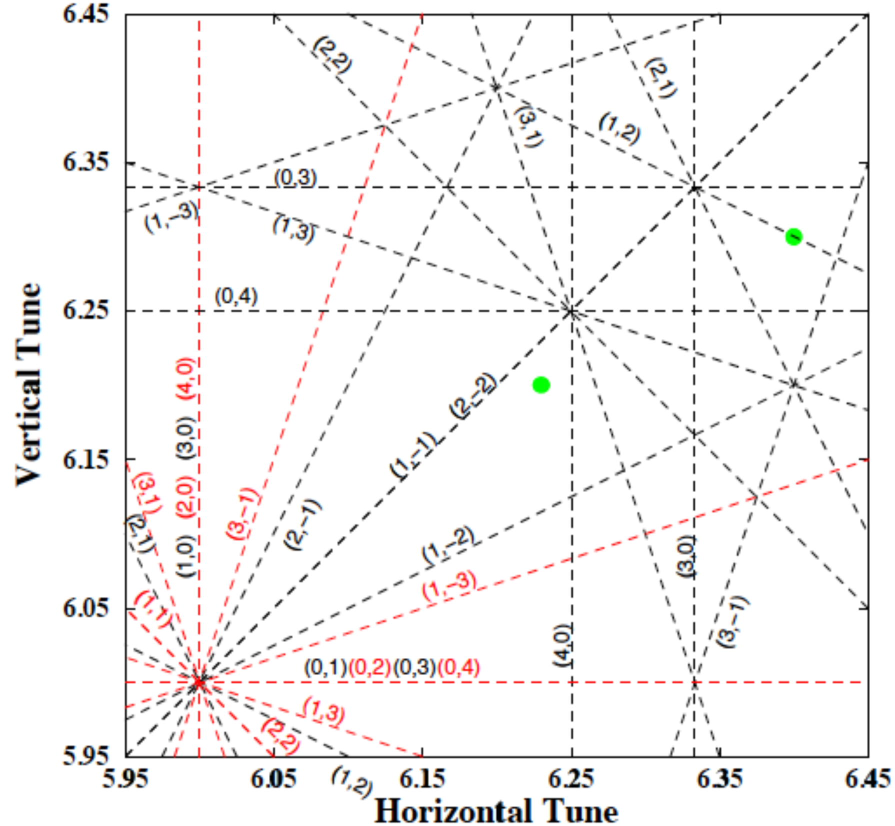
General resonance conditions

- The general resonance conditions is $m + q_x \nu_{0x} + q_y \nu_{0y} = \nu_{0x}$ or $m + q'_x \nu_{0x} + q_y \nu_{0y} = 0$, with order $|q_x| + |q_y| + 1$
- The same condition can be obtained in the vertical plane
- For all the polynomial field terms of a $2n$-pole, the **main** excited resonances satisfy the condition $q'_x + q_y = n$ but there are also **sub-resonances** for which $q'_x + q_y < n$
- For **normal** (erect) multi-poles, the main resonances are $(q'_x, q_y) = (n, 0), \ (n-2, \pm 2), \ldots$ whereas for **skew** multi-poles $(q'_x, q_y) = (n-1, \pm 1), \ (n-3, \pm 3), \ldots$
- If perturbation is large, **all** resonances can be potentially excited
- The resonance conditions form lines in the frequency space and fill it up as the order grows (the rational numbers form a dense set inside the real numbers)



Systematic and random resonances

- If lattice is made out of N identical cells, and the perturbation follows the same periodicity, resulting in a reduction of the resonance conditions to the ones satisfying $q_x \nu_{0x} + q_y \nu_{0y} = jN$
- These are called **systematic** resonances
- Practically, any (linear) lattice perturbation breaks super-periodicity and any **random** resonance can be excited
- Careful choice of the working point is necessary



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Summary

- Accelerator performance depends heavily on the understanding and control of non-linear effects
- The ability to integrate differential equations has a deep impact to the dynamics of the system
- Phase space is the natural space to study this dynamics
- Perturbation theory helps integrate iteratively differential equations and reveals appearance of resonances
- Periodic perturbations drive infinite number of resonances
- There is an amplitude dependent tune-shift at 1st order for even multi-poles
- Periodicity of the lattice very important for reducing number of lines excited at first order

Magnetic multipole expansion

- From Gauss law of magnetostatics, a vector potential exist

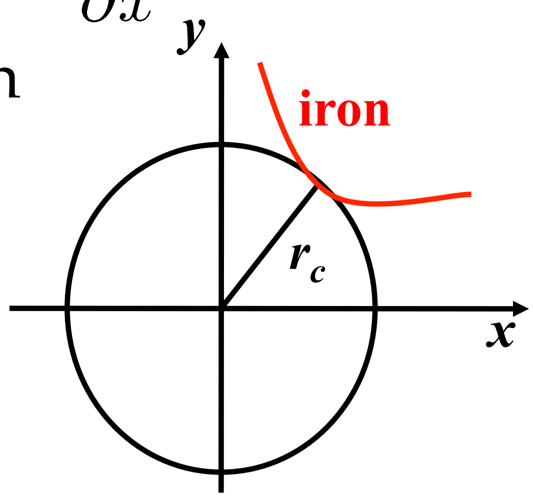
$$\nabla \cdot \mathbf{B} = 0 \quad \rightarrow \quad \exists \mathbf{A} : \quad \mathbf{B} = \nabla \times \mathbf{A}$$

- Assuming transverse 2D field, vector potential has only one component A_s. The Ampere's law in vacuum (inside the beam pipe) $\nabla \times \mathbf{B} = 0 \quad \rightarrow \quad \exists V : \quad \mathbf{B} = -\nabla V$
- Using the previous equations, the relations between field components and potentials are

$$B_x = -\frac{\partial V}{\partial x} = \frac{\partial A_s}{\partial y} \, , \quad B_y = -\frac{\partial V}{\partial y} = -\frac{\partial A_s}{\partial x}$$

i.e. Riemann conditions of an analytic function

Exists complex potential of $z = x + iy$ with power series expansion convergent in a circle with radius $|z| = r_c$ (distance from iron yoke)



$$\mathcal{A}(x + iy) = A_s(x, y) + iV(x, y) = \sum_{n=1}^{\infty} \kappa_n z^n = \sum_{n=1}^{\infty} (\lambda_n + i\mu_n)(x + iy)^n$$

Multipole expansion II

- From the complex potential we can derive the fields

$$B_y + iB_x = -\frac{\partial}{\partial x}(A_s(x,y) + iV(x,y)) = -\sum_{n=1}^{\infty} n(\lambda_n + i\mu_n)(x+iy)^{n-1}$$

- Setting $b_n = -n\lambda_n \ , \quad a_n = n\mu_n$

$$B_y + iB_x = \sum_{n=1}^{\infty}(b_n - ia_n)(x+iy)^{n-1}$$

- Define normalized coefficients

$$b'_n = \frac{b_n}{10^{-4}B_0} r_0^{n-1} \ , \ a'_n = \frac{a_n}{10^{-4}B_0} r_0^{n-1}$$

on a reference radius r_0, 10^{-4} of the main field to get

$$B_y + iB_x = 10^{-4}B_0 \sum_{n=1}^{\infty}(b'_n - ia'_n)(\frac{x+iy}{r_0})^{n-1}$$

- **Note:** $n' = n - 1$ is the US convention