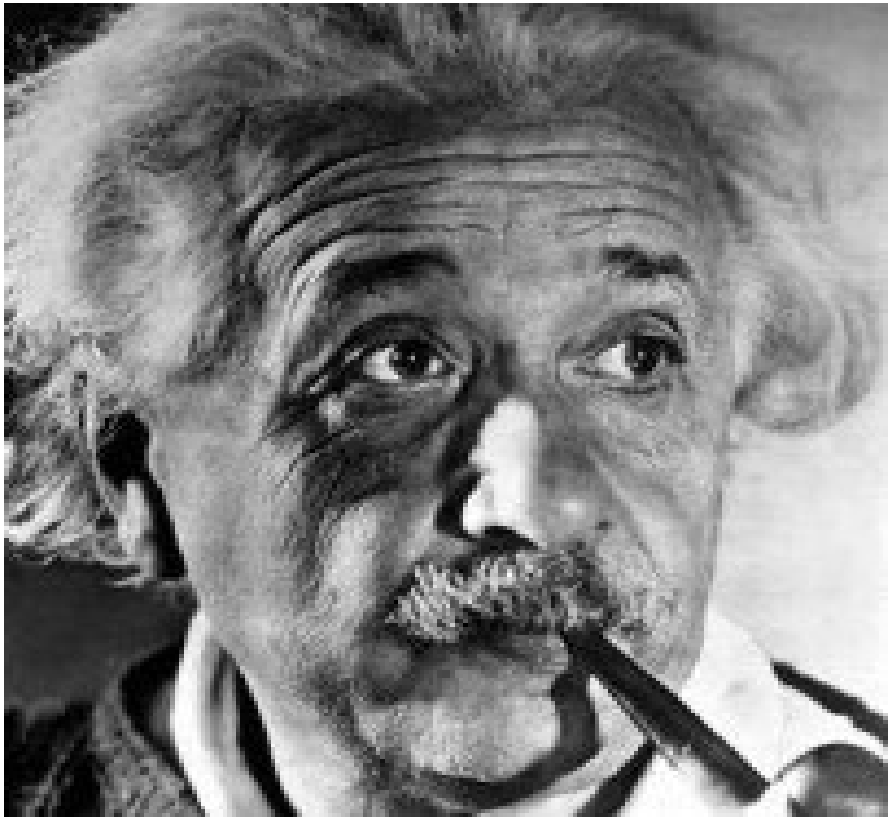
Review of Special Relativity



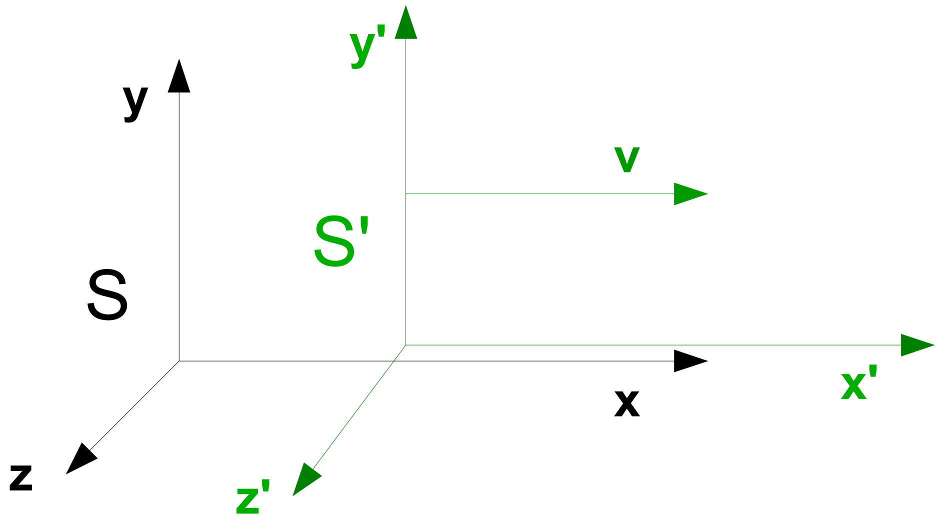
This review is not meant to teach the subject, but to repeat and to refresh, at least partially, what you have learnt at university.

Why was „Special Relativity“ needed?

Mechanical laws (Newton's laws) are the same for all inertial systems.

They are invariant under a Galilean transformation (G-T):

$$x' = x - vt, \quad y' = y, \quad z' = z, \quad t' = t$$



Example: Man walking in train (S'), observer at rest (S).

El.-mag. laws are not invariant under a G-T,

take e.g. the wave equation $$[\frac{\partial^2}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}]\Phi = 0$$

it transforms to

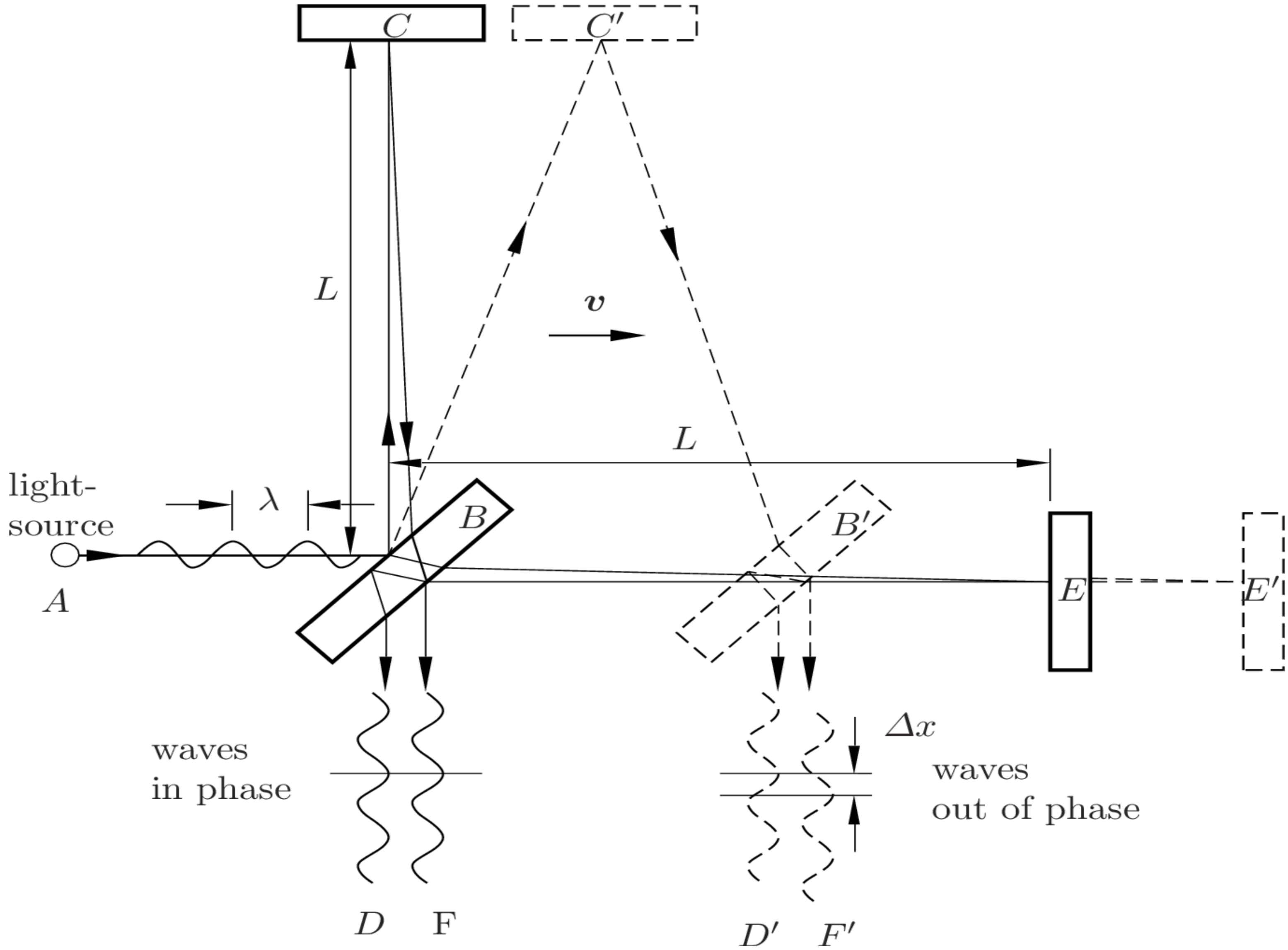
$$[\frac{\partial^2}{\partial x'^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t'^2} - \frac{v^2}{c^2}\frac{\partial^2}{\partial x'^2} - 2\frac{v}{c^2}\frac{\partial^2}{\partial x'\partial t'}]\Phi = 0$$

Moreover, el.-mag. laws predict the speed of light as equal in all reference systems.
This contradicted the deep belief in a supporting media (ether) for the waves. If there were an ether, c would be different in different reference systems.

Many experiments tried to prove el.-mag. theory wrong.
They all failed!

Michelson-Morley interferometer experiment (1887)

showed that c is a constant and that there exists no „ether".



travel time from B to E' $\quad c\,t_1 = L + v t_1$

travel time from E' to B' $\quad c\,t_2 = L - v t_2$

round trip travel time $\quad t_\parallel = t_1 + t_2 = \gamma^2\, 2\frac{L}{c}, \quad \gamma = \frac{1}{\sqrt{1-(v/c)^2}}$

travel time B to C' $\quad c\,t_3 = \sqrt{L^2 + (v t_3)^2}$

round trip travel time $\quad t_\perp = 2 t_3 = \gamma\, 2\frac{L}{c} = t_\parallel / \gamma$

Since $t_\perp = t_\parallel / \gamma$ there should be interference between D' and F'. But no interference was observed.

Conclusion: $L_\parallel$ appears shorter than $L_\perp$ by $1/\gamma$

→ Newton-Galileo concept of space and time had to be modified

Relativistic Kinematics

Einstein based his theory on two postulates:

1. All inertial frames are equivalent w.r.t. all laws of physics.
2. The speed of light is equal in all reference frames.

Consequence of 1st postulate:

Space is isotropic (all directions are equivalent)
Space is homogeneous (all points are equivalent)

Lorentz Transformation

Homogeneity of space and form-invariance of laws under transformation require a linear transformation.

$$ct' = a_{00}\,ct + a_{01}\,x + a_{02}\,y + a_{03}\,z$$
$$x' = a_{10}\,ct + a_{11}\,x + a_{12}\,y + a_{13}\,z$$
$$y' = a_{20}\,ct + a_{21}\,x + a_{22}\,y + a_{23}\,z$$
$$z' = a_{30}\,ct + a_{31}\,x + a_{32}\,y + a_{33}\,z$$

Successive use of homogeneity, isotropy and the speed of light determines the constants. E.g.:

$a_{02} = a_{03} = 0$ events at $y = \pm y_0$ or $z = \pm z_0$ have to take place at equal times in S'

$a_{20} = a_{30} = 0$ origin, $x = y = z = 0$, has to stay on x'-axis

$a_{12} = a_{13} = a_{21} = a_{23} = a_{31} = a_{32} = 0$

x-,x'- and y-,y'- and z-, z'-axis have should stay parallel

etc.

The final result is the Lorentz-Transformation (L-T) :

$$ct' = \gamma(ct - \beta x) \qquad y' = y$$
$$x' = \gamma(x - \beta ct) \qquad z' = z$$

$$\beta = \frac{v}{c}, \qquad \gamma = \frac{1}{\sqrt{1-\beta^2}}$$

Inverse transformation:

replace primed variables by unprimed,
unprimed by primed
β by $-\beta$

$$ct = \gamma(ct' + \beta x') \qquad y = y'$$
$$x = \gamma(x' + \beta ct') \qquad z = z'$$

We write the L-T as

$$\begin{bmatrix} ct' \\ x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} ct \\ x \\ y \\ z \end{bmatrix}$$

$$\underline{L} = \begin{bmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \underline{L}^{-1} = \begin{bmatrix} \gamma & \beta\gamma & 0 & 0 \\ \beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \underline{L}\,\underline{L}^{-1} = \underline{1}$$

L-T is an affine transformation. It preserves the rectilinearity and parallelisme of straight lines.

Consequences following from L-T

Time dilation:

Two events in S' *at* t_1', t_2' and *at location* $x_1' = x_2'$
appear at times t_1, t_2 in S

$$c(t_2 - t_1) = \gamma c(t_2' - t_1') + \gamma\beta(x_2' - x_1')$$

$$\rightarrow \quad \Delta t = \gamma \Delta t'$$

Two events in S *at* $t_{1,} t_2$ and *at location* $x_1 = x_2$
appear at times $t'_{1,} t'_2$ in S'

$$c(t_2' - t_1') = \gamma c(t_2 - t_1) - \gamma\beta(x_2 - x_1)$$

$$\rightarrow \quad \Delta t' = \gamma \Delta t$$

Length contraction:

$$A\,distance\; x_2' - x_1' \text{ in } S'\, measured \text{ in } S\, at\; t_1 = t_2$$

$$x_2 - x_1 = \gamma(x_2' - x_1') + \gamma\beta c(t_2' - t_1') =$$

$$= \gamma(x_2' - x_1') + \gamma^2\beta c^2(t_2 - t_1) - \gamma^2\beta^2(x_2 - x_1) =$$

$$= \gamma(x_2' - x_1') - \gamma^2\beta^2(x_2 - x_1)$$

$$\rightarrow \quad \Delta x = \frac{1}{\gamma}\Delta x'$$

$$A\,distance\; x_2 - x_1 \text{ in } S\, measured \text{ in } S'\, at\; t_1' = t_2'$$

$$x_2' - x_1' = \gamma(x_2 - x_1) - \gamma\beta c(t_2 - t_1) =$$

$$= \gamma(x_2 - x_1) - \gamma^2\beta c^2(t_2' - t_1') - \gamma^2\beta^2(x_2' - x_1') =$$

$$= \gamma(x_2 - x_1) - \gamma^2\beta^2(x_2' - x_1')$$

$$\rightarrow \quad \Delta x' = \frac{1}{\gamma}\Delta x$$

Time intervals and distances depend on the motion of the observer.

$$\Delta t \Leftarrow \gamma \Delta t' \quad and \quad \Delta x \Leftarrow \frac{1}{\gamma} \Delta x'$$

$$(not\ standard\ equations!!)$$

Perpendicular dimensions remain: $\Delta y = \Delta y', \ \Delta z = \Delta z'$

Example length contraction: Michelson-Morley

Example: Muons created in upper atmosphere, v=0.994c

Lifetime in restframe of muons

$$T'_{1/2} = 1.5 \mu s \quad \rightarrow \quad l' = 450\text{m}$$

Lifetime on earth (with time dilation)

$$\gamma = 9, \quad T_{1/2} = 13.5 \mu s \quad \rightarrow \quad l = 4\text{km}$$

Transformation of velocity

A particle moving with velocity u' in S' has velocity u in S

$$u_x = \frac{dx}{dt} = \frac{dx}{dt'}\frac{dt'}{dt} = \gamma\left(\frac{dx'}{dt'} + \beta c\right)\frac{dt'}{dt} = \gamma\left(u'_x + v\right)\frac{dt'}{dt}$$

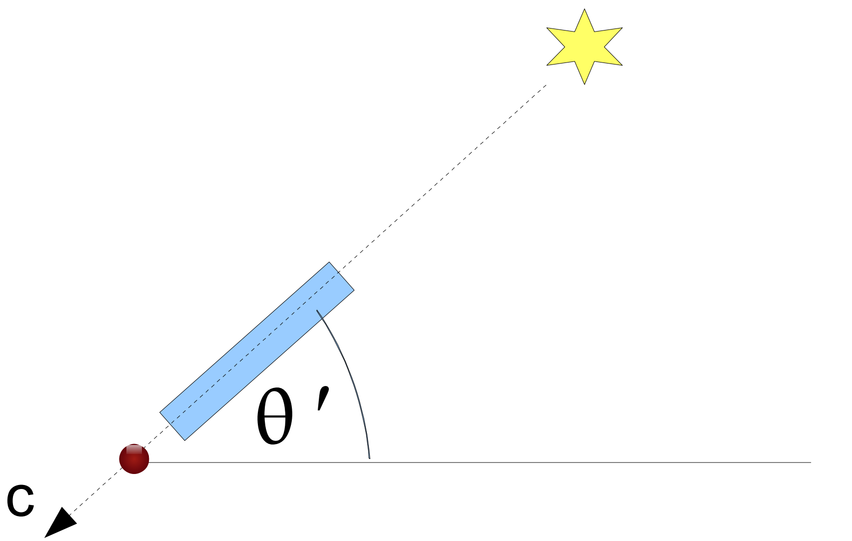
$$\frac{dt}{dt'} = \gamma\left(1 + \frac{\beta}{c}\frac{dx'}{dt'}\right) = \gamma\left(1 + \frac{v}{c^2}u'_x\right)$$

$$u_x = \frac{u'_x + v}{1 + vu'_x/c^2}, \quad u_y = \frac{u'_y}{\gamma\left(1 + vu'_x/c^2\right)}, \quad u_z = \frac{u'_z}{\gamma\left(1 + vu'_x/c^2\right)}$$

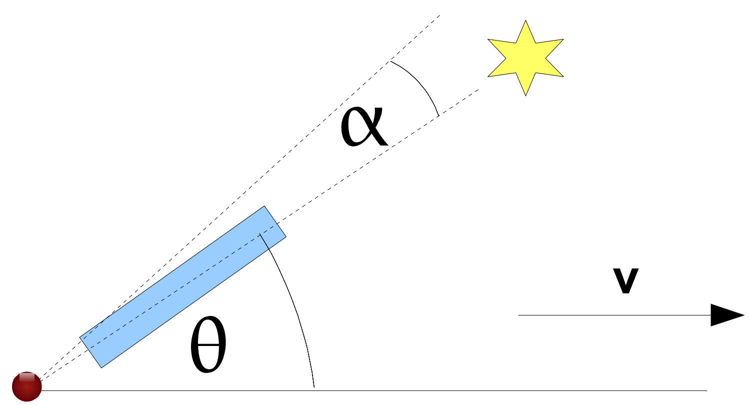
Example: Light aberration

A star appears on earth under an angle different than its real position.

Earth at rest:



Earth moving with v:



In rest system of earth the star moves with -v and u' is c, then

$$u'_x = -c\cos(\vartheta'), \qquad u'_y = -c\sin(\vartheta')$$

velocity transformation gives

$$u_x = -c\cos\vartheta = \frac{-c\cos(\vartheta') - v}{1 + \beta\cos(\vartheta')}$$

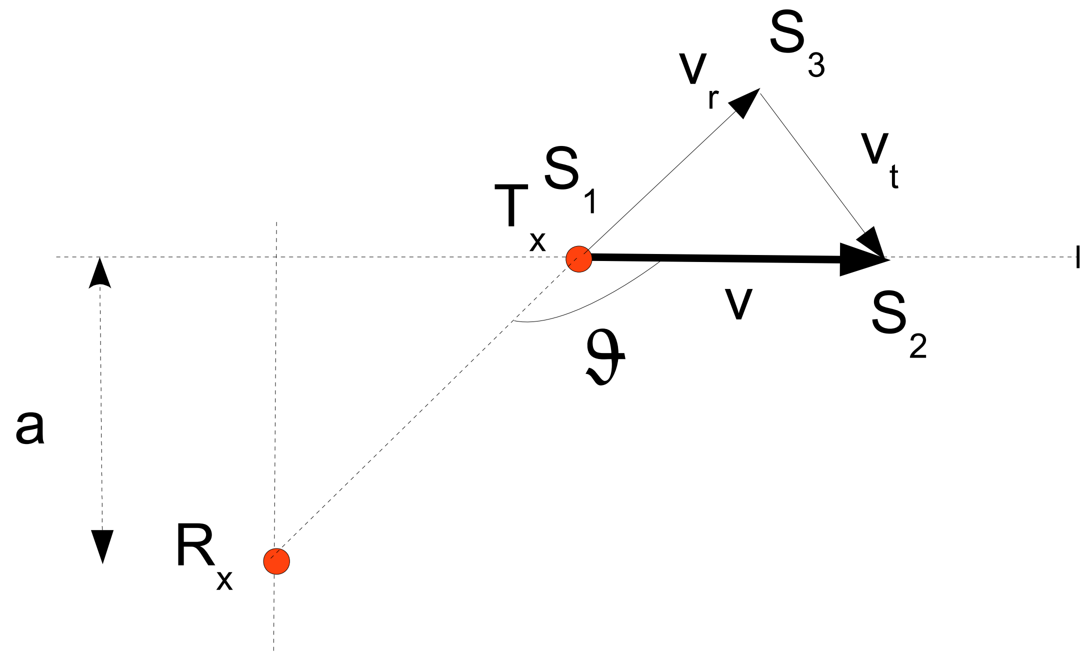
$$u_y = -c\sin\vartheta = \frac{-c\sin(\vartheta')}{\gamma(1 + \beta\cos(\vartheta'))}$$

and using $\tan\left(\frac{\vartheta}{2}\right) = \frac{\sin(\vartheta)}{1 + \cos(\vartheta)}$ finally

$$\tan\left(\frac{\vartheta}{2}\right) = \frac{\sin(\vartheta')}{\gamma(1+\beta)(1 + \cos(\vartheta'))} = \sqrt{\frac{1-\beta}{1+\beta}}\tan\left(\frac{\vartheta'}{2}\right)$$

Example: Doppler effect

Emitter T_x is moving with v, receiver R_x at rest.



$$v_r = v\cos(\pi - \vartheta)$$
$$= -v\cos(\vartheta)$$

A signal emitted at S_1 reaches R_x at time

$$t_1 = \frac{1}{c}\overline{R_x S_1} = \frac{a}{c\sin(\vartheta)}$$

T_x moves from S_1 to S_2 in an RF period T_0.

At S_2 it emitts a signal which reaches R_x at the time

$$t_2 = \gamma T_0 + \frac{1}{c}\overline{R_x S_3} = \gamma T_0 + \frac{1}{c}\left[\frac{a}{\sin(\vartheta)} - v\gamma T_0 \cos(\vartheta)\right]$$

where T_0 has been dilated by γ and $\overline{R_x S_2} = \overline{R_x S_3}$ for $vT_0 \ll \overline{R_x S_1}$ has been used.

The RF period T experienced by R_x is

$$T = t_2 - t_1 = \gamma(1 - \beta\cos(\vartheta))T_0$$

therefore

$$\frac{f}{f_0} = \frac{T_0}{T} = \frac{1}{\gamma(1-\beta\cos(\vartheta))} = \frac{\sqrt{1-\beta^2}}{1-\beta\cos(\vartheta)}$$

$$\vartheta \to \pi : \quad \frac{f}{f_0} \to \sqrt{\frac{1-\beta}{1+\beta}}$$ longitudinal Doppler effect

$$\vartheta = \frac{\pi}{2} : \quad \frac{f}{f_0} = \sqrt{1-\beta^2}$$ transverse Doppler effect

Example: Astronomy

Transverse Doppler « longitudinal Doppler ($v \approx v_r$)

$$\frac{f}{f_0} = \frac{\lambda_0}{\lambda} = \sqrt{\frac{1-\beta}{1+\beta}} \quad \to \quad \beta = \frac{1-(\lambda_0/\lambda)^2}{1+(\lambda_0/\lambda)^2}$$

$$\lambda > \lambda_0 \quad \to \quad \textit{redshift}$$

Transformation of acceleration

A particle moving with u' in S' and experiencing an acceleration a' has an acceleration a in S

$$a_x = \frac{du_x}{dt} = \frac{du_x}{dt'}\frac{dt'}{dt} = \frac{d}{dt'}\frac{u'_x + v}{1 + vu'_x/c^2}\frac{dt'}{dt} =$$

$$= \frac{a'_x}{\gamma^3 (1 + vu'_x/c^2)^3}$$

$$a_y = \frac{a'_y}{\gamma^2 (1 + vu'_x/c^2)^2} - \frac{(vu'_y/c^2)\, a'_x}{\gamma^2 (1 + vu'_x/c^2)^3}$$

$$a_z = \frac{a'_z}{\gamma^2 (1 + vu'_x/c^2)^2} - \frac{(vu'_z/c^2)\, a'_x}{\gamma^2 (1 + vu'_x/c^2)^3}$$

Acceleration in an inertial system is possible !!

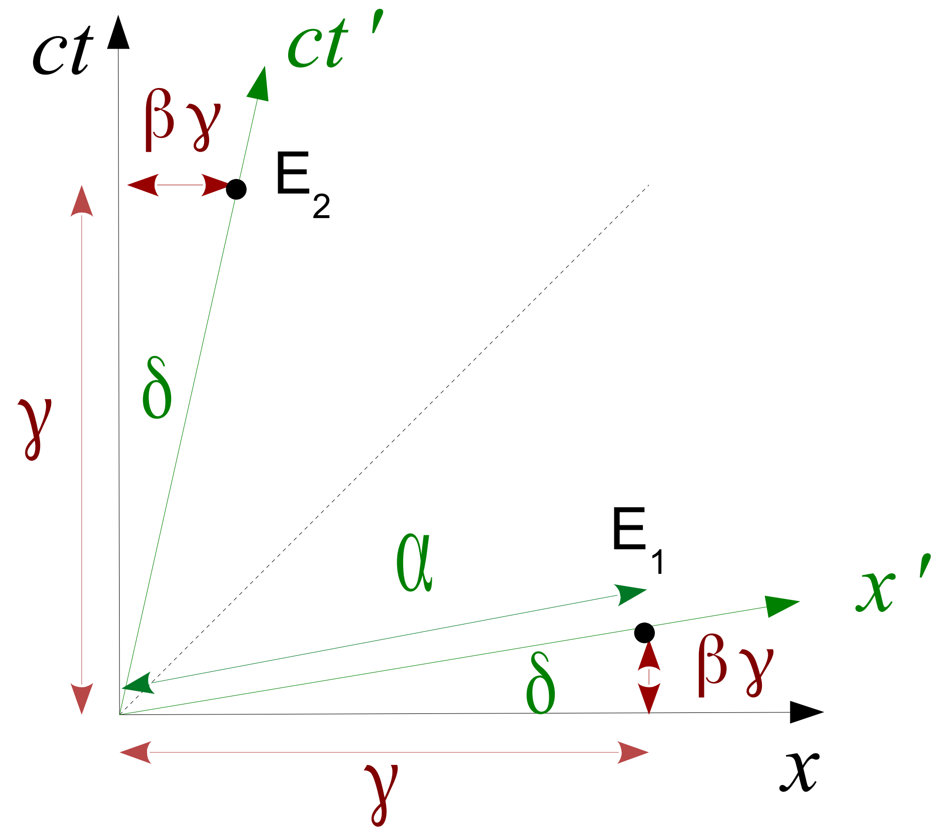
Minkowski diagram

Shows the coordinates in S' for an event E in S and vice versa.

$$ct = \gamma(ct' + \beta x')$$
$$x = \gamma(x' + \beta ct')$$

$$E_1(ct'=0, x'=1) \rightarrow ct = \beta\gamma,\ x = \gamma$$
$$E_2(ct'=1, x'=0) \rightarrow ct = \gamma,\ x = \beta\gamma$$

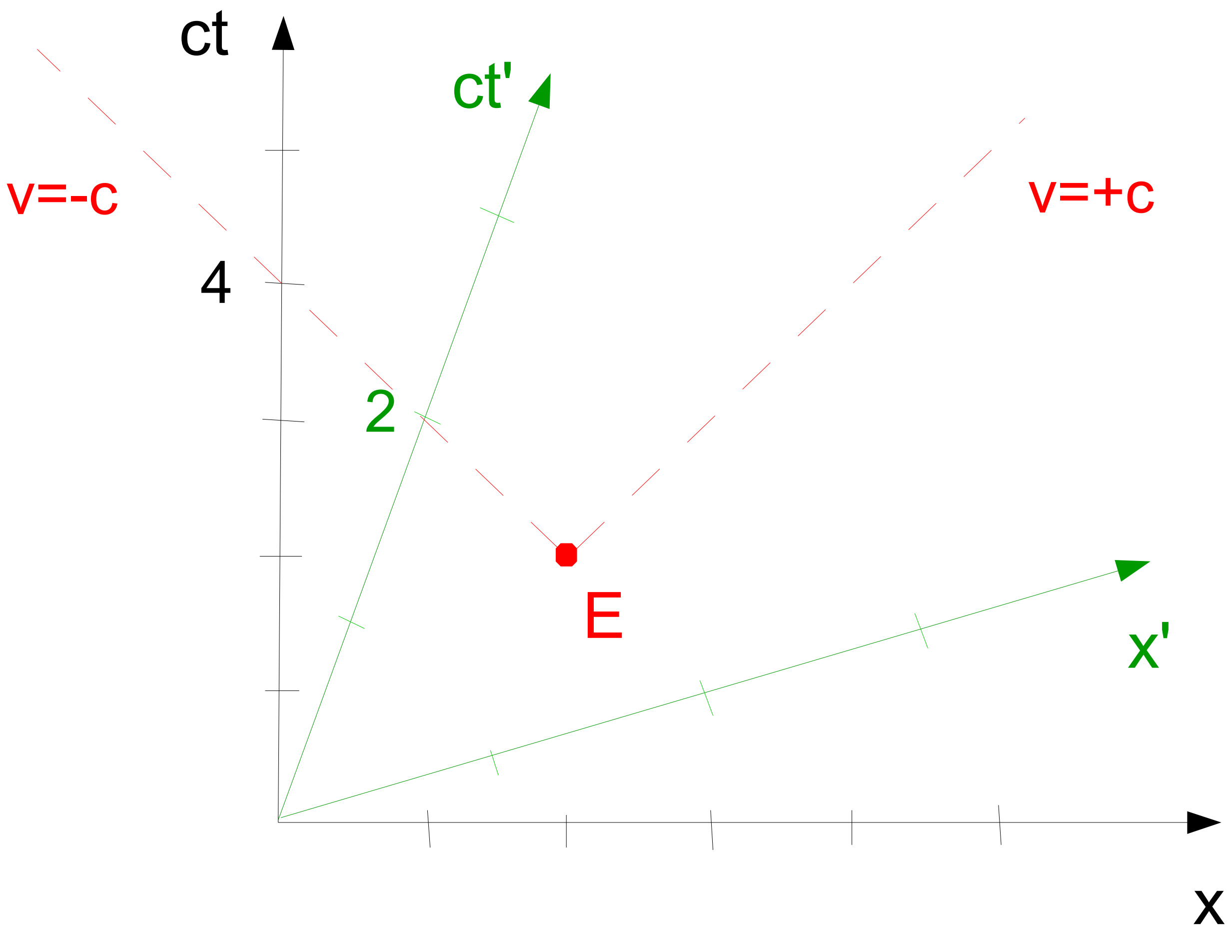
$$\tan(\delta) = \beta$$



Scale in S': $\alpha = \sqrt{\gamma^2 + \beta^2\gamma^2} = \sqrt{\dfrac{1+\beta^2}{1-\beta^2}}$

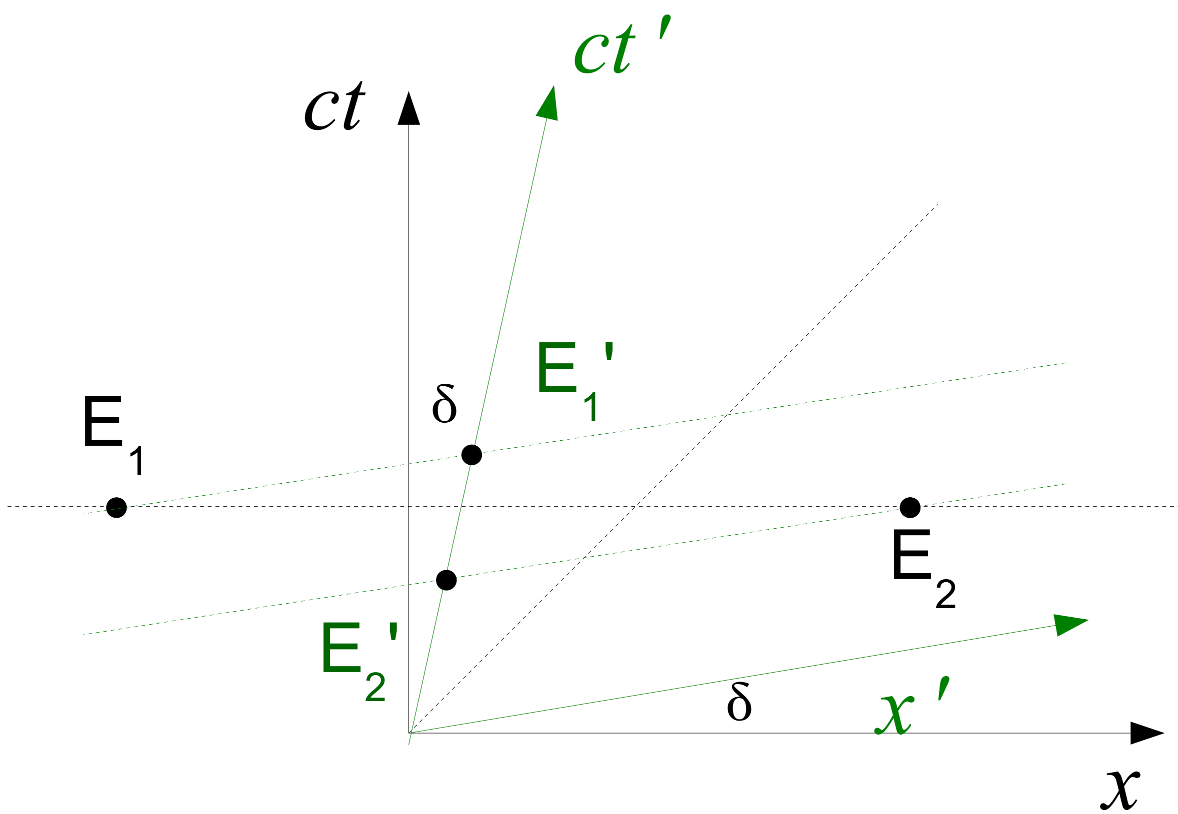
Light pulse

Event E is a spherical light shell expanding in all directions. The light reaches an observer at x=0 at the time ct=4 and an observer at x'=0 at time ct'=2.



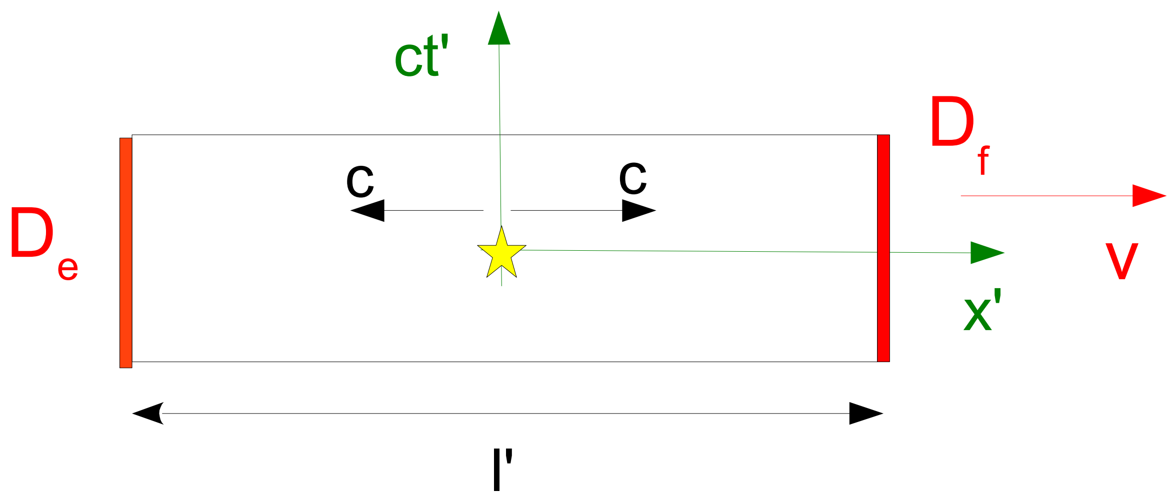
Simultaneity

Events E_1, E_2 which are simultaneous in S are not simultaneous in S'.



Example: Light flash in rocket

Rocket is moving with v (frame S'). Light flash is emitted at the center and reaches the front and end detector at the same time. In S the times are different.

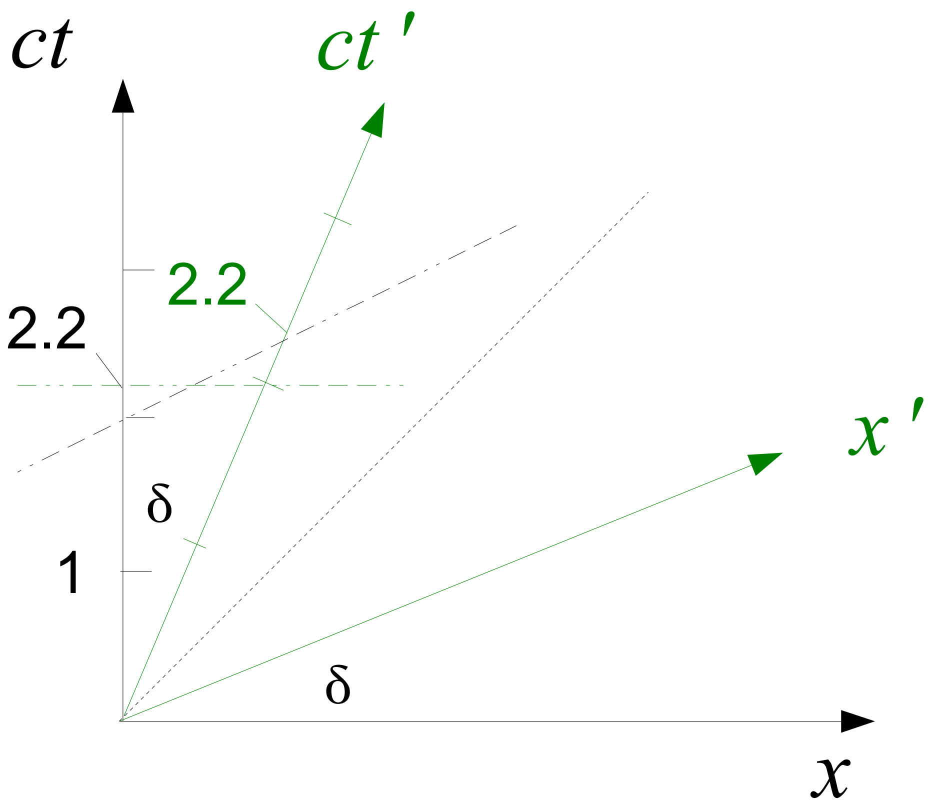


in space craft: $$ct_f' = ct_e' = \frac{l'}{2}$$

on earth: $$ct_f = \frac{l}{2} + vt_f, \quad ct_e = \frac{l}{2} - vt_e, \quad t_f - t_e = \gamma^2 \beta \frac{l}{c} = \gamma \beta \frac{l'}{c}$$

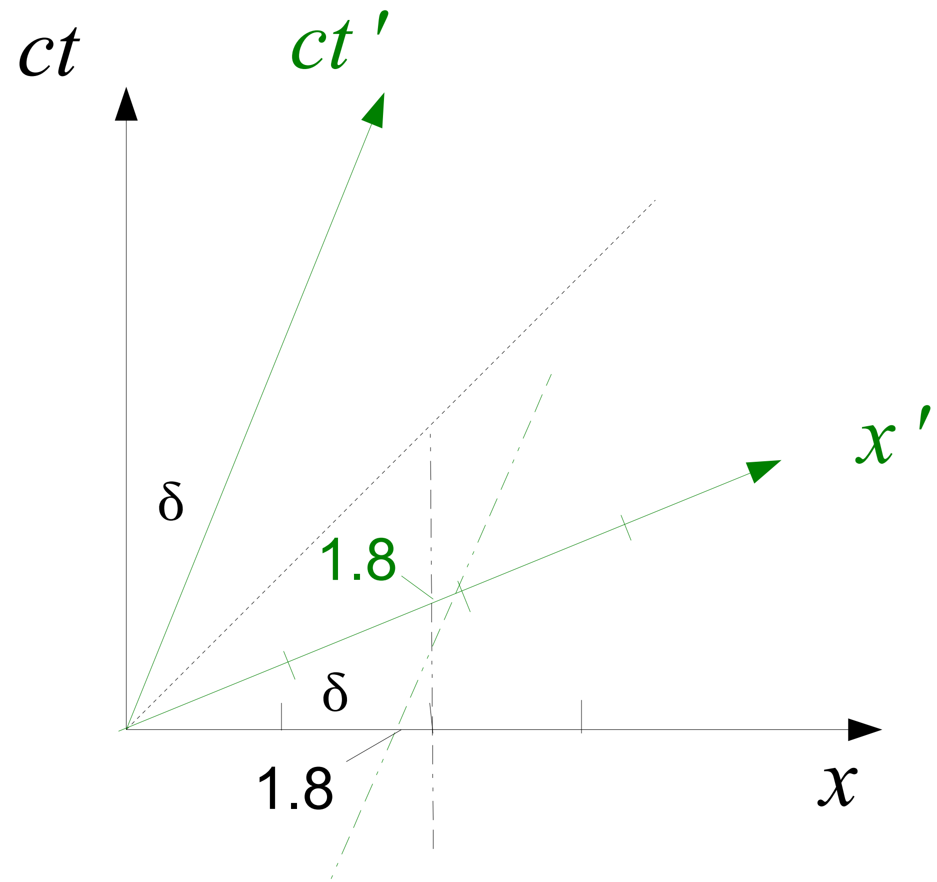
Time dilation:

β=0.42, δ=22.8°, γ=1.1, α=1.2

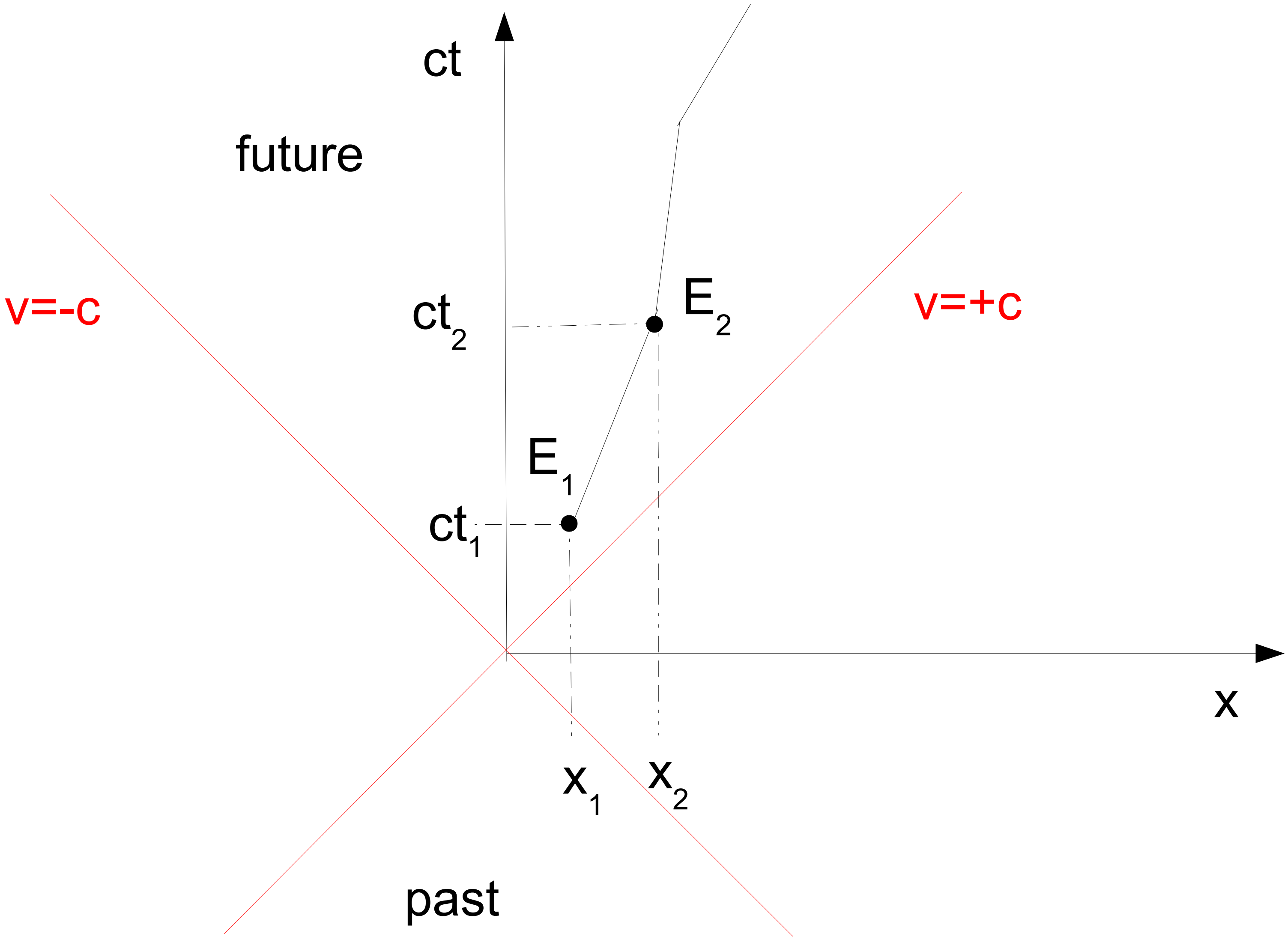


Length contraction:

β=0.42, δ=22.8°, γ=1.1, α=1.2



World-line (path-time diagram)



Relativistic Dynamics

Based on two principles:

1. Conservation of linear momentum   ($p=mu$)

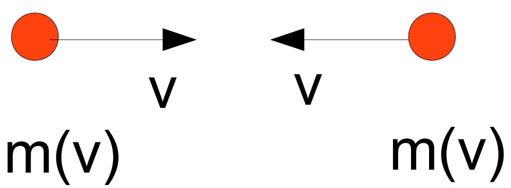
2. Conservation of energy   ($E=mc^2$)

Moving mass

Because of $E=mc^2$ we choose as ansatz $m=m(v)$ and calculate the function $m(v)$ by an experiment.

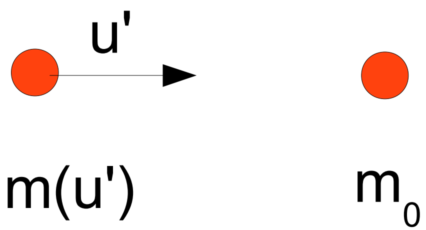
Inelastic collision between 2 identical particles:

In laboratory frame S:



composite particle at rest after collision

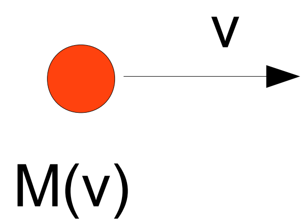
In rest frame S' of right particle:



S moves with v to the right w.r.t. S'.
The left particle has velocity u=v in the moving frame S and u' in the rest frame S'.
Therefore

$$u'=\frac{u+v}{1+vu/c^2} \quad \rightarrow \quad u'=\frac{2v}{1+\beta^2} \qquad (1)$$

After collision composite particle moving with v in S'



Conservation of momentum $\quad m(u')u'=M(v)v \qquad (2)$

Conservation of energy $\quad m(u')c^2+m_0c^2=M(v)c^2 \qquad (3)$

From (1), (2), (3) after eliminating M

$$m(u')=\frac{m_0}{\sqrt{1-(u'/c)^2}}=\gamma_{u'}m_0 \qquad (4)$$

Mass

$$m(u')=\gamma_{u'}m_0$$

From (1), (2), (4)

$$M(v)=\frac{2\mathrm{m}_0}{1-(v/c)^2}=\gamma M_0, \quad M_0=2\gamma m_0$$

Rest mass not conserved: $M_0-2\mathrm{m}_0=2\mathrm{m}_0(\gamma-1)>0$

E_{kin} completely converted into mass:

$$2E_{kin}=2(E-E_0)=2(\gamma m_0 c^2-m_0 c^2)=M_0 c^2-2\mathrm{m}_0 c^2$$

Momentum $$\vec{p}(u)=m(u)\vec{u}=\gamma_u m_0 \vec{u}$$

Force $$\vec{f}=\frac{d\vec{p}}{dt}=m_0\frac{d\gamma_u}{dt}\vec{u}+m_0\gamma_u\frac{d\vec{u}}{dt}$$

with $$\frac{d\gamma_u}{dt}=\frac{d}{dt}\frac{1}{\sqrt{1-\vec{u}\cdot\vec{u}/c^2}}=\frac{\gamma_u^3}{c^2}(\vec{u}\cdot\vec{a})$$ we get

Force $$\vec{f}=\gamma_u^3\frac{m_0}{c^2}(\vec{u}\cdot\vec{a})\vec{u}+\gamma_u m_0\vec{a}$$

$\vec{f},\vec{u},\vec{a}$ *are not colinear !*

Using $\vec{f}\cdot\vec{u}=\gamma_u^3 m_0(\vec{u}\cdot\vec{a})$ one can solve for $\vec{a}$

$$\gamma_u m_0\vec{a}=\vec{f}-\frac{1}{c^2}(\vec{f}\cdot\vec{u})\vec{u}$$

In linear motion (linac) with $\vec{u}=(u,0,0)$, $\vec{f}=(f,0,0)$, $\vec{a}=(a,0,0)$ it is

$$f=\gamma_u^3 m_0 a$$

and one speaks of

$$\text{longitudinal mass}=\gamma_u^3 m_0$$

In circular motion (synchrotron) with $\vec{f}\perp\vec{u}$ it is

$$f=\gamma_u m_0 a$$

and one speaks of

$$\text{transverse mass}=\gamma_u m_0$$

Energy

A particle moves with $\vec{u}=(u,0,0)$ and *experiences a force* f_x.

Work done at path dx is

$$dE_{kin}=f_x dx=\gamma_u^3 m_0 a_x dx=\gamma_u^3 m_0 \frac{du}{dt} dx=\gamma_u^3 m_0 u\, du$$

$$E_{kin}=m_0 c^2 \int_o^{\beta_u} \frac{\beta_u d\beta_u}{\left(1-\beta_u^2\right)^{3/2}}=\gamma_u m_0 c^2-m_0 c^2=E-E_0$$

Power absorbed by the particle

$$P=\frac{dE_{kin}}{dt}=\frac{d\left(m-m_0\right)}{dt}c^2=\frac{dm}{dt}c^2$$

$$\vec{f}=\frac{d\vec{p}}{dt}=\frac{dm}{dt}\vec{u}+m\frac{d\vec{u}}{dt}=\frac{1}{c^2}\frac{dE_{kin}}{dt}\vec{u}+\gamma_u m_0\vec{a}$$

before $\gamma_u m_0\vec{a}=\vec{f}-\frac{1}{c^2}\left(\vec{f}\cdot\vec{u}\right)\vec{u}$

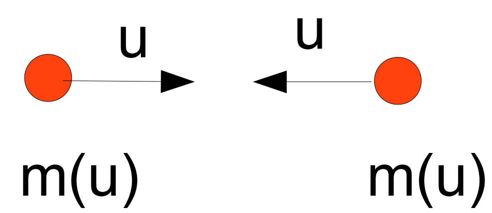
$$\rightarrow \quad P=\frac{dE_{kin}}{dt}=\vec{f}\cdot\vec{u}$$

The temporal change of E_{kin} *of a body, or the power it absorbs, is the scalar product of* $\vec{f}$ and $\vec{u}$.

Example: Collider

1) 3.5 TeV head-on p-p collider

before collision



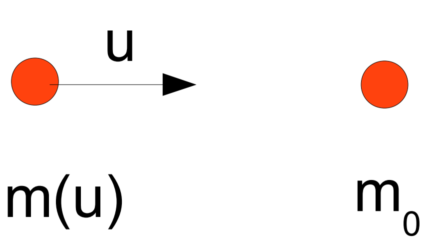
composite particle at rest after collision

transparency 29:

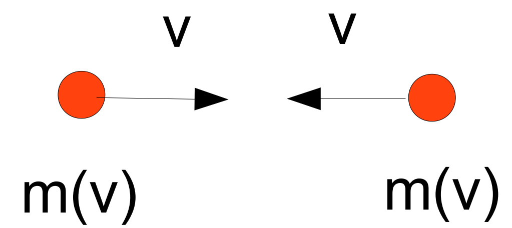
$$M_0 = 2\gamma_u m_0 \quad \rightarrow \quad E_{CM} = M_0 c^2 = 2E = 7\,TeV$$

2) 3.5 TeV fixed target p-machine

In laboratory frame S
before collision



Center of mass frame S'
($\Sigma p=0$) moves to the right with v



W.r.t. S', S moves with v to the left and
the left particle has velocity u'=v in S' (transp. 13)

$$u' = v = \frac{u-v}{1-uv/c^2} \qquad \textit{therefore}$$

$$\beta_v = \frac{1}{\beta_u}\left(1-\sqrt{1-\beta_u^2}\right), \qquad \gamma_v = \sqrt{\frac{1}{2}(1+\gamma_u)}$$

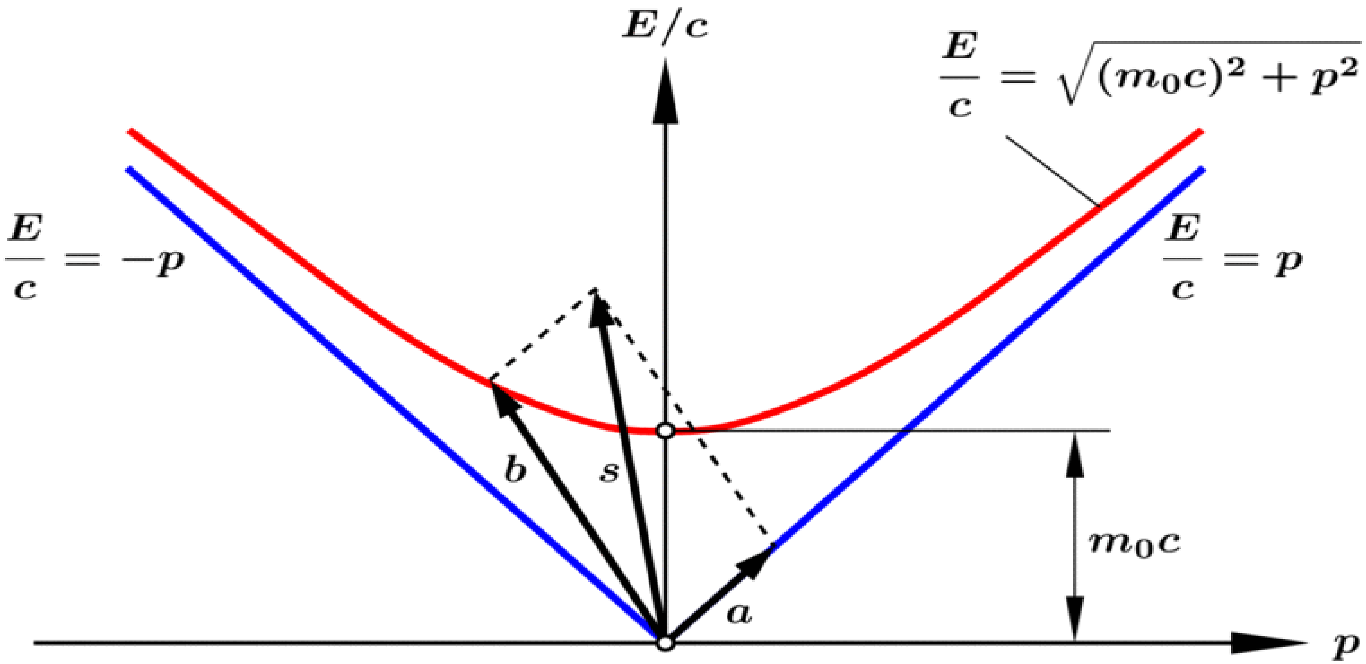
$$E_{CM} = 2\gamma_v E_0 = \sqrt{2(1+\gamma_u)}\,E_0 =$$

$$= \sqrt{2(E_0+E)E_0} = 81\,GeV \qquad (E_{p0} = 938\,MeV)$$

Energy-momentum equation and diagram

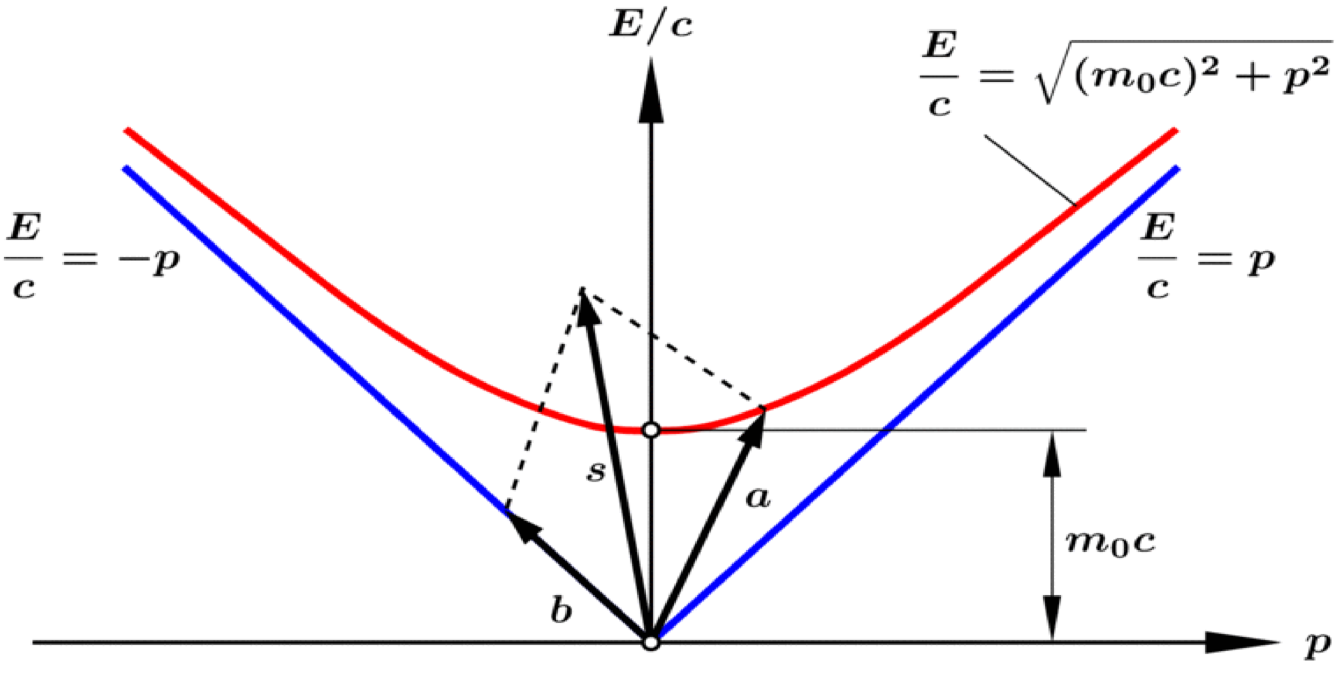
$$E^2=(mc^2)^2=(m_0c^2)^2\gamma^2=(m_0c^2)^2\frac{(1-\beta^2)+\beta^2}{1-\beta^2}=E_0^2+(pc)^2$$

$$\frac{E}{c}=\sqrt{(m_0c)^2+p^2},\qquad \textit{mass-less particle}:\quad E=|p|c$$

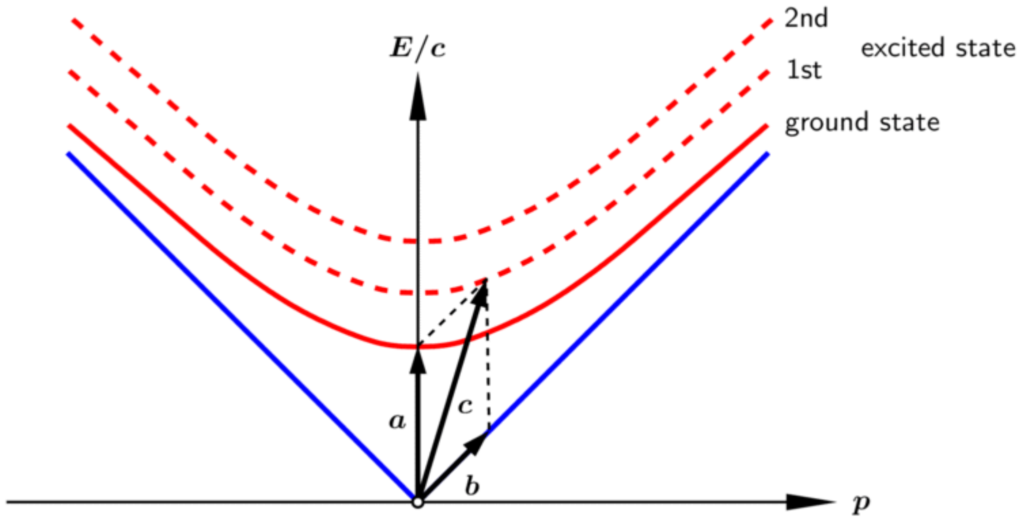


Since conserved quantities are plotted, arrows can be added like vectors.

All interactions are allowed in which energy-momentum vectors a, b after interaction add to vector s.

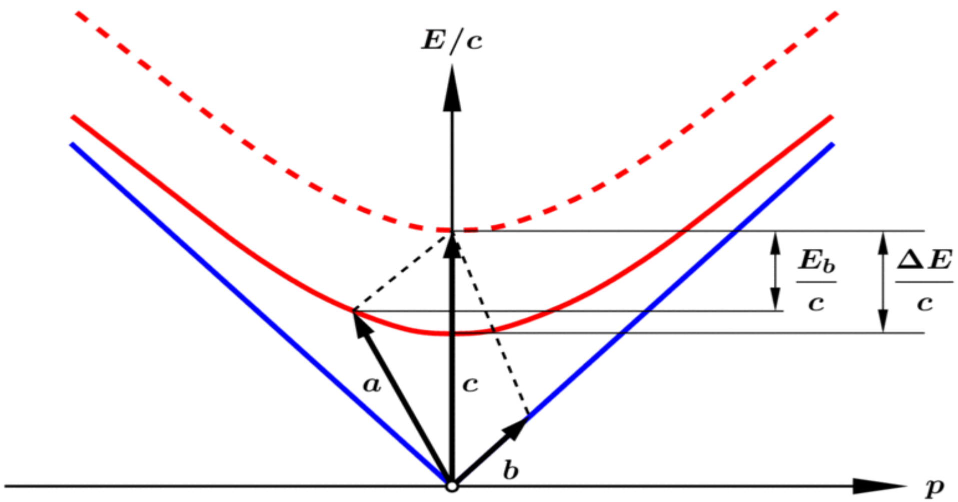


Example: Photon absorption by a particle at rest



Absorption only for composite particles with excited states.

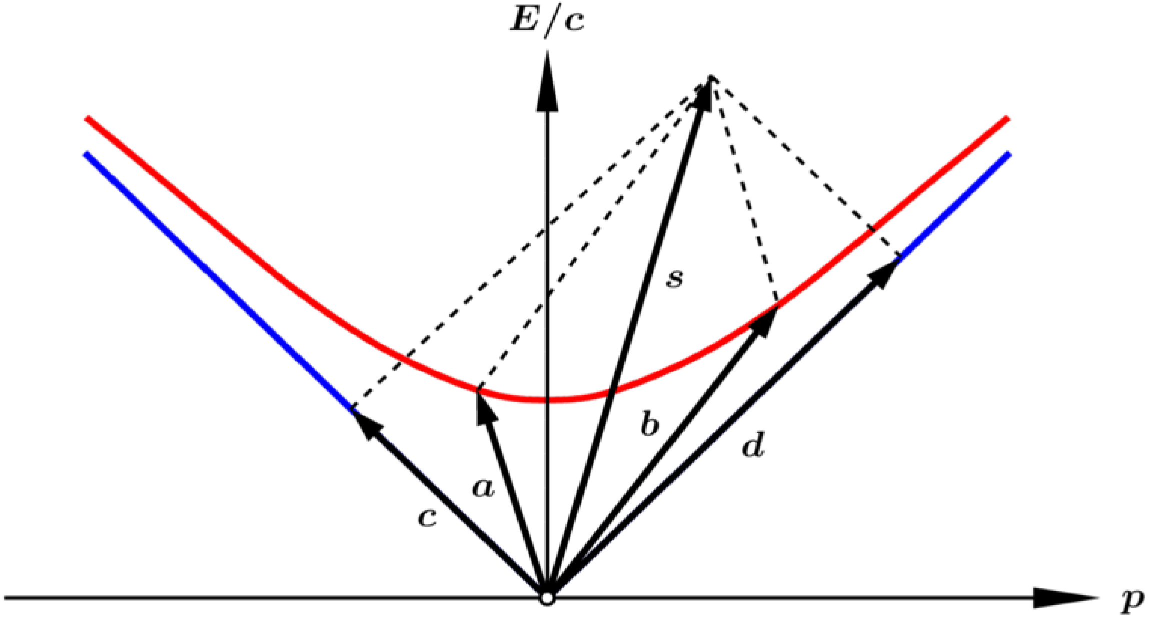
Example: Photon emission by a composite particle at rest



$\Delta E > E_b$

difference in recoil of particle

Example: Pair annihilation



4-Vectors

Normal, 3-dimensional vectors are defined by a linear transformation. They are invariant against translation and rotation of the coordinate system.

Similar we define 4-vectors by the Lorentz-Transformation:
Any quadruple which transforms with an L-T is a 4-vector.

Let us define contra- and covariant 4-vectors:

contravariant $X^{\mu}=(X^0, X^1, X^2, X^3)$

covariant $X_{\mu}=(X_0, -X_1, -X_2, -X_3)$

Using Einsteins summation rule the scalar product is

$$X^{\mu} X_{\mu} = X^{0} X_{0} - X^{1} X_{1} - X^{2} X_{2} - X^{3} X_{3} = X'^{\mu} X'_{\mu}$$

It is invariant under an L-T.

In general:

The scalar product of any two 4-vectors is L-T invariant.

$$A^{\mu} B_{\mu} = A'^{\mu} B'_{\mu}$$

Position 4-vector

The quadruple

$X^{\mu}=(ct, x, y, z)$

transforms with an L-T and is called position 4-vector

Velocity 4-vector

$U^{\mu} = \frac{dX^{\mu}}{dt}$ is not a 4-vector, since dt is not invariant. Try to find a quantity with dimension of time and which is invariant.

An event which moves by (dx, dy, dz) in dt has the Lorentz invariant space-time interval ds

$$dX^{\mu} dX_{\mu} = ds^2 = (cdt)^2 - dx^2 - dy^2 - dz^2 = dX'^{\mu} dX'_{\mu}$$

We write

$$ds = cdt \sqrt{1 - \frac{1}{c^2}\left(\frac{dx^2}{dt^2} + \frac{dy^2}{dt^2} + \frac{dz^2}{dt^2}\right)} = c\,dt \sqrt{1 - \left(\frac{v}{c}\right)^2} =$$

$$= c\frac{dt}{\gamma} = c\,d\tau \qquad \rightarrow \qquad dt = \gamma\, d\tau$$

Now, $d\tau$ is the time interval an observer moving with v would measure. τ is called proper time and is Lorentz invariant (Lorentz scalar). Using τ instead of t, we get the velocity 4-vector

$$U^{\mu} = \frac{dX^{\mu}}{d\tau} = \frac{dX^{\mu}}{dt}\frac{dt}{d\tau} = \left(c, \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\right)\frac{dt}{d\tau} =$$

$$= \gamma_u (c, u_x, u_y, u_z) \quad \rightarrow \quad U^{\mu} U_{\mu} = U'^{\mu} U'_{\mu} = c^2$$

U^{μ} is not a measurable quantity. dX^{μ} is space-time distance between two events in one frame, while $d\tau$ is time increment in a different frame in which both events take place at the same location.

But U^{μ} helps to facilitate calculations.

Example: Man in aircraft and on ground

Energy-momentum 4-vector

A particle with m_0 moves in S with $\vec{u}=(u,0,0)$

$$E=\gamma_u m_0 c^2, \qquad p_x=\gamma_u m_0 u=E\frac{u}{c^2}$$

In S' it's velocity, energy and momentum are

$$u'=\frac{u-v}{1-uv/c^2} \quad \rightarrow \quad \gamma_{u'}=\frac{1}{\sqrt{1-(u'/c)^2}}=\gamma_u\gamma(1-\beta_u\beta)$$

$$\frac{E'}{c}=\gamma_{u'}m_0 c=\gamma\left(\frac{E}{c}-\beta p_x\right)$$

$$p_x'=\gamma_{u'}m_0 u'=\gamma\left(p_x-\beta\frac{E}{c}\right), \quad p_y'=p_y, \quad p_z'=p_z$$

$(E/c, p_x, p_y, p_z)$ transforms with L-T and is called energy-momentum 4-vector.

1. Derivation of energy momentum equation:

In *a frame where momentum does not vanish*:

$$P^{\mu} P_{\mu} = (E/c)^2 - p_x^2 - p_y^2 - p_z^2 = (E/c)^2 - p^2$$

In *a frame with vanishing momentum*: $P'^{\mu} P'_{\mu} = (E_0/c)^2$

$$P^{\mu} P_{\mu} = P'^{\mu} P'_{\mu} \quad \rightarrow \quad E^2 = E_0^2 + (pc)^2$$

2. Derivation of Planck's hypothesis E=hν

A photon with energy E' in S' travels in -x' direction

$$p_x' = -\frac{E'}{c}, \qquad \frac{E}{c} = \gamma\left(\frac{E'}{c} + \beta p_x'\right) = \sqrt{\frac{1-\beta}{1+\beta}} \frac{E'}{c}$$

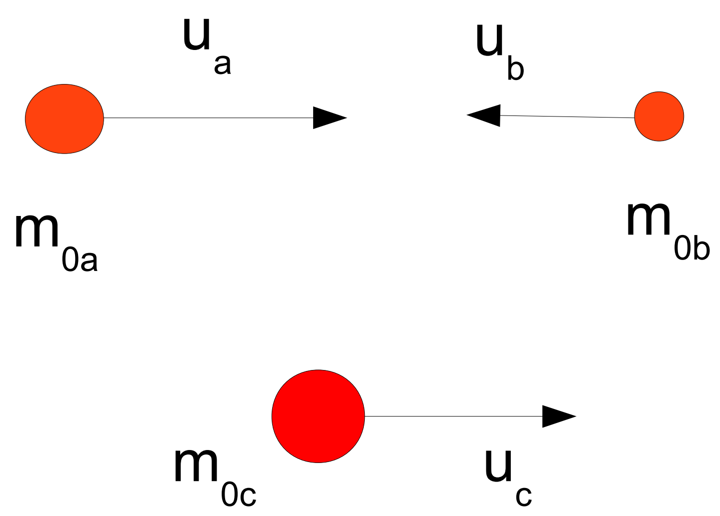
Frequency Doppler shift (transp. 17, θ=180°)

$$\nu = \sqrt{\frac{1-\beta}{1+\beta}} \nu' \quad \rightarrow \quad \frac{E}{\nu} = \frac{E'}{\nu'} = const. = h$$

3. Inelastic collision:

before collision

after collision



Energy, momentum conservation:

$$E_a + E_b = E_c, \quad \vec{p}_a + \vec{p}_b = \vec{p}_c$$

$$\rightarrow \quad P_a^\mu + P_b^\mu = P_c^\mu \qquad \cdot \left(P_{a\mu} + P_{b\mu} = P_{c\mu}\right)$$

$$P_a^\mu P_{a\mu} + 2 P_a^\mu P_{b\mu} + P_b^\mu P_{b\mu} = P_c^\mu P_{c\mu} \qquad (1)$$

Rest frames for (a), (b), (c) $(E/c=m_o c)$:

$$P_a^\mu P_{a\mu}=(m_{0a}c)^2, \qquad P_b^\mu P_{b\mu}=(m_{0b}c)^2$$
$$P_c^\mu P_{c\mu}=(m_{0c}c)^2 \qquad (2)$$

Laboratory frame:

$$P_a^\mu=(\gamma_a m_{0a}c,\gamma_a m_{0a}u_a,0,0)$$
$$P_b^\mu=(\gamma_b m_{0b}c,-\gamma_b m_{0b}u_b,0,0)$$
$$2P_a^\mu P_{b\mu}=2\gamma_a\gamma_b m_{0a}m_{0b}(c^2+u_a u_b) \qquad (3)$$

(2), (3) substituted in (1)

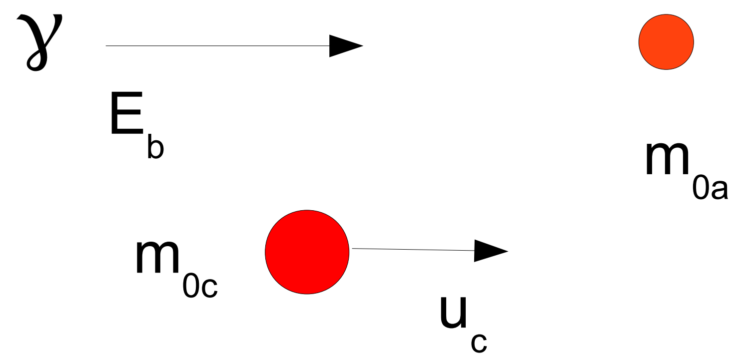
$$m_{0c}=\sqrt{m_{0a}^2+m_{0b}^2+2m_{0a}m_{0b}\gamma_a\gamma_b\left(1+\frac{u_a u_b}{c^2}\right)}\geq m_{0a}+m_{0b}$$

Rest mass of particle (c) is larger than the rest masses of (a) and (b).

4. Absorption of a photon by an atom at rest

before absorption

after absorption



see example 3: $$P_a^\mu P_{a\mu} + 2P_a^\mu P_{b\mu} + P_b^\mu P_{b\mu} = P_c^\mu P_{c\mu} \qquad (1)$$

rest frame of (a) before absorption

$$P_a^\mu = (m_{0a} c, 0, 0, 0), \quad P_b^\mu = \left(\frac{E_b}{c}, p_{bx}, 0, 0\right) = \left(\frac{h\nu}{c}, \frac{h\nu}{c}, 0\right)$$

rest frame of (c) after absorption $$P_c^\mu = (m_{0c} c, 0, 0, 0)$$

$$P_a^{\mu} P_{a\mu} = (m_{0a} c)^2, \qquad P_b^{\mu} P_{b\mu} = 0$$
$$P_c^{\mu} P_{c\mu} = (m_{0c} c)^2, \qquad P_a^{\mu} P_{b\mu} = m_{0a} h\nu \qquad (2)$$

(2) substituted in (1):

$$(m_{0a} c)^2 + 2 m_{0a} c \frac{h\nu}{c} + 0 = (m_{0c} c)^2$$

$$m_{oc} = \sqrt{m_{0a}^2 + 2 m_{0a} \frac{h\nu}{c^2}} = m_{0a} \sqrt{1 + 2 \frac{h\nu}{m_{0a} c^2}}$$

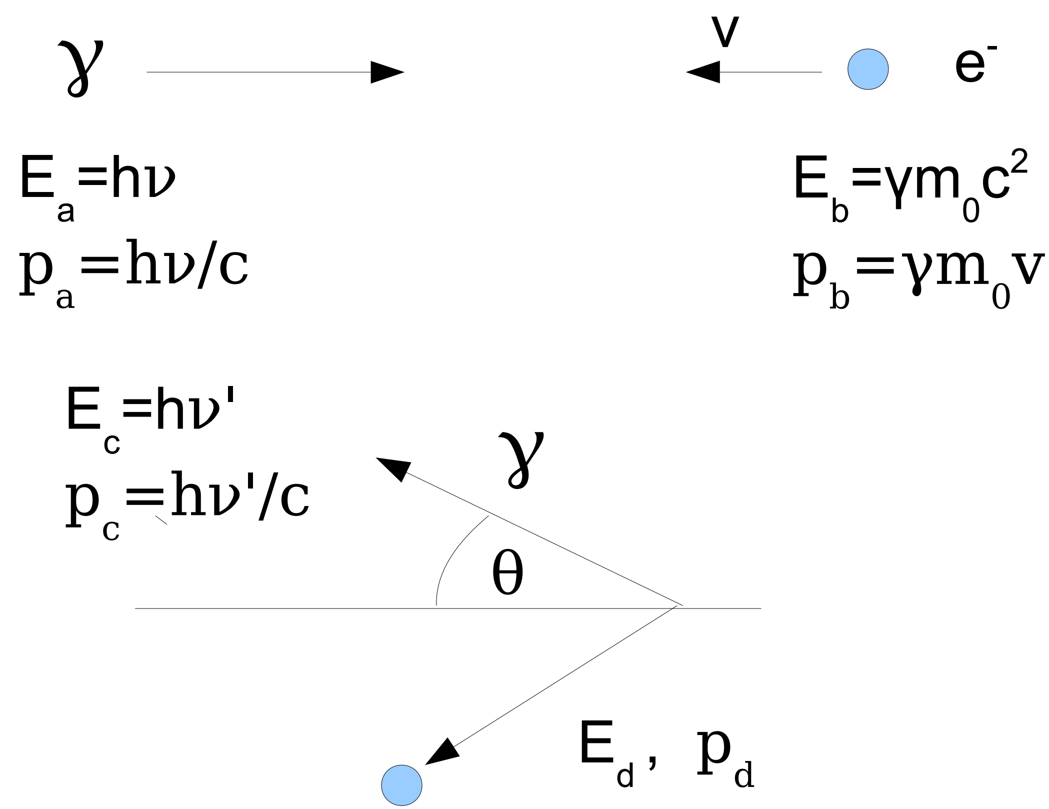
$$\text{If } m_{0a} c^2 \gg h\nu \quad \rightarrow \quad m_{oc} c^2 \approx m_{oa} c^2 + h\nu$$

Rest energy of particle (c) equals rest energy of particle (a) plus photon energy.

5. Compton effect (photon scattered at electron)

before collision

after collision



Energy, momentum conservation:

$$P_a^\mu + P_b^\mu = P_c^\mu + P_d^\mu \qquad (1)$$

Scalar product of (1) with itself $\rightarrow$

$$P_a^\mu P_{a\mu} + 2P_a^\mu P_{b\mu} + P_b^\mu P_{b\mu} = P_c^\mu P_{c\mu} + 2P_c^\mu P_{d\mu} + P_d^\mu P_{d\mu} \qquad (2)$$

for photons: $P_a^\mu P_{a\mu} = P_c^\mu P_{c\mu} = 0$

in rest frames of (a), (b): $P_b^\mu P_{b\mu} = P_d^\mu P_{d\mu}$

substituted in (2): $P_a^\mu P_{b\mu} = P_c^\mu P_{d\mu} \qquad (3)$

multiplication of (1) with $P_{c\mu}$ and use of (3):

$$P_a^\mu P_{c\mu} + P_b^\mu P_{c\mu} = P_c^\mu P_{c\mu} + P_d^\mu P_{c\mu} = P_a^\mu P_{b\mu} \qquad (4)$$

in laboratory frame:

$$P_a^\mu = \left(\frac{h\nu}{c}, \frac{h\nu}{c}, 0, 0\right)$$

$$P_b^\mu = \left(\gamma m_0 c, -\gamma m_0 v, 0, 0\right)$$

$$P_c^\mu = \left(\frac{h\nu'}{c}, -\frac{h\nu'}{c}\cos(\vartheta), \frac{h\nu'}{c}\sin(\vartheta), 0\right)$$

substituted in (4):

$$\left(\frac{h}{c}\right)^2 \nu\nu'(1+\cos(\vartheta))+\gamma m_0 h\nu'(1-\beta\cos(\vartheta))=$$
$$=\gamma m_0 h\nu(1+\beta)$$

$$\frac{\nu'}{\nu}=\frac{1+\beta}{1-\beta\cos(\vartheta)+(1+\cos(\vartheta))E_a/E_b}$$

Electron at rest, β=0, θ=180°-φ :

$$\frac{\nu'}{\nu}=\frac{1}{1+(1-\cos\varphi)h\nu/m_0 c^2}$$

$$\lambda'-\lambda=\frac{h}{m_0 c}(1-\cos\varphi)$$ Compton equation

$$\frac{h}{m_o c}=2{,}42.10^{-12}m$$ Compton wavelength

Astronomy: Microwave background radiation with $E_\gamma \approx 10^{-3}$ eV is scattered at high energy electrons $\gamma \gg 10^8$

$$\vartheta \approx 0, \quad 1+\beta \approx 2,$$

$$\beta = \sqrt{1-\frac{1}{\gamma^2}} \quad \rightarrow \quad 1-\beta \approx \frac{1}{2\gamma^2}$$

$$\frac{\nu'}{\nu} \approx \frac{4\gamma^2}{1+4\gamma^2 E_a/E_b} \approx \gamma \frac{E_{e0}}{E_\gamma}, \qquad E_a = h\nu, \quad E_b = \gamma E_{e0}$$

$$\rightarrow \quad E'_\gamma = h\nu' \approx \gamma E_{e0} = \gamma\, 511\, keV$$

Dramatic increase of photon energy !

Acceleration 4-vector

We use again the proper time τ

$$A^{\mu}=\frac{dU^{\mu}}{d\tau}=\frac{dU^{\mu}}{dt}\frac{dt}{d\tau}=$$

$$=\gamma_u\left[\frac{d\gamma_u}{dt}(c,u_x,u_y,u_z)+\gamma_u\frac{d}{dt}(c,u_x,u_y,u_z)\right]$$

$$=\frac{\gamma_u^4}{c^2}(\vec{u}\cdot\vec{a})(c,\vec{u})+\gamma_u^2(0,\vec{a})$$

$$A^{\mu}A_{\mu}=-\frac{\gamma_u^6}{c^2}(\vec{u}\cdot\vec{a})^2-\gamma_u^4 a^2 \qquad (1)$$

U^{μ} and A^{μ} are perpendicular $\rightarrow$ $U^{\mu}A_{\mu}=0$

For A^μ the same remarks are valid as for U^μ. But it is useful to calculate the proper acceleration α.

In an instantaneous rest frame S' of a particle with u'=0, $\gamma_{u'}=1$:

$$A'^\mu=(0,\vec{\alpha}), \quad A'^\mu A'_\mu=-\alpha^2 \qquad (2)$$

Linear acceleration, $\vec{u}\,||\,\vec{a}$, and *use of* (1), (2):

$$\alpha^2=\gamma_u^6\beta_u^2 a^2+\gamma_u^4 a^2=\gamma_u^6 a^2 \quad \rightarrow \quad \alpha=\gamma_u^3 a$$

The same result follows from transp. 19 with $u'_x=0$, v=u

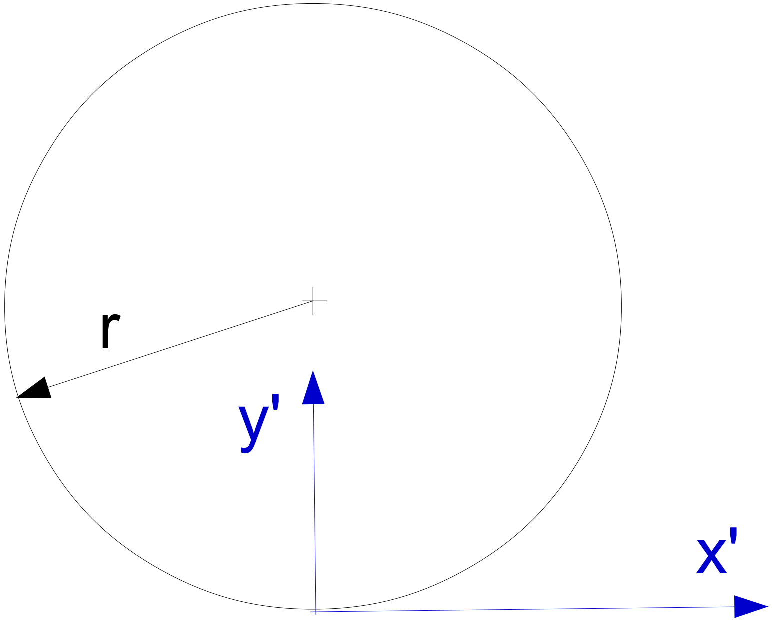
$$a_x=\frac{a'_x}{\gamma_u^3} \quad \rightarrow \quad a'_x=\alpha=\gamma_u^3 a_x$$

In case of circular motion, $\vec{u} \perp \vec{a}$, it follows from (1),(2):

$$\alpha^2 = \gamma_u^4 a^2 \quad \rightarrow \quad \alpha = \gamma_u^2 a$$

Which is identical to the result from transp. 19 for $u_x'=u_y'=0$ and v=u, i.e. a'_y in an instantaneous system S'

$$a_y = \frac{a'_y}{\gamma_u^2} \quad \rightarrow \quad a'_y = \alpha = \gamma_u^2 a_y$$



Frequency-wavenumber 4-vector

Plane wave: $\vec{E}=\vec{E}_0 \sin(\omega t-\vec{k}\cdot\vec{r}), \quad |\vec{k}|=\frac{2\pi}{\lambda}=\frac{\omega}{c}$

Phase at a fixed position must be the same for all reference systems:

$$\Phi=\omega t-\vec{k}\cdot\vec{r}=\omega t-(k_x x+k_y y+k_z z)=\Phi'$$

Φ can be written as

$$K^\mu X_\mu=K'^\mu X'_\mu$$

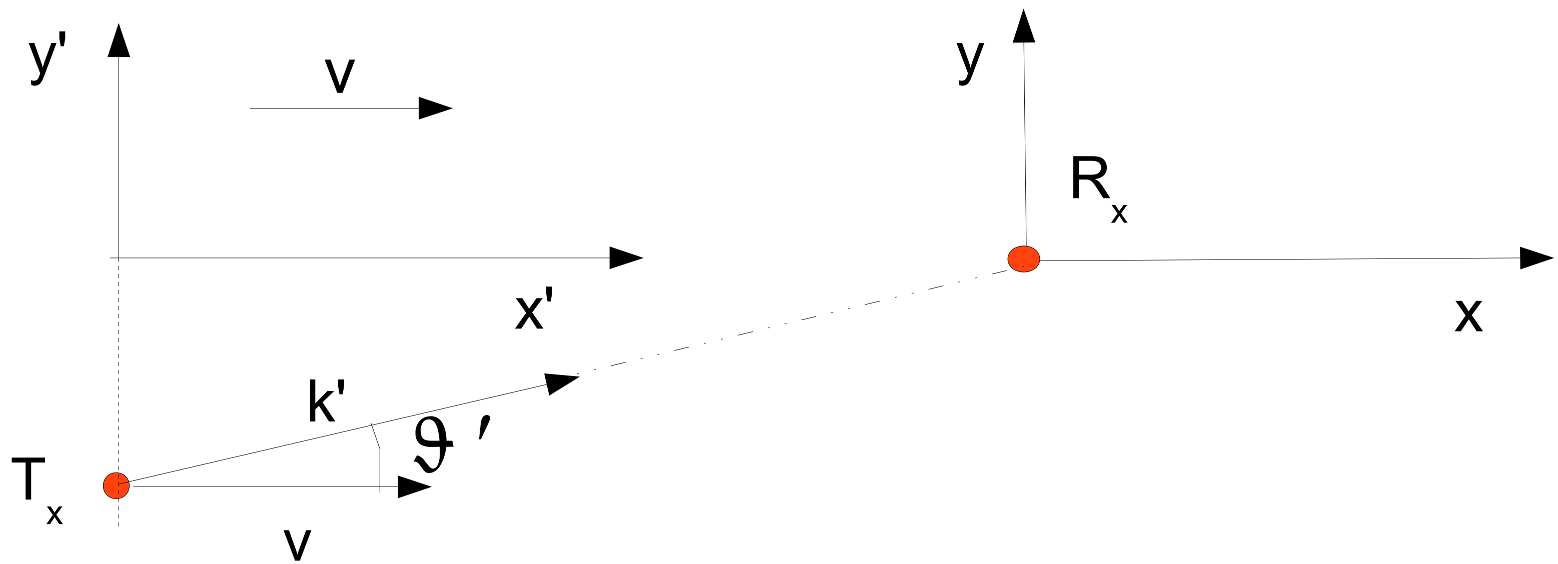
with the 4-vectors

$$K^\mu=\left(\frac{\omega}{c},k_x,k_y,k_z\right), \quad X^\mu=(ct,x,y,z)$$

Since E=hν=ħω and E=pc for photons, it is p=ħω/c=ħk and

$$P^\mu=\left(\frac{E}{c},p_x,p_y,p_z\right)=\hbar K^\mu$$

6.Doppler effect



$$K'^{\mu} = \left(\frac{\omega'}{c}, k'_x, k'_y, k'_z\right) = \left(\frac{\omega'}{c}, \frac{\omega'}{c}\cos\vartheta', \frac{\omega'}{c}\sin\vartheta', 0\right)$$

L-T of K'^{μ} yields frequency in S

$$\frac{\omega}{c} = \gamma\left(\frac{\omega'}{c} + \beta k'_x\right) = \gamma\left(1 + \beta\cos(\vartheta')\right)\frac{\omega'}{c} \qquad (1)$$

and the wave number in S

$$k_x = \frac{\omega}{c}\cos(\vartheta) = \gamma(\beta + \cos\vartheta')\frac{\omega'}{c}$$

$$k_y = \frac{\omega}{c}\sin(\vartheta) = \frac{\omega'}{c}\sin\vartheta', \qquad k_z = 0 \qquad (2)$$

using $\tan(\frac{\vartheta}{2}) = \frac{\sin(\vartheta)}{1+\cos(\vartheta)}$ together with (1) and (2):

$$\tan(\frac{\vartheta}{2}) = \frac{\sin(\vartheta')}{\gamma(1+\beta)(1+\cos(\vartheta'))} = \sqrt{\frac{1-\beta}{1+\beta}}\tan(\frac{\vartheta'}{2})$$

The wave appears under a smaller angle in S than in S', see transp. 15.

Charge-current 4-vector

Going from S to S' charge must be conserved

$$\rho_0 \, dx \, dy \, dz = \rho' \, dx' \, dy' \, dz', \quad dx' = \frac{dx}{\gamma_u} \quad \rightarrow \quad \rho' = \gamma_u \rho_0$$

Moving charge density: $\rho = \gamma_u \rho_0$

Current density: $\vec{j} = \rho \vec{u} = \gamma_u \rho_0 \vec{u}$

Then

$$J^\mu = (\rho c, j_x, j_y, j_z) = \gamma_u \rho_0 (c, u_x, u_y, u_z) = \rho_0 U^\mu$$

is the charge-current 4-vector, since ρ_0 is a Lorentz scalar and U^μ a 4-vector.

Power-force 4-vector (Minkowski force)

Ansatz with proper time τ:

$$F^{\mu}=\frac{dP^{\mu}}{d\tau}=\frac{dP^{\mu}}{dt}\frac{dt}{d\tau}=\gamma_u\left(\frac{1}{c}\frac{dE}{dt},\frac{dp_x}{dt},\frac{dp_y}{dt},\frac{dp_z}{dt}\right)$$

$$\frac{dE}{dt}=\frac{dE_{kin}}{dt}=\vec{f}\cdot\vec{u} \qquad \rightarrow \qquad F^{\mu}=\gamma_u\left(\frac{1}{c}\vec{f}\cdot\vec{u},f_x,f_y,f_z\right)$$

F^{μ} is the power-force 4-vector, since P^{μ} is a 4-vector and τ a Lorentz scalar.

Relativistic Newton's 2nd law: $F^{\mu}=m_0 A^{\mu}$

Remarks

Linear motion:

$$F^{\mu} = \gamma_u \left(\frac{1}{c} f_x u_x, f_x, 0, 0 \right) \qquad \text{L-T: } f'_x = f_x$$

and

$$f_x = f'_x = \alpha_x m_0 = \gamma_u^3 m_0 a_x \qquad \text{(transp. 31, 56)}$$

Circular motion, $\vec{f} \cdot \vec{u} = 0$, and f'_y in an instantaneous system S' moving with v=u:

$$F^{\mu} = \gamma_u \left(0, 0, f_y, 0 \right) \qquad \text{L-T: } f'_y = \gamma_u f_y$$

and

$$f_y = \frac{1}{\gamma_u} f'_y = \frac{1}{\gamma_u} \alpha_y m_0 = \gamma_u m_0 a_y \qquad \text{(transp. 31, 57)}$$

Transformation of electromagnetic fields

Lorentz force $\qquad \vec{f}=q(\vec{E}+\vec{u}\times\vec{B})$

Power-force 4-vector

$$F^{\mu}=\gamma_u(\frac{1}{c}\vec{f}\cdot\vec{u},f_x,f_y,f_z)=\gamma_u q(\frac{1}{c}\vec{E}\cdot\vec{u},\vec{E}+\vec{u}\times\vec{B})$$

in components

$$\begin{bmatrix} F^0 \\ F^1 \\ F^2 \\ F^3 \end{bmatrix}=\frac{q}{c}\begin{bmatrix} 0 & E_x & E_y & E_z \\ E_x & 0 & cB_z & -cB_y \\ E_y & -cB_z & 0 & cB_x \\ E_z & cB_y & -cB_x & 0 \end{bmatrix}\begin{bmatrix} \gamma_u c \\ \gamma_u u_x \\ \gamma_u u_y \\ \gamma_u u_z \end{bmatrix}=\frac{q}{c}\underline{T}\,U^{\mu}$$

With the Lorentz-transformation from S to S'
and the inverse transformation

$$\underline{L}=\begin{bmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \underline{L}^{-1}=\begin{bmatrix} \gamma & \beta\gamma & 0 & 0 \\ \beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

we get

$$F'^{\mu}=\underline{L}\,F^{\mu}=\frac{q}{c}\underline{L}\,\underline{T}\,U^{\mu}=\frac{q}{c}\underline{L}\,\underline{T}\,\underline{L}^{-1}U'^{\mu}=\frac{q}{c}\underline{T}^{-1}U'^{\mu}$$

evaluating $\underline{T}^{-1} = \underline{L}\,\underline{T}\,\underline{L}^{-1}$ and comparing $\underline{T}^{-1}$ with $\underline{T}$:

$$E'_x = E_x \qquad B'_x = B_x$$

$$E'_y = \gamma(E_y - vB_z) \qquad B'_y = \gamma\left(B_y + \frac{v}{c^2} E_z\right)$$

$$E'_z = \gamma(E_z + vB_y) \qquad B'_z = \gamma\left(B_z - \frac{v}{c^2} E_y\right)$$

which can be written as

$$\vec{E}'_{\parallel} = \vec{E}_{\parallel} \qquad \vec{B}'_{\parallel} = \vec{B}_{\parallel}$$

$$\vec{E}'_{\perp} = \gamma(\vec{E}_{\perp} + \vec{v} \times \vec{B}_{\perp}) \qquad \vec{B}'_{\perp} = \gamma\left(\vec{B}_{\perp} - \frac{1}{c^2}\vec{v} \times \vec{E}_{\perp}\right)$$

7. Uniformly moving charge

Point charge at rest in origin of S'

$$\vec{E}\,'=\frac{q}{4\pi\epsilon_0\left(x'^2+y'^2+z'^2\right)^{3/2}}\left(x',y',z'\right),\qquad \vec{B}\,'=0$$

The point P=(0,a,0) in S has coordinates P'=(-vt',a,0) in S', yielding

$$\vec{E}\,'_P(t')=\frac{q}{4\pi\epsilon_0\left(v^2t'^2+a^2\right)^{3/2}}\left(-vt',a,o\right)$$

Transformation of t' $\quad t'=\gamma\left(t-\frac{v}{c^2}x\right)=\gamma t \quad for \quad x=0$

$$\vec{E}\,'_P(t)=\frac{q}{4\pi\epsilon_0\left(v^2\gamma^2t^2+a^2\right)^{3/2}}\left(-v\gamma t,a,o\right)$$

Transformation of fields

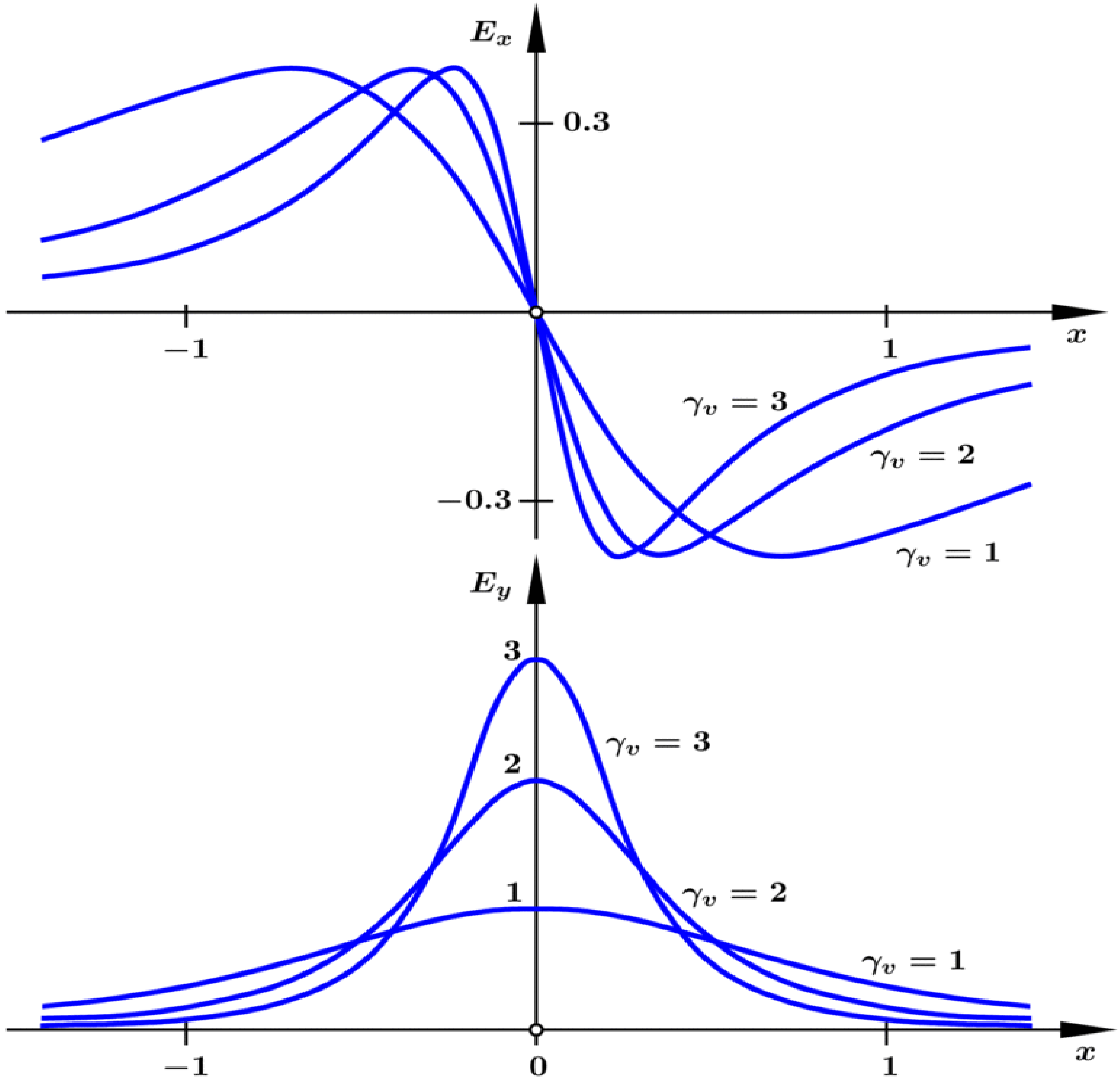
$$E_{Px} = E_{Px}', \qquad B_{Px} = 0$$

$$E_{Py} = \gamma E_{Py}', \qquad B_{Py} = -\gamma \frac{v}{c^2} E_{Pz}'$$

$$E_{Pz} = \gamma E_{Pz}', \qquad B_{Pz} = +\gamma \frac{v}{c^2} E_{Py}'$$

$$\vec{E}_P(t) = \frac{q}{4\pi\epsilon_0 (\gamma^2 v^2 t^2 + a^2)^{3/2}} (-\gamma v t, \gamma a, 0)$$

$$\vec{B}_P(t) = \frac{q}{4\pi\epsilon_0 (\gamma^2 v^2 t^2 + a^2)^{3/2}} \left(0, 0, \gamma \frac{v}{c^2} a\right)$$



E_x
0.3
−0.3
−1
1
x
$\gamma_v = 3$
$\gamma_v = 2$
$\gamma_v = 1$
E_y
3
2
1
$\gamma_v = 3$
$\gamma_v = 2$
$\gamma_v = 1$
−1
0
1
x

Literature:

R. P. Feynman, R. B. Leighton, M. Sands: Lectures on physics. Vol. I. Addison & Wesley, 1963

A. P. French: Special relativity. W. W. Norton & Company, 1966

J. Freund: Special relativity for beginners. World Scientific, 2008

Obligatory homeworks

Experience has shown that not all students have learned „Special Relativity“ at university. Therefore, you receive the lecture notes ahead of the course, so that you can prepare yourself. The exercises are obligatory and will be collected at the beginning of the course.

Exercise 1

Prove that the scalar product of any two 4-vectors A^μ and B^μ is Lorentz invariant.

Exercise 2

Prove the relativistic 2nd law of Newton $F^\mu = m_0 A^\mu$ by comparing the components.

Exercise 3

A space craft travels away from earth with β=0.8. At a distance d= $2.16 \cdot 10^8$ km a radio signal from earth is transmitted to the space craft. How long does the signal need to reach the space craft in the system of the earth? Solve it by using a Minkowski diagram.

Exercise 4

A charge q is at rest. At t=0 an electric field E_x is turned on. Calculate the velocity in the instantaneous rest frame S' of the charge where it experiences a constant acceleration $\alpha = q E_x / m_0$.

Exercise 5

A particle moves in S with velocity $\vec{u}$ and experiences a force $\vec{f}$. What is the force $\vec{f}\,'$ in S'?.

Exercise 6

An electron with velocity u collides with a nucleus at rest. After the collision a photon (bremsstrahlung) is emitted and the electron and nucleus are moving together with u'. What is the energy of the emitted photon?

Exercise 7

A point charge q moves with velocity v parallel to a current carrying wire. Calculate the force on the charge in its rest frame S' by transforming the e.-m. fields.