Introduction to Transverse Beam Dynamics

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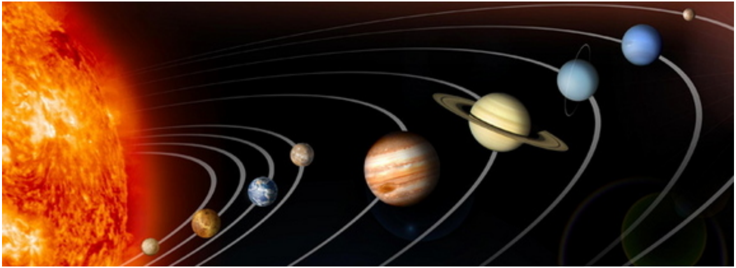
Luminosity run of a typical storage ring

In a storage ring: the protons are accelerated and stored for $\sim$ 12 hours

The distance traveled by particles running at *nearly* the speed of light, $v \approx c$, for 12 hours is

$$\textbf{distance} \approx \mathbf{12 \times 10^{11}}\ \textbf{km}$$

$\rightarrow$ this is about 100 times the distance from Sun to Pluto and back !



Introduction and basic ideas

It's a circular machine: we need a transverse deflecting force→the Lorentz force

$$\vec{F} = q \cdot \left(\vec{E} + \vec{v} \wedge \vec{B} \right)$$

where, in high energy machines, $|\vec{v}| \approx c \approx 3 \cdot 10^8$ m/s. Usually there is no electric field, and the transverse deflection is given by a magnetic field only.

Example

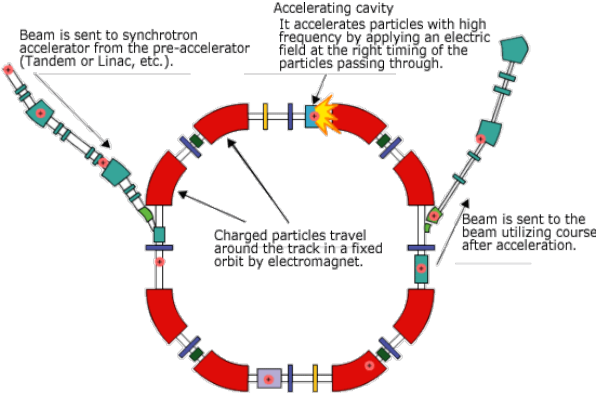
$$B = 1 \text{ T} \rightarrow \quad \begin{aligned} F &= q \cdot 3 \cdot 10^8 \ \frac{m}{s} \cdot 1 \text{ T} \\ &= q \cdot 3 \cdot 10^8 \ \frac{m}{s} \cdot 1 \ \frac{Vs}{m^2} \\ &= q \cdot 300 \ \frac{MV}{m} \end{aligned}$$

Notice that there is a technical limit for an electric field:

$$E \lesssim 1 \ \frac{MV}{m}$$

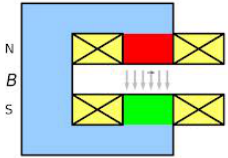
Therefore **in an accelerator, use magnetic fields wherever it's possible**

$$
\left.\begin{array}{lll}
\text{Lorentz force} & F_L & = qvB \\
\text{Centrifugal force} & F_{\text{centr}} & = \dfrac{\gamma m_0 v^2}{\rho} \\
 & \dfrac{\gamma m_0 v^{\not 2}}{\rho} & = q\not v B
\end{array}\right\}
\qquad
\begin{array}{c}
P = m_0 \gamma v = mv \text{ "momentum"} \\
\dfrac{P}{q} = B\rho \\
B\rho = \text{"beam ridigity"}
\end{array}
$$



Dipole magnets: the magnetic guide

- Dipole magnets:
  - define the ideal orbit
  - in a homogeneous field created by two flat pole shoes, $B = \frac{\mu_0 n I}{h}$



- Normalise magnetic field to momentum:

$$\boxed{\frac{P}{q} = B\rho \quad \Rightarrow \quad \frac{1}{\rho} = \frac{qB}{P}} \qquad B = [T]; \quad \text{P}=\left[\frac{GeV}{c}\right]; \quad 1\ \text{T}=\frac{1\ V \cdot 1\ s}{1\ m^2}$$

- Example: the LHC, accelerating protons (q=1 e)

$$\left.\begin{aligned} B &= 8.3\ \text{T} \\ p &= 7000\ \frac{GeV}{c} \end{aligned}\right\} \quad \begin{aligned} \frac{1}{\rho} &= \text{e}\frac{8.3\ \frac{Vs}{m^2}}{7000 \cdot 10^9\ \frac{eV}{c}} = \frac{8.3\text{s}\ \cdot 3 \cdot 10^8\ \frac{m}{s}}{7000 \cdot 10^9\ \text{m\^{}2}} = \\ &= 0.333 \cdot \frac{8.3}{7000}\ \frac{1}{m} = \frac{1}{2.53}\ \frac{1}{km} \end{aligned}$$

Dipole magnets: the magnetic guide

Very important rule of thumb:

$$\frac{1}{\rho\ [m]} \approx 0.3 \frac{B\ [T]}{P\ [GeV/c]}$$

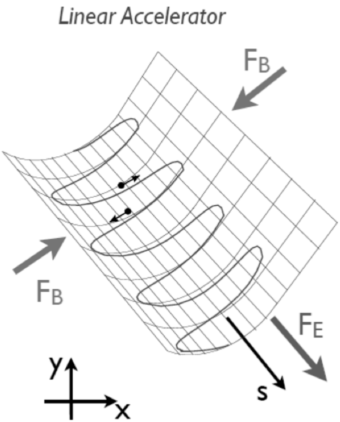
In the LHC, $\rho = 2.53$ km. The circumference $2\pi\rho = 15.9$ km $\approx 60\%$ of the entire LHC.

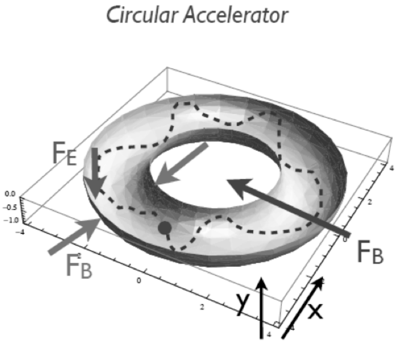
The field B is $\approx 1 \ldots 8$ T

which is a sort of "normalised bending strength", normalised to the momentum of the particles.

The focusing force

$$\vec{F} = q \cdot \left(\vec{E} + \vec{v} \wedge \vec{B} \right)$$





Remember the 1d harmonic oscillator: $F = -k\,x$

Quadrupole magnets: the focusing force

Quadrupole magnets are required to keep the trajectories in vicinity of the ideal orbit

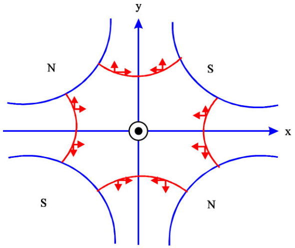
They exert a linearly increasing Lorentz force, thru a linearly increasing magnetic field:

$$\begin{matrix} B_x = gy \\ B_y = gx \end{matrix} \Rightarrow \begin{matrix} F_x = -qv_z gx \\ F_y = qv_z gy \end{matrix}$$

Gradient of a quadrupole magnet:

$$g = \frac{2\mu_0 nI}{r^2} \left[\frac{T}{m}\right] = \frac{B_{\text{poles}}}{r_{\text{aperture}}} \left[\frac{T}{m}\right]$$

- LHC main quadrupole magnets: $g \approx 25 \ldots 235$ T/m



the arrows show the force exerted on a particle

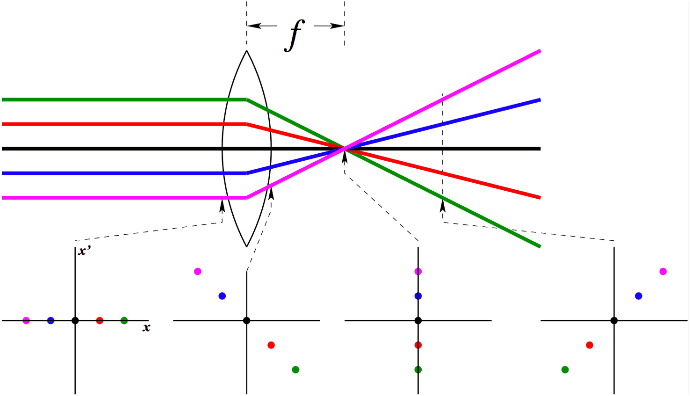
Divide by p/q to find the nornalised focusing strength, k:

$$k = \frac{g}{P/q} \left[m^{-2}\right]; \quad \Rightarrow \quad g = \left[\frac{T}{m}\right]; \quad q = [e]\,; \quad \frac{P}{q} = \left[\frac{\text{GeV}}{c \cdot e}\right] = \left[\frac{GV}{c}\right] = [T\ m]$$

A simple rule: $k\left[m^{-2}\right] \approx 0.3 \frac{g\ [T/m]}{P/q\ [GeV/c/e]}$.

Focal length of a quadrupole

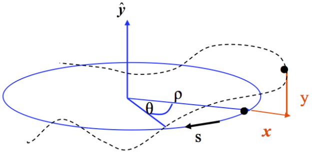
The focal length of a quadrupole is $f = \frac{1}{k \cdot L}$ [m], where L is the quadrupole length:



Towards the equation of motion

Linear approximation:

- the ideal particle ⇒ stays on the **design orbit** (i.e. $x, y, P_x, P_y = 0$; $P = P_0$)
- any other particle ⇒ has coordinates x, y
  - which are small quantities: $x, y \ll \rho$
  - P_x, P_y are small, and $P \neq P_0$
- only linear terms in x and y of B are taken into account



Let's recall some useful relativistic formulæ and definitions:

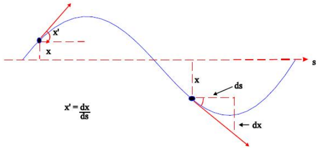
$P_0 = m\,\gamma\,v_0$	reference momentum
$P = P_0\,(1+\delta)$	total momentum
$\delta = (P - P_0)\,/\,P_0$	relative momentum offset
$E = \sqrt{P^2c^2 + m^2c^4} = m\,\gamma\,c^2 = K + m\,c^2$	total energy
$K = E - m\,c^2$	kinetic energy
$\beta = \frac{v}{c} = \frac{Pc}{E}$; $\quad \gamma = \frac{1}{\sqrt{1-\beta^2}} = \frac{E}{m\,c^2}$	relativistic beta and gamma

Phase-space coordinates

The state of a particle is represented with a 6-dimensional phase-space vector:

$$(x,\ x',\ y,\ y',\ z,\ \delta)$$

where x' and y' are the transverse angles:



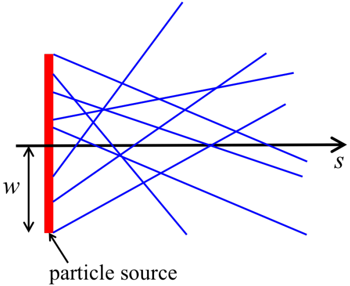
with

$$
\begin{aligned}
& x & & \text{[m]} \\
& x' = \frac{\mathrm{d}x}{\mathrm{d}s} = \frac{\mathrm{d}x}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}s} = \frac{V_x}{V_z} = \frac{P_x}{P_z} \approx \frac{P_x}{P} & & \text{[rad]} \\
& y & & \text{[m]} \\
& y' = \frac{\mathrm{d}y}{\mathrm{d}s} = \frac{\mathrm{d}y}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}s} = \frac{V_y}{V_z} = \frac{P_y}{P_z} \approx \frac{P_y}{P} & & \text{[rad]} \\
& z & & \text{[m]} \\
& \delta = \frac{\Delta P}{P_0} & & \text{[\#]}
\end{aligned}
$$

Exercise: Phase space representations

1. Consider a source at position s_0 with radius w emitting particles. Make a drawing of this setup in configuration space and in phase space. Which part of phase space can be occupied by the emitted particles?

   Hint: the particle source in the configuration space



2. Any real beam emerging from a source like the one above will be clipped by aperture limitations of the vacuum chamber. This can be modelled by assuming that a distance d away from the source there is an iris with an opening with radius $R = w$. Make a drawing of this setup in configuration and phase space. Which part of phase space is occupied by the beam at a location after the iris?

Towards the equation of motion

Taylor expansion of the B_y field:

$$B_y(x) = B_{y0} + \frac{\partial B_y}{\partial x} x + \frac{1}{2} \frac{\partial^2 B_y}{\partial x^2} x^2 + \frac{1}{3!} \frac{\partial^3 B_y}{\partial x^3} x^3 + \dots$$

Now we drop the suffix 'y' and normalise to the magnetic rigidity $p/q = B\rho$

$$\frac{B(x)}{P/q} = \frac{B_0}{B_0 \rho} + \frac{g}{P/q} x + \frac{1}{2} \frac{g'}{P/q} x^2 + \frac{1}{3!} \frac{g''}{P/q} x^3 + \dots$$

$$= \frac{1}{\rho} + kx + \frac{1}{2} m x^2 + \frac{1}{3!} n x^3 + \dots$$

In the linear approximation, only the terms linear in x and y are taken into account:

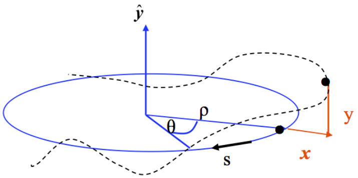
- dipole fields, $1/\rho$
- quadrupole fields, k

It is more practical to use "separate function" magnets, rather than combined ones:

- split the magnets and optimise them regarding their function
  - bending
  - focusing, etc.

The equation of motion in radial coordinates

Let's consider a local segment of one particle's trajectory:



and recall the radial centrifugal acceleration: $a_r = \frac{d^2\rho}{dt^2} - \rho\left(\frac{d\theta}{dt}\right)^2 = \frac{d^2\rho}{dt^2} - \rho\omega^2$.

- For an ideal orbit: $\rho = \text{const} \Rightarrow \frac{d\rho}{dt} = 0$

  $\Rightarrow$the force is
  $$\begin{aligned} F_{\text{centrifugal}} &= -m\rho\omega^2 = -mv^2/\rho \\ F_{\text{Lorentz}} &= qB_y v = -F_{\text{centrifugal}} \end{aligned} \quad \Rightarrow \quad \frac{P}{q} = B_y\rho$$

- For a general trajectory: $\rho \rightarrow \rho + x$:

  $$F_{\text{centrifugal}} = m\,a_r = -F_{\text{Lorentz}} \quad \Rightarrow \quad m\left[\frac{d^2}{dt^2}(\rho + x) - \frac{v^2}{\rho + x}\right] = -qB_y v$$

$$F = \underbrace{m\frac{\mathrm{d}^2}{\mathrm{d}t^2}(\rho + x)}_{\text{term 1}} - \frac{mv^2}{\underbrace{\rho + x}_{\text{term 2}}} = -qB_y v$$

- Term 1: As $\rho =$const...

$$m\frac{\mathrm{d}^2}{\mathrm{d}t^2}(\rho + x) = m\frac{\mathrm{d}^2}{\mathrm{d}t^2}x$$

- Term 2: Remember: $x \approx$ mm whereas $\rho \approx$ m $\rightarrow$ we develop for small x

remember

$$\frac{1}{\rho + x} \approx \frac{1}{\rho}\left(1 - \frac{x}{\rho}\right)$$

Taylor expansion:

$$f(x) = f(x_0) +$$
$$+ (x - x_0) f'(x_0) + \tfrac{(x-x_0)^2}{2!} f''(x_0) + \cdots$$

$$m\frac{d^2x}{\mathrm{d}t^2} - \frac{mv^2}{\rho}\left(1 - \frac{x}{\rho}\right) = -qB_y v$$

The guide field in linear approximation $B_y = B_0 + x\frac{\partial B_y}{\partial x}$

$$m\frac{\mathrm{d}^2x}{\mathrm{d}t^2} - \frac{mv^2}{\rho}\left(1-\frac{x}{\rho}\right) = -qv\left\{B_0 + x\frac{\partial B_y}{\partial x}\right\} \qquad \text{let's divide by } m$$

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} - \frac{v^2}{\rho}\left(1-\frac{x}{\rho}\right) = -\frac{qvB_0}{m} - x\frac{qvg}{m}$$

Independent variable: $t \to s$

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{\mathrm{d}x}{\mathrm{d}s}\frac{\mathrm{d}s}{\mathrm{d}t} = x'v$$

$$\frac{d^2x}{\mathrm{d}t^2} = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\underbrace{\frac{\mathrm{d}x}{\mathrm{d}s}}_{x'}\underbrace{\frac{\mathrm{d}s}{\mathrm{d}t}}_{v}\right) = \frac{\mathrm{d}}{\mathrm{d}t}\left(x'v\right) =$$

$$= \frac{\mathrm{d}}{\mathrm{d}s}\underbrace{\frac{\mathrm{d}s}{\mathrm{d}t}}_{v}\left(x'v\right) = \frac{\mathrm{d}}{\mathrm{d}s}\left(x'v^2\right) = x''v^2 + \cancel{x'2v\frac{\mathrm{d}v}{\mathrm{d}s}}$$

$$x''v^2 - \frac{v^2}{\rho}\left(1-\frac{x}{\rho}\right) = -\frac{qvB_0}{m} - x\frac{vg}{m} \qquad \text{let's divide by } v^2$$

$$x'' - \frac{1}{\rho}\left(1 - \frac{x}{\rho}\right) = -\frac{qB_0}{mv} - x\frac{qg}{mv}$$

$$x'' - \frac{1}{\rho} + \frac{x}{\rho^2} = -\frac{B_0}{P/q} - \frac{xg}{P/q}$$

$$x'' \cancel{-\frac{1}{\rho}} + \frac{x}{\rho^2} = \cancel{-\frac{1}{\rho}} - kx$$

Remember:

$$mv = p$$

Normalise to the momentum of the particle:

$$\frac{1}{\rho} = \frac{B_0}{P/q}\,[\mathrm{m}^{-1}]; \quad k = \frac{g}{P/q}\,[\mathrm{m}^{-2}]$$

$$x'' + x\left(\frac{1}{\rho^2} + k\right) = 0$$

Equation for the vertical motion

- $\frac{1}{\rho^2} = 0$ usually there are not vertical bends
- $k \longleftrightarrow -k$ quadrupole field changes sign

$$y'' - ky = 0$$

Remarks

- "Weak" focusing:

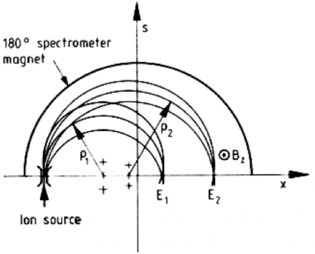
$$x'' + \left(\frac{1}{\rho^2} + k\right) x = 0$$

there is a focusing force, $\frac{1}{\rho^2}$, even without a quadrupole gradient,

$$k = 0 \quad \Rightarrow \quad x'' = -\frac{1}{\rho^2} x$$

even without quadrupoles there is retrieving force (focusing) in the bending plane of the dipole magnets

- In large machine this effect is very weak...



Mass spectrometer: particles are separated according to their energy and focused due to the $1/\rho$ effect of the dipole

Fringe fields

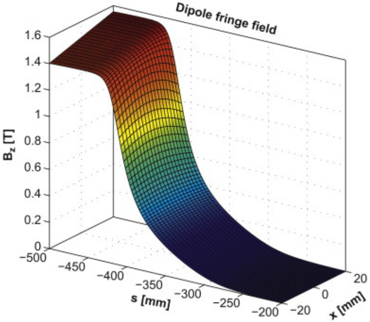
- Hard-edge model:

$$x'' + \left(\frac{1}{\rho^2} + k\right) x = 0$$

  this equation is not really correct

- Bending and focusing forces -even inside a magnet- depend on the position s

$$x''(s) + \left\{\frac{1}{\rho^2(s)} + k(s)\right\} x(s) = 0$$



Fringe field of a dipole magnet (in this case: a combined dipole + quadrupole magnet, notice the slope of the field along the x axis)

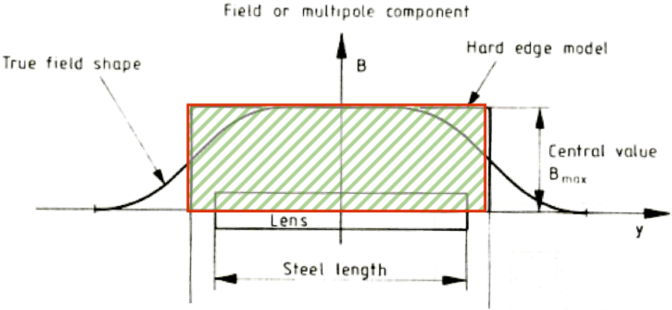
But still: inside the magnet the focusing properties hold:

$$\frac{1}{\rho} = const$$

$$k = const$$

Effective length

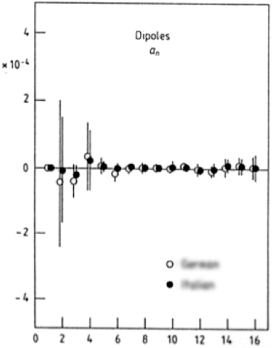
$$B \cdot L_{eff} = \int_0^{l_{mag}} B ds$$



Multipolar moments

Taylor expansion of the B field:

$$B_y(x) = B_{y0} + \frac{\partial B_y}{\partial x} x + \frac{1}{2} \frac{\partial^2 B_y}{\partial x^2} x^2 + \frac{1}{3!} \frac{\partial^3 B_y}{\partial x^3} x^3 + \dots \qquad \text{divide by } B_{y0}$$



Multipole coefficients:

- *divide by the main field to get the relative error contribution*

Solution of the trajectory equations: focusing quadrupole

Definition:

$$\left.\begin{array}{rll} \text{horizontal plane} & K & = {}^{1}/_{\rho^2} + k \\ \text{vertical plane} & K & = -k \end{array}\right\} \quad x'' + Kx = 0$$

This is the differential equation of a harmonic oscillator ... with spring constant K. We make an ansatz:

$$x(s) = a_1 \cos(\omega s) + a_2 \sin(\omega s)$$

General solution: a linear combination of two independent solutions:

$$x'(s) = -a_1 \omega \sin(\omega s) + a_2 \omega \cos(\omega s)$$

$$x''(s) = -a_1 \omega^2 \cos(\omega s) + a_2 \omega^2 \sin(\omega s) = -\omega^2 x(s) \quad \rightarrow \quad \omega = \sqrt{K}$$

General solution, for $K > 0$:

$$x(s) = a_1 \cos\left(\sqrt{K} s\right) + a_2 \sin\left(\sqrt{K} s\right)$$

We determine a_1, a_2 by imposing the following boundary conditions:

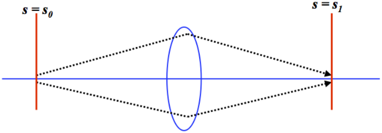
$$s = 0 \quad \rightarrow \quad \begin{cases} x(0) = x_0, & a_1 = x_0 \\ x'(0) = x'_0, & a_2 = \frac{x'_0}{\sqrt{K}} \end{cases}$$

Horizontal focusing quadrupole, $K > 0$:

$$x(s) = x_0 \cos\left(\sqrt{K}s\right) + x'_0 \frac{1}{\sqrt{K}} \sin\left(\sqrt{K}s\right)$$
$$x'(s) = -x_0\sqrt{K} \sin\left(\sqrt{K}s\right) + x'_0 \cos\left(\sqrt{K}s\right)$$

For convenience we can use a matrix formalism:

$$\begin{pmatrix} x \\ x' \end{pmatrix}_{s_1} = M_{foc} \begin{pmatrix} x_0 \\ x'_0 \end{pmatrix}_{s_0}$$



For a quadrupole of length L:

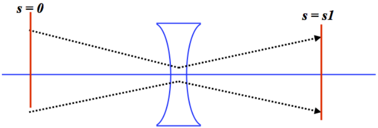
$$M_{\text{foc}} = \begin{pmatrix} \cos\left(\sqrt{K}L\right) & \frac{1}{\sqrt{K}} \sin\left(\sqrt{K}L\right) \\ -\sqrt{K} \sin\left(\sqrt{K}L\right) & \cos\left(\sqrt{K}L\right) \end{pmatrix}$$

Defocusing quadrupole

The equation of motion is

$$x'' + Kx = 0$$

with $K < 0$



Remember: $f(s) = \cosh(s)$
$f'(s) = \sinh(s)$

The solution is in the form:

$$x(s) = a_1 \cosh(\omega s) + a_2 \sinh(\omega s)$$

with $\omega = \sqrt{|K|}$. For a quadrupole of length L the transfer matrix reads:

$$M_{\text{defoc}} = \begin{pmatrix} \cosh\left(\sqrt{|K|}L\right) & \frac{1}{\sqrt{|K|}} \sinh\left(\sqrt{|K|}L\right) \\ \sqrt{|K|} \sinh\left(\sqrt{|K|}L\right) & \cosh\left(\sqrt{|K|}L\right) \end{pmatrix}$$

Notice that for a drift space, i.e. when $K = 0 \rightarrow M_{\text{drift}} = \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix}$

Summary of the transfer matrices

- Focusing quad, $K > 0$

$$M_{\text{foc}} = \begin{pmatrix} \cos\left(\sqrt{K}L\right) & \frac{1}{\sqrt{K}}\sin\left(\sqrt{K}L\right) \\ -\sqrt{K}\sin\left(\sqrt{K}L\right) & \cos\left(\sqrt{K}L\right) \end{pmatrix}$$

- Defocusing quad, $K < 0$

$$M_{\text{defoc}} = \begin{pmatrix} \cosh\left(\sqrt{|K|}L\right) & \frac{1}{\sqrt{|K|}}\sinh\left(\sqrt{|K|}L\right) \\ \sqrt{|K|}\sinh\left(\sqrt{|K|}L\right) & \cosh\left(\sqrt{|K|}L\right) \end{pmatrix}$$

- Drift space, $K = 0$

$$M_{\text{drift}} = \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix}$$

With the assumptions we have made, the motion in the horizontal and vertical planes is independent: "... the particle motion in x and y is uncoupled"

Thin-lens approximation of a quadrupole magnet

When the focal length f of the quadrupolar lens is much bigger than the length of the magnet itself, L_Q

$$f = \frac{1}{k \cdot L} \qquad \gg L_Q$$

we can derive the limit for $L \to 0$ while keeping constant f, i.e. $k \cdot L_Q = \text{const.}$

The transfer matrices are

$$M_x = \begin{pmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{pmatrix} \qquad M_y = \begin{pmatrix} 1 & 0 \\ \frac{1}{f} & 1 \end{pmatrix}$$

focusing, and defocusing respectively.

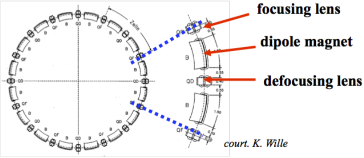
This approximation (yet quite accurate, in large machines) is useful for fast calculations... (e.g. for the guided studies!)

Transformation through a system of lattice elements

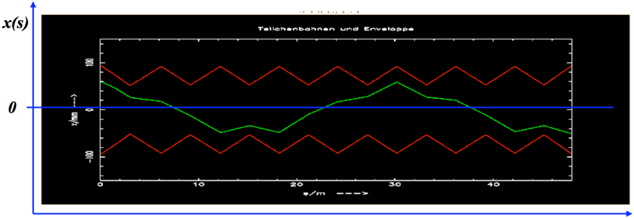
One can compute the solution of a system of elements, by multiplying the matrices of each single element:

$$M_{\text{total}} = M_{\text{QF}} \cdot M_{\text{D}} \cdot M_{\text{Bend}} \cdot M_{\text{D}} \cdot M_{\text{QD}} \cdot \quad \cdots$$

$$\begin{pmatrix} x \\ x' \end{pmatrix}_{s_2} = M_{s_1 \rightarrow s_2} \cdot M_{s_0 \rightarrow s_1} \cdot \begin{pmatrix} x \\ x' \end{pmatrix}_{s_0}$$



In each accelerator element the particle trajectory corresponds to the movement of a harmonic oscillator.



...typical values are:

$$x \approx \text{mm}$$

$$x' \leq \text{mrad}$$

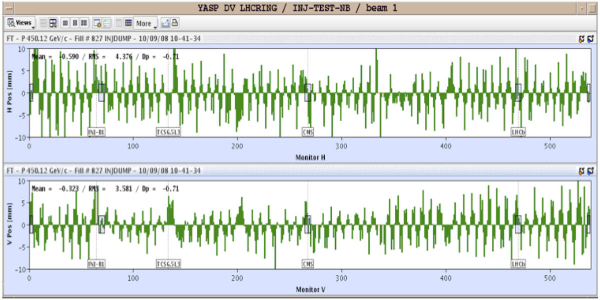
Orbit and tune

Tune: the number of oscillations per turn.

Example:

64.31

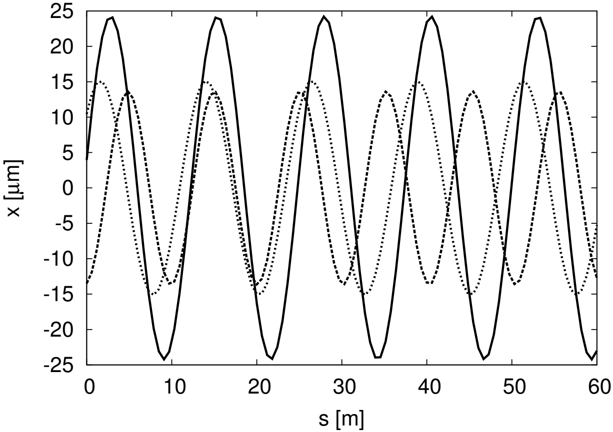
59.32



Relevant for beam stability studies is : the non-integer part

Exercise

The following plot represents the trajectories of three particles traveling in a transfer line with constant focusing strength.



Among the three particles, one is significantly off-momentum. Which one is it (full, small-dot or large-dot line)? Is its rigidity higher or lower than the on-momentum particles?

Summary

beam rigidity: $B\rho = \frac{P}{q}$

bending strength of a dipole: $\frac{1}{\rho}\ [m^{-1}] = \frac{0.2998 \cdot B_0\ [\mathrm{T}]}{P\ [\mathrm{GeV/c}]}$

focusing strength of a quadruple: $k\ [m^{-2}] = \frac{0.2998 \cdot g}{P\ [\mathrm{GeV/c}]}$

focal length of a quadrupole: $f = \frac{1}{k \cdot L_{\mathrm{Q}}}$

equation of motion: $x'' + \left(\frac{1}{\rho^2} + k\right) x = 0$

solution of the eq. of motion: $x_{s_2} = M \cdot x_{s_1}$ ...with $M \equiv \begin{pmatrix} C & S \\ C' & S' \end{pmatrix}$

e.g.: $M_{\mathrm{QF}} = \begin{pmatrix} \cos\left(\sqrt{K}L\right) & \frac{1}{\sqrt{K}}\sin\left(\sqrt{K}L\right) \\ -\sqrt{K}\sin\left(\sqrt{K}L\right) & \cos\left(\sqrt{K}L\right) \end{pmatrix},$

$$M_{\mathrm{QD}} = \begin{pmatrix} \cosh\left(\sqrt{|K|}L\right) & \frac{1}{\sqrt{|K|}}\sinh\left(\sqrt{|K|}L\right) \\ \sqrt{|K|}\sinh\left(\sqrt{|K|}L\right) & \cosh\left(\sqrt{|K|}L\right) \end{pmatrix}, \quad M_{\mathrm{D}} = \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix}$$

Extra: Summary of momenta and angles definitions

$P = P_0\,(1+\delta)$ total momentu w.r.t. reference momentum

$P = \sqrt{P_x^2 + P_y^2 + P_z^2}$ total momentum

- General convention: lower-case momenta: normalised to P_0

$$p = \frac{P}{P_0} = 1+\delta$$

$$p_x = \frac{P_x}{P_0}$$

$$p_y = \frac{P_y}{P_0}$$

$$p_z = \frac{P_z}{P_0} = \frac{\sqrt{P^2 - P_x^2 - P_y^2}}{P_0} = \sqrt{(1+\delta)^2 - p_x^2 - p_y^2} \approx$$

$$\approx (1+\delta)\left(1 - \frac{1}{2}\frac{p_x^2 + p_y^2}{(1+\delta)^2}\right) =$$

$$= 1+\delta - \frac{1}{2}\frac{p_x^2+p_y^2}{1+\delta} \approx 1+\delta \text{ for small } p_x \text{ and } p_y$$

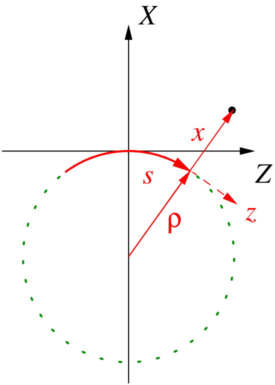
$$x' = \frac{\mathrm{d}x}{\mathrm{d}s} = \frac{V_x}{V_z} = \frac{P_x}{P_z} \approx \frac{P_x}{P_0\,(1+\delta)}$$

$$y' = \frac{\mathrm{d}y}{\mathrm{d}s} = \frac{V_y}{V_z} = \frac{P_y}{P_z} \approx \frac{P_y}{P_0\,(1+\delta)}$$

Extra: From a Cartesian to a curved reference system

We use a Curved Reference System: the Frenet–Serret rotating frame

Curvilinear → Cartesian $(x, y, z) \to (X, Y, Z)$	**Cartesian → Curvilinear** $(X, Y, Z) \to (x, y, z)$
$z = s - \beta ct$	$s = \rho \arctan \frac{Z}{X+\rho}$
$X = (\rho + x)\cos\frac{s}{\rho} - \rho$ $Y = y$ $Z = (\rho + x)\sin\frac{s}{\rho}$	$x = \sqrt{(X+\rho)^2 + Z^2} - \rho$ $y = Y$ $z = s - \beta ct$
$P_x = P_X \cos\frac{s}{\rho} + P_Z \sin\frac{s}{\rho}$ $P_y = P_Y$	$P_X = P_x \cos\frac{s}{\rho} - P_z \sin\frac{s}{\rho}$ $P_Y = P_y$

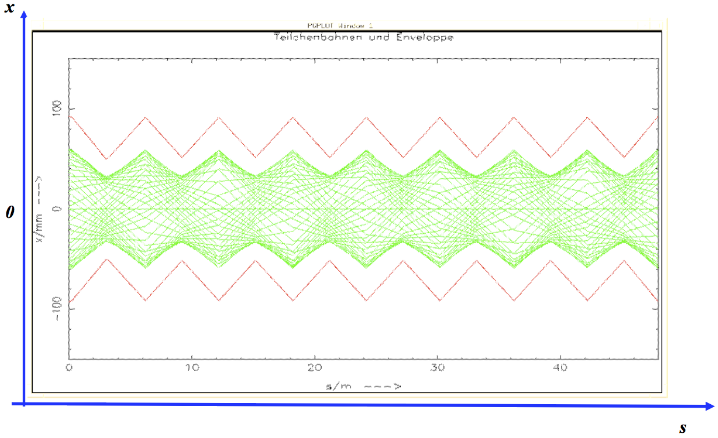


The y and Y axes are parallel and ortogonal to this page.

Envelope

We have studied the motion of a particle.
Question: what will happen, if the particle performs a second turn ?

- ... or a third one or ... 10^{10} turns ...



The Hill's equation

In 19th century George William Hill, one of the greatest master of celestial mechanics of his time, studied the differential equation for "motions with periodic focusing properties": the "Hill's equation"

$$x''(s) + K(s)\,x(s) = 0$$

with:

- a restoring force $\neq$ const
- $K(s)$ depends on the position s
- $K(s+L) = K(s)$ periodic function, where L is the "lattice period"

We expect a solution in the form of a quasi harmonic oscillation: amplitude and phase will depend on the position s in the ring.

The beta function

General solution of Hill's equation:

$$x(s) = \sqrt{\varepsilon}\sqrt{\beta(s)}\cos(\mu(s) + \mu_0) \qquad (1)$$

ε, μ_0 =integration constants determined by initial conditions

$\beta(s)$ is a periodic function given by the focusing properties of the lattice $\leftrightarrow$ quadrupoles

$$\beta(s + L) = \beta(s)$$

Inserting Eq. (1) in the equation of motion, we get (Floquet's theorem) the following result

$$\mu(s) = \int_0^s \frac{\mathrm{d}s}{\beta(s)}$$

$\mu(s)$ is the "phase advance" of the oscillation between the points 0 and s along the lattice. For one complete revolution, $\mu(s)$ is the number of oscillations per turn, or "tune" when normalised to 2π

$$Q = \frac{1}{2\pi}\oint \frac{\mathrm{d}s}{\beta(s)}$$

ε is the Courant-Snyder invariant.

The beam ellipse

General solution of the Hill's equation

$$\begin{cases} x(s) = \sqrt{\varepsilon}\sqrt{\beta(s)}\cos(\mu(s)+\mu_0) & (1) \\ x'(s) = -\frac{\sqrt{\varepsilon}}{\sqrt{\beta(s)}}\{\alpha(s)\cos(\mu(s)+\mu_0)+\sin(\mu(s)+\mu_0)\} & (2) \end{cases}$$

From Eq. (1) we get

$$\cos(\mu(s)+\mu_0) = \frac{x(s)}{\sqrt{\varepsilon}\sqrt{\beta(s)}}$$

$$\alpha(s) = -\frac{1}{2}\beta'(s)$$

$$\gamma(s) = \frac{1+\alpha(s)^2}{\beta(s)}$$

Insert into Eq. (2) and solve for ε

$$\varepsilon = \gamma(s)\,x(s)^2 + 2\alpha(s)\,x(s)\,x'(s) + \beta(s)\,x'(s)^2$$

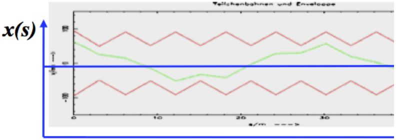
- ε is a constant of the motion, independent of s, i.e. the Courant-Snyder invariant
- it is a parametric representation of an ellipse in the xx' space
- the shape and the orientation of the ellipse are given by α, β, and γ $\Rightarrow$ these are the Twiss parameters

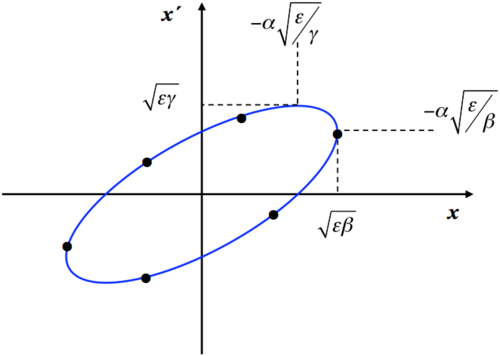
Learning from the phase-space ellipse

$$\varepsilon = \gamma(s)\,x(s)^2 + 2\alpha(s)\,x(s)\,x'(s) + \beta(s)\,x'(s)^2$$

Liouville: in an ideal storage ring, if there is no beam energy change, the area of the ellipse in the phase space $x - x'$ is constant

$$A = \pi \cdot \varepsilon = \text{const}$$





The area of ellipse, $\pi \cdot \varepsilon$, is an intrinsic beam parameter and cannot be changed by the focal properties.

Learning from the phase-space ellipse

Given the particle trajectory:

$$x(s) = \sqrt{\varepsilon}\sqrt{\beta(s)}\cos(\mu(s) + \mu_0)$$

- the max. amplitude is:

$$\hat{x}(s) = \sqrt{\varepsilon\beta}$$

- the corresponding angle, in $\hat{x}(s)$, can be found putting $\hat{x}(s) = \sqrt{\varepsilon\beta}$ in Eq.

$$\varepsilon = \gamma(s)\,x(s)^2 + 2\alpha(s)\,x(s)\,x'(s) + \beta(s)\,x'(s)^2$$

  and solving for x':

$$\varepsilon = \gamma \cdot \epsilon\beta + 2\alpha\sqrt{\varepsilon\beta} \cdot x' + \beta x'^2$$

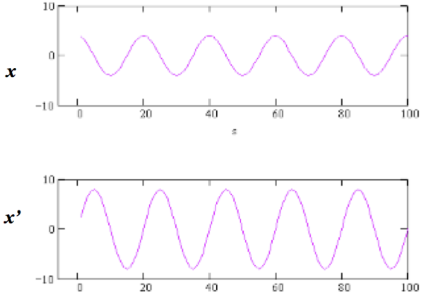
$$\rightarrow \quad \hat{x}' = -\alpha\sqrt{\frac{\varepsilon}{\beta}} \quad \leftarrow$$

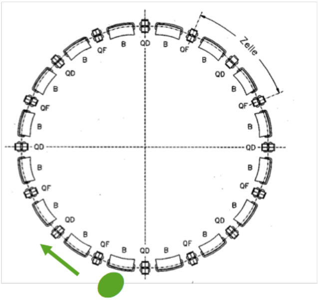
Important remarks:

- A large β-function corresponds to a large beam size and a small beam divergence
- wherever β reaches a maximum or a minimum, $\alpha = 0$ (and $x' = 0$)

Particle tracking in a storage ring

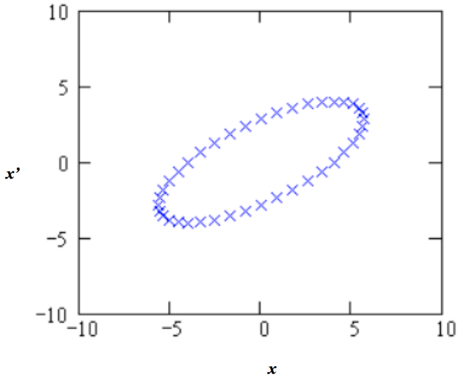
Computation of x and x' for each linear element, according to matrix formalism. We plot x and x' as a function of s

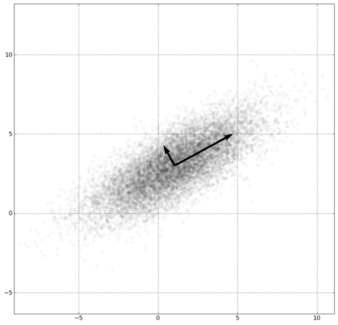




Particle tracking and beam ellipse

For each turn x, x' at a given position s_1 and plot in the phase-space diagram



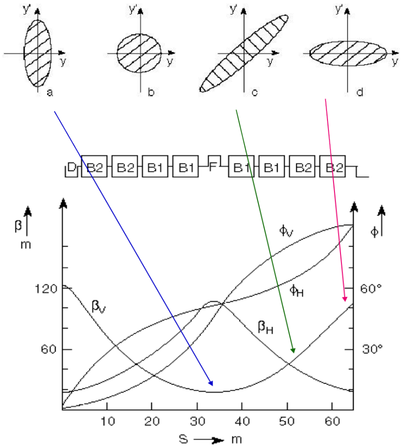


Plane: $x - x'$

Evolution of the phase-space ellipse

Let's repeat the remarks:

- A large β-function corresponds to a large beam size and a small beam divergence
- In the middle of a quadrupole, β is maximum, and $\alpha = 0 \Rightarrow x' = 0$



[VIDEOS!]

Patricles distribution, beam matrix, and emittance

To track a beam of particles, let's assume with Gaussian distribution, the beam ellipse can be characterised by a "beam matrix" Σ

The equation of an ellipse can be written in matrix form:

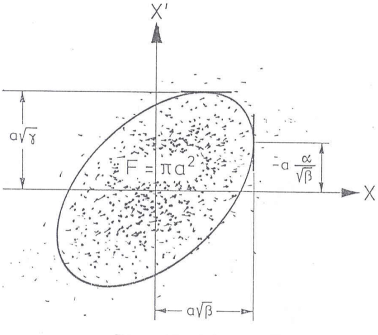
$$X^T \Omega^{-1} X = \varepsilon$$

with $X = \begin{pmatrix} x \\ x' \end{pmatrix}$ and $\Omega = \begin{pmatrix} \beta & -\alpha \\ -\alpha & \gamma \end{pmatrix}$.

For many particles we can define Σ as:

$$\Sigma = \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{pmatrix} = \begin{pmatrix} \langle x^2 \rangle & \langle x x' \rangle \\ \langle x' x \rangle & \langle x'^2 \rangle \end{pmatrix} = \epsilon\, \Omega$$

the covariance matrix of the particles distribution represents an ellipse.



- Given a particles distribution, we define the *geometric emittance* ϵ as a function of the ellipse area:

$$\epsilon = \sqrt{\det \Sigma} = \sqrt{\det\left(\mathrm{cov}(\mathbf{x}, \mathbf{x}')\right)} = \text{Area of the ellipse}/\pi$$

  with slope $r_{21} = \sigma_{21}/\sqrt{\sigma_{11}\sigma_{22}}$

- The emittance ϵ is the area covered by the particles in the transverse x-x' phase-space, and it is preserved along the beam line (Liouville's theorem)

Geometric and Normalised Emittance

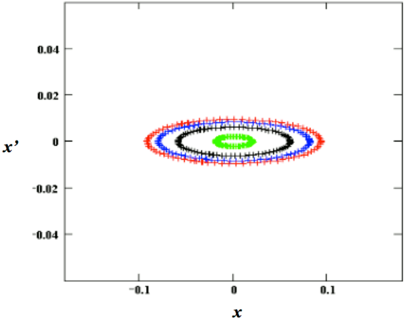
Example: LHC

beam parameters in the arc

$\beta(x) \approx 180\ m$

$\varepsilon \approx 5*10^{-10}\ rad \cdot m \qquad (\leftrightarrow 1\sigma)$

$\sigma = \sqrt{\varepsilon\beta} \approx 0.3\ mm$



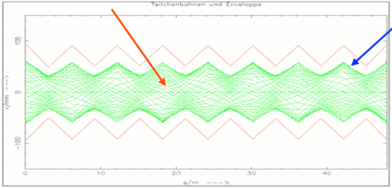
ϵ is the *geometric emittance*. It's a constant of motion only if there is no acceleration, i.e. $P_z =$ constant. If $P_z \to P_z + \Delta P_z$,

$$x' = \frac{P_x}{P_z} \quad \to \quad x' = \frac{P_x}{P_z + \Delta P_z}$$

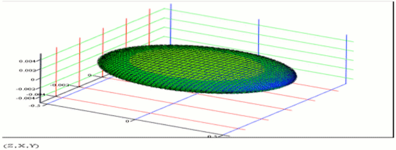
The *normalised* emittance, $\epsilon_N \stackrel{\text{def}}{=} \epsilon_{\text{geom}} \cdot \beta_{\text{relativistic}} \cdot \gamma_{\text{relativistic}}$, is a constant of motion even in case of acceleration.

Emittance of an ensemble of particles

$$x(s) = \sqrt{\varepsilon}\sqrt{\beta(s)} \cdot \cos(\Psi(s) + \phi) \qquad \hat{x}(s) = \sqrt{\varepsilon}\sqrt{\beta(s)}$$



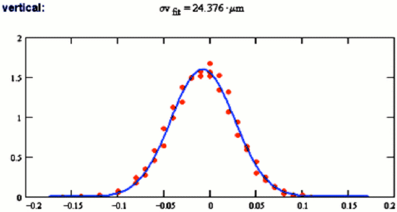
single particle trajectories, $N \approx 10^{11}$ per bunch

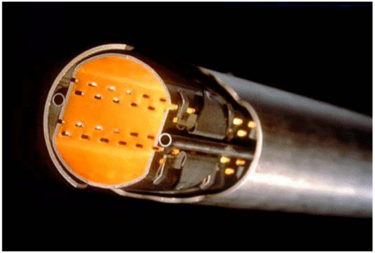


Gauss Particle Distribtion:

$$\rho(x) = \frac{N \cdot e}{\sqrt{2\pi}\sigma_x} \cdot e^{-\frac{1}{2}\frac{x^2}{\sigma_x^2}}$$

particle at distance 1 σ from centre ↔ 68.3 % of all beam particles





LHC: $\sigma = \sqrt{\varepsilon * \beta} = \sqrt{5*10^{-10}m*180m} = 0.3\,mm$

aperture requirements: $r_0 \geq 10 * \sigma$

The transfer matrix M

As we have already seen, a general solution of the Hill's equation is:

$$x(s) = \sqrt{\varepsilon\beta(s)}\cos(\mu(s) + \mu_0)$$
$$x'(s) = -\sqrt{\frac{\varepsilon}{\beta(s)}}\left[\alpha(s)\cos(\mu(s) + \mu_0) + \sin(\mu(s) + \mu_0)\right]$$

Let's remember some trigonometric formulæ:

$$\sin(a \pm b) = \sin a \cos b \pm \cos a \sin b,$$
$$\cos(a \pm b) = \cos a \cos b \mp \sin a \sin b, \ \ldots$$

then,

$$x(s) = \sqrt{\varepsilon\beta(s)}\left(\cos\mu(s)\cos\mu_0 - \sin\mu(s)\sin\mu_0\right)$$
$$x'(s) = -\sqrt{\frac{\varepsilon}{\beta(s)}}\left[\alpha(s)\left(\cos\mu(s)\cos\mu_0 - \sin\mu(s)\sin\mu_0\right) + \right.$$
$$\left. + \sin\mu(s)\cos\mu_0 + \cos\mu(s)\sin\mu_0\right]$$

At the starting point, $s(0) = s_0$, we put $\mu(0) = 0$. Therefore we have

$$\cos\mu_0 = \frac{x_0}{\sqrt{\varepsilon\beta_0}}$$
$$\sin\mu_0 = -\frac{1}{\sqrt{\varepsilon}}\left(x_0'\sqrt{\beta_0} + \frac{\alpha_0 x_0}{\sqrt{\beta_0}}\right)$$

If we replace this in the formulæ, we obtain:

$$\underline{x(s)} = \sqrt{\frac{\beta_s}{\beta_0}}\left\{\cos\mu_s + \alpha_0\sin\mu_s\right\}\underline{x_0} + \left\{\sqrt{\beta_s\beta_0}\sin\mu_s\right\}\underline{x_0'}$$
$$\underline{x'(s)} = \frac{1}{\sqrt{\beta_s\beta_0}}\left\{(\alpha_0-\alpha_s)\cos\mu_s - (1+\alpha_0\alpha_s)\sin\mu_s\right\}\underline{x_0} + \sqrt{\frac{\beta_0}{\beta_s}}\left\{\cos\mu_s - \alpha_s\sin\mu_s\right\}\underline{x_0'}$$

The linear map follows easily,

$$\begin{pmatrix} x \\ x' \end{pmatrix}_s = M \begin{pmatrix} x \\ x' \end{pmatrix}_0 \rightarrow M = \begin{pmatrix} \sqrt{\frac{\beta_s}{\beta_0}}(\cos\mu_s + \alpha_0\sin\mu_s) & \sqrt{\beta_s\beta_0}\sin\mu_s \\ \frac{(\alpha_0-\alpha_s)\cos\mu_s-(1+\alpha_0\alpha_s)\sin\mu_s}{\sqrt{\beta_s\beta_0}} & \sqrt{\frac{\beta_0}{\beta_s}}(\cos\mu_s - \alpha_s\sin\mu_s) \end{pmatrix}$$

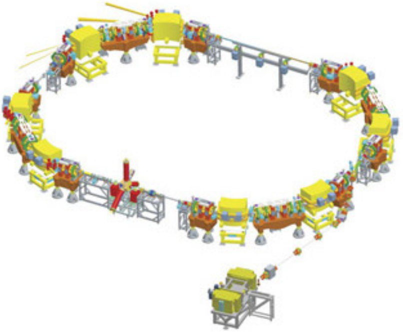
- We can compute the single particle trajectories between two locations in the ring, if we know the α, β, and γ at these positions!
- Exercise: prove that $\det(M) = 1$

Periodic lattices

The transfer matrix for a particle trajectory

$$M = \begin{pmatrix} \sqrt{\frac{\beta_s}{\beta_0}}\left(\cos\mu_s + \alpha_0 \sin\mu_s\right) & \sqrt{\beta_s\beta_0}\sin\mu_s \\ \frac{(\alpha_0-\alpha_s)\cos\mu_s-(1+\alpha_0\alpha_s)\sin\mu_s}{\sqrt{\beta_s\beta_0}} & \sqrt{\frac{\beta_0}{\beta_s}}\left(\cos\mu_s - \alpha_s\sin\mu_s\right) \end{pmatrix}$$

simplifies considerably if we consider one complete turn...



$$M = \begin{pmatrix} \cos\mu_L + \alpha_s\sin\mu_L & \beta_s\sin\mu_L \\ -\gamma_s\sin\mu_L & \cos\mu_L - \alpha_s\sin\mu_L \end{pmatrix}$$

where μ_L is the phase advance per period

$$\mu_L = \int_s^{s+L} \frac{ds}{\beta(s)}$$

Remember: the tune is the phase advance in units of 2π:

$$Q = \frac{1}{2\pi}\oint \frac{ds}{\beta(s)} = \frac{\mu_L}{2\pi}$$

Stability condition

Question: Given a periodic lattice with generic transport map M,

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

under which condition the matrix M provides stable motion after N turns (with $N \to \infty$)?

The answer is simple: the motion is stable when all elements of M^N are finite.

But... what does this imply, for M ?
Remember:

- $\det(M) = ad - bc = 1$
- $\mathrm{trace}(M) = a + d$

If we diagonalize M, we can rewrite it as:

$$M = U \cdot \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \cdot U^T$$

where U is some unitary matrix, λ_1 and λ_2 are the eigenvalues.

Stability condition (cont.)

What happens if we consider N turns?

$$M^N = U \cdot \begin{pmatrix} \lambda_1^N & 0 \\ 0 & \lambda_2^N \end{pmatrix} \cdot U^T$$

Notice that λ_1 and λ_2 can be complex numbers. Given that $\det(M) = 1$, then

$$\lambda_1 \cdot \lambda_2 = 1 \quad \rightarrow \lambda_1 = \frac{1}{\lambda_2} \quad \rightarrow \lambda_{1,2} = e^{\pm i\, x}$$

$\Rightarrow$ to have a stable motion, x must be <u>real</u>: $x \in \mathrm{R}$.

Now we can find the eigenvalues through the characteristic equation:

$$\det(M - \lambda I) = \det \begin{pmatrix} a - \lambda & b \\ c & d - \lambda \end{pmatrix} = 0$$

$$\lambda^2 - (a + d)\,\lambda + (ad - bc) = 0$$
$$\lambda^2 - \mathrm{trace}\,(M)\,\lambda + 1 = 0$$

$$\mathrm{trace}\,(M) = \lambda + 1/\lambda =$$
$$= e^{ix} + e^{-ix} = 2\cos x$$

From which derives the stability condition:

$$\text{since } x \in \mathrm{R} \quad \rightarrow \quad |\mathrm{trace}\,(M)| \leq 2$$

Stability condition (example)

Matrix for 1 turn:

$$M = \begin{pmatrix} \cos\mu_L + \alpha\sin\mu_L & \beta\sin\mu_L \\ -\gamma\sin\mu_L & \cos\mu_L - \alpha\sin\mu_L \end{pmatrix}$$

The condition is satisfied: $|\mathrm{tr}\,(M) = 2\,\cos\mu_L| \leq 2$.

Demonstration for N turns:

$$M = \cos\mu_L \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}_{\mathbf{I}} + \sin\mu_L \underbrace{\begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix}}_{\mathbf{J}}$$

Given that:

$$\begin{aligned}
\mathbf{I}^2 &= \mathbf{I} \\
\mathbf{IJ} &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix} = \mathbf{J} \\
\mathbf{JI} &= \begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix} = \mathbf{J} \\
\mathbf{J}^2 &= \begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix}\begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix} = \begin{pmatrix} \alpha^2 - \gamma\beta & \alpha\beta - \beta\alpha \\ -\gamma\alpha + \alpha\gamma & \alpha^2 - \gamma\beta \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = -\mathbf{I}
\end{aligned}$$

one can compute that:

$$M^N = \mathbf{I}\cos(N\mu_L) + \mathbf{J}\sin(N\mu_L)$$

which indeeds provides stable motion:

$$\left|\mathrm{tr}\left(M^N\right) = 2\,\cos N\mu_L\right| \leq 2$$

Exercise: stability condition

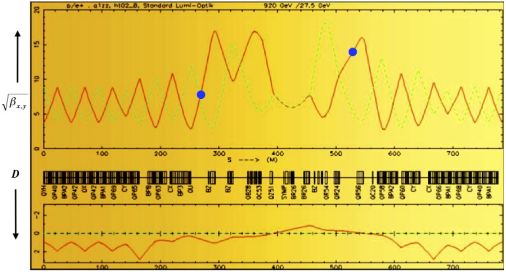
Consider a lattice composed by a single defocusing quadrupole, with $f = 1$ m, and $L_{\text{quad}} = 2$ m.

- Prove that such a lattice is not stable
- Prove that if the quadrupole is focusing, then the lattice is stable

The transformation for α, β, and γ

Consider two positions in the storage ring: s_0, s

$$\begin{pmatrix} x \\ x' \end{pmatrix}_s = M \begin{pmatrix} x \\ x' \end{pmatrix}_{s_0} \quad \text{with} \quad \begin{array}{l} M = M_{\mathrm{QF}} \cdot M_{\mathrm{D}} \cdot M_{\mathrm{Bend}} \cdot M_{\mathrm{D}} \cdot M_{\mathrm{QD}} \cdot \quad \cdots \\ M = \begin{pmatrix} C & S \\ C' & S' \end{pmatrix} \qquad M^{-1} = \begin{pmatrix} S' & -S \\ -C' & C \end{pmatrix} \end{array}$$



Since the Liouville theorem holds, $\varepsilon = \text{const}$:

$$\varepsilon = \beta x'^2 + 2\alpha x x' + \gamma x^2$$

$$\varepsilon = \beta_0 x_0'^2 + 2\alpha_0 x_0 x_0' + \gamma_0 x_0^2$$

We express x_0 and x'_0 as a function of x and x':

$$\begin{pmatrix} x \\ x' \end{pmatrix}_{s_0} = M^{-1} \begin{pmatrix} x \\ x' \end{pmatrix}_{s} \quad \Rightarrow \quad \begin{aligned} x_0 &= S'x - Sx' \\ x'_0 &= -C'x + Cx' \end{aligned}$$

Inserting into ϵ we obtain:

$$\begin{aligned} \varepsilon &= \beta x'^2 + 2\alpha x x' + \gamma x^2 \\ \varepsilon &= \beta_0 \left(-C'x + Cx'\right)^2 + 2\alpha_0 \left(S'x - Sx'\right)\left(-C'x + Cx'\right) + \gamma_0 \left(S'x - Sx'\right)^2 \end{aligned}$$

We need to sort by x and x':

$$\begin{aligned} \beta(s) &= C^2\beta_0 - 2SC\alpha_0 + S^2\gamma_0 \\ \alpha(s) &= -CC'\beta_0 + \left(SC' + S'C\right)\alpha_0 - SS'\gamma_0 \\ \gamma(s) &= C'^2\beta_0 - 2S'C'\alpha_0 + S'^2\gamma_0 \end{aligned}$$

The transformation for α, β, and γ

The beam ellipse transformation in matrix notation:

$$T_{0\rightarrow s} = \left(\begin{array}{ccc} C^2 & -2SC & S^2 \\ -CC' & SC' + S'C & -SS' \\ C'^2 & -2S'C' & S'^2 \end{array} \right)$$

$$\left(\begin{array}{c} \beta \\ \alpha \\ \gamma \end{array} \right)_s = T_{0\rightarrow s} \left(\begin{array}{c} \beta \\ \alpha \\ \gamma \end{array} \right)_0$$

This expression is important, and useful:

1. given the twiss parameters α, β, γ at any point in the lattice we can transform them and compute their values at any other point in the ring
2. the transfer matrix is given by the focusing properties of the lattice elements, the elements of M are just those that we used to compute single particle trajectories

Beam ellipse transformation (another approach)

Let's start from the equation of Σ seen before, now for x_0:

$$X_0^T \Sigma_0^{-1} X_0 = \varepsilon \qquad \text{with:} \quad \Sigma_0 = \begin{pmatrix} \beta_0 & -\alpha_0 \\ -\alpha_0 & \gamma_0 \end{pmatrix}$$

At a later point if the lattice the coordinates of an individual particle are given using the transfer matrix M from s_0 to s_1:

$$X_1 = M \cdot X_0$$

Solving for X_0 , i.e. $X_0 = M^{-1} \cdot X_1$, and inserting in the first equation above, one obtains:

$$\left(M^{-1} \cdot X_1\right)^T \Sigma_0^{-1} \left(M^{-1} \cdot X_1\right) = \varepsilon$$

$$\left(X_1^T \cdot \left(M^T\right)^{-1}\right) \Sigma_0^{-1} \left(M^{-1} \cdot X_1\right) = \varepsilon$$

$$X_1^T \cdot \underbrace{\left(M^T\right)^{-1} \Sigma_0^{-1} M^{-1}}_{\Sigma_1^{-1}} \cdot X_1 = \varepsilon$$

Which gives:

$$\Sigma_1 = M \cdot \Sigma_0 \cdot M^T$$

Beam matrix and Twiss parameters

The beam matrix is the covariance matrix of the particle distribution

$$\Sigma = \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{pmatrix} = \begin{pmatrix} \langle x^2 \rangle & \langle x x' \rangle \\ \langle x' x \rangle & \langle x'^2 \rangle \end{pmatrix}$$

this matrix can be also expressed in terms of Twiss parameters α, β, γ and of the emittance ϵ:

$$\Sigma = \begin{pmatrix} \langle x^2 \rangle & \langle x x' \rangle \\ \langle x' x \rangle & \langle x'^2 \rangle \end{pmatrix} = \epsilon \begin{pmatrix} \beta & -\alpha \\ -\alpha & \gamma \end{pmatrix}$$

Given $M = \begin{pmatrix} C & S \\ C' & S' \end{pmatrix}_{0 \to s}$, we can transport the beam matrix, or the twiss parameters, from 0 to s in two equivalent ways:

- Twiss 3×3 transport matrix:

$$\begin{pmatrix} \beta \\ \alpha \\ \gamma \end{pmatrix}_s = \begin{pmatrix} C^2 & -2SC & S^2 \\ -CC' & SC' + S'C & -SS' \\ C'^2 & -2S'C' & S'^2 \end{pmatrix} \begin{pmatrix} \beta \\ \alpha \\ \gamma \end{pmatrix}_0$$

- Recalling that $\Sigma_s = M \, \Sigma_0 \, M^T$:

$$\begin{pmatrix} \beta & -\alpha \\ -\alpha & \gamma \end{pmatrix}_s = M \cdot \begin{pmatrix} \beta & -\alpha \\ -\alpha & \gamma \end{pmatrix}_0 \cdot M^T$$

Exercise: Twiss transport matrix, T

Compute the Twiss transport matrix, T,

$$T = \begin{pmatrix} C^2 & -2SC & S^2 \\ -CC' & SC' + S'C & -SS' \\ C'^2 & -2S'C' & S'^2 \end{pmatrix}$$

$$\begin{pmatrix} \beta \\ \alpha \\ \gamma \end{pmatrix}_s = T \begin{pmatrix} \beta \\ \alpha \\ \gamma \end{pmatrix}_0$$

for:

1. the identity matrix: $M = \pm \mathbf{I}$
2. a thin quadrupole with focal length $\pm f$
3. a drift of length L

Summary

Hill's equation: $x''(s) + K(s)\,x(s) = 0, \quad K(s) = K(s+L)$

general solution of the Hill's equation: $x(s) = \sqrt{\varepsilon\beta(s)}\cos(\mu(s) + \mu_0)$

phase advance & tune: $\mu_{12} = \int_{s_1}^{s_2} \frac{ds}{\beta(s)}, \quad Q = \frac{1}{2\pi}\oint \frac{ds}{\beta(s)}$

beam ellipse: $\varepsilon = \gamma(s)\,x(s)^2 + 2\alpha(s)\,x(s)\,x'(s) + \beta(s)\,x'(s)^2$

beam emittance: $\epsilon = \text{Area of the ellipse}/\pi = \sqrt{\det\left(\text{cov}(\mathbf{x}, \mathbf{x}')\right)}$

transfer matrix $s_1 \to s_2$: $M = \begin{pmatrix} \sqrt{\frac{\beta_s}{\beta_0}}(\cos\mu_s + \alpha_0 \sin\mu_s) & \sqrt{\beta_s\beta_0}\sin\mu_s \\ \frac{(\alpha_0-\alpha_s)\cos\mu_s-(1+\alpha_0\alpha_s)\sin\mu_s}{\sqrt{\beta_s\beta_0}} & \sqrt{\frac{\beta_0}{\beta_s}}(\cos\mu_s - \alpha_s\sin\mu_s) \end{pmatrix}$

stability criterion: $|\text{trace}(M)| \leq 2$

The transfer matrix M

- Transformation of particle coordinates:

$$\begin{pmatrix} x \\ x' \end{pmatrix}_s = M_{2\times 2} \begin{pmatrix} x \\ x' \end{pmatrix}_0$$

- using matrix notation in terms of the focusing strength K:

$$M = \begin{pmatrix} \cos\left(\sqrt{K}L\right) & \frac{1}{\sqrt{K}}\sin\left(\sqrt{K}L\right) \\ -\sqrt{K}\sin\left(\sqrt{K}L\right) & \cos\left(\sqrt{K}L\right) \end{pmatrix} = \begin{pmatrix} C & S \\ C' & S' \end{pmatrix}$$

- in Twiss form, and for a periodic lattice (over a period):

$$M(s) = \begin{pmatrix} \sqrt{\frac{\beta_s}{\beta_0}}\left(\cos\mu + \alpha_0 \sin\mu\right) & \sqrt{\beta_s\beta_0}\sin\mu \\ \frac{(\alpha_0-\alpha_s)\cos\mu-(1+\alpha_0\alpha_s)\sin\mu}{\sqrt{\beta_s\beta_0}} & \sqrt{\frac{\beta_0}{\beta_s}}\left(\cos\mu - \alpha_s\sin\mu\right) \end{pmatrix}$$

for a period: (1) phase advance: $\cos\mu = \frac{1}{2}\text{trace}(M)$; (2) stability condition: $|\text{trace}(M)| \leq 2$

- Transport of Twiss parameters:

$$\begin{pmatrix} \beta \\ \alpha \\ \gamma \end{pmatrix}_s = \begin{pmatrix} C^2 & -2SC & S^2 \\ -CC' & SC' + S'C & -SS' \\ C'^2 & -2S'C' & S'^2 \end{pmatrix} \begin{pmatrix} \beta \\ \alpha \\ \gamma \end{pmatrix}_0$$

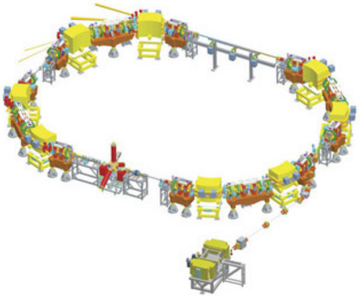
Lattice design in particle accelerators

Or..."how to build a storage ring"

High energy accelerators — are mostly circular machines
we need to juxtapose a number of **dipole** magnets,
to bend the design orbit to a closed ring, then add
quadrupole magnets (FODO cells) to focus the beam
transversely

The geometry of the system is determined by the following equality

$$\text{centrifugal force} = \text{Lorentz force}$$



$$\begin{aligned} \text{Lorentz force} \quad F_L &= evB \\ \text{Centrifugal force} \quad F_{\text{centr}} &= \frac{\gamma m v^2}{\rho} \\ \frac{\gamma m v^{\not 2}}{\rho} &= e \not v B \end{aligned}$$

$$\frac{P}{q} = B\rho$$

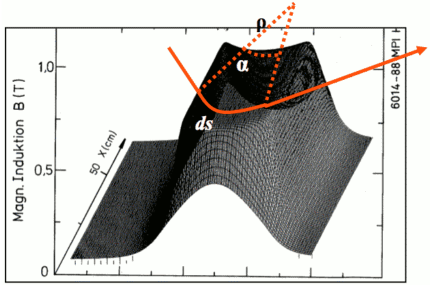
$B\rho$ is the well known beam ridigity

Lattice design: the magnetic guide

$$B\rho = P/q$$

Circular orbit: the dipole magnets define the geometry

$$\theta = \frac{ds}{\rho} \approx \frac{B\,L}{B\rho}$$



field map of a storage ring dipole magnet

The angle spanned in one revolution must be 2π, so, for a full circle:

$$\theta = \frac{\int B dl}{B\rho} = 2\pi \quad \rightarrow \quad \int B dl \approx N L_{\text{Bend}} B = 2\pi \frac{P}{q}$$

this defines the integrated dipole field around the machine.

Note that usually $\frac{\Delta B}{B} \approx 10^{-4}$ is required!

7000 GeV proton storage ring

$N = 1232$ **dipole magnets**

$L_{\text{Bend}} = 15$ **m**

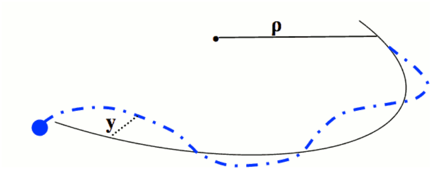
$q = +e$

$$\int B\,dl \approx N L_{\text{Bend}} B = 2\pi p/e$$

$$B \approx \frac{2\pi \cdot 7000 \cdot 10^9\ \textbf{eV}}{1232 \cdot 15\ \textbf{m} \cdot 3 \cdot 10^8\ \frac{m}{s}\, e} = 8.3\ \textbf{T}$$

Focusing forces for single particles

$$x'' + Kx = 0$$



$K = 1/\rho^2 + k$ hor. plane

$K = -k$ vert. plane

$$\left.\begin{array}{lll} \text{dipole magnet} & \frac{1}{\rho} & = \frac{B}{P/q} \\ \text{quadrupole magnet} & k & = \frac{g}{P/q} \end{array}\right\}$$

Example: the LHC ring

Bending radius: $\rho = 2.53$ km

Quad gradient: $g = 220$ T/m

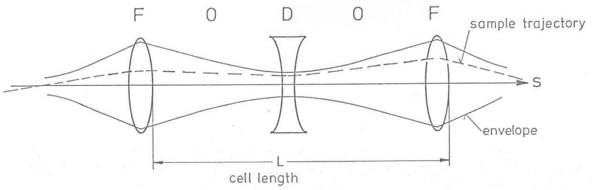
$k = 9.4 \cdot 10^{-3}$ m^{-2}

$1/\rho^2 = 1.3 \cdot 10^{-7}$ m^{-2}

For estimates, in large accelerators, the weak focusing term $1/\rho^2$ can in general be neglected

The FODO lattice

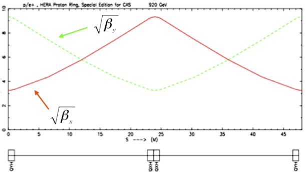
- Most high energy accelerators or storage rings have a periodic sequence of quadrupole magnets of alternating polarity in the arcs

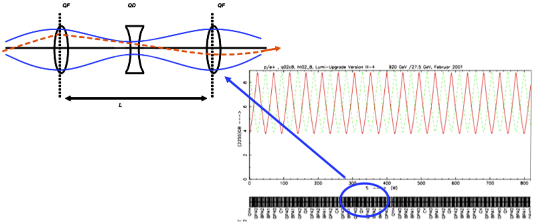


- A magnet structure consisting of focusing and defocusing quadrupole lenses in alternating order with "nothing" in between

- Nota bene: "nothing" here means the elements that can be neglected on first sight: drift, bending magnet, RF structures ... and experiments...

Periodic solution in a FODO Cell





Output of MAD-X

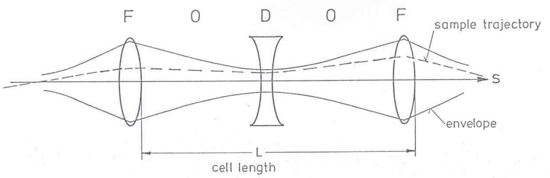
Nr	Type	Length	Strength	β_x	α_x	φ_x	β_z	α_z	φ_z
		m	1/m2	m		$1/2\pi$	m		$1/2\pi$
0	IP	0,000	0,000	11,611	0,000	0,000	5,295	0,000	0,000
1	QFH	0,250	-0,541	11,228	1,514	0,004	5,488	-0,781	0,007
2	QD	3,251	0,541	5,488	-0,781	0,070	11,228	1,514	0,066
3	QFH	6,002	-0,541	11,611	0,000	0,125	5,295	0,000	0,125
4	IP	6,002	0,000	11,611	0,000	0,125	5,295	0,000	0,125

QX= 0,125 QZ= 0,125

$0.125 * 2\pi = 45^0$

The FODO cell

The transfer matrix gives all the information we need.



In thin-lens approximation, we have:

$$M_{\mathrm{F}} = \begin{pmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{pmatrix}; \qquad M_{\mathrm{O}} = \begin{pmatrix} 1 & L/2 \\ 0 & 1 \end{pmatrix}; \qquad M_{\mathrm{D}} = \begin{pmatrix} 1 & 0 \\ +\frac{1}{f} & 1 \end{pmatrix}$$

the transformation matrix of the cell is:

$$M_{\mathrm{FODO}} = M_{\mathrm{F}} \cdot M_{\mathrm{O}} \cdot M_{\mathrm{D}} \cdot M_{O}$$

(notice that you can also write $M = M_{\mathrm{F}/2} \cdot M_{\mathrm{O}} \cdot M_{\mathrm{D}} \cdot M_{O} \cdot M_{\mathrm{F}/2}$, or other permutations), which corresponds to

$$M_{\mathrm{FODO}} = \begin{pmatrix} 1 + \frac{L}{2f} & L + \frac{L^2}{4f} \\ -\frac{2L}{f^2} & 1 - \frac{L}{2f} - \frac{L^2}{4f^2} \end{pmatrix}$$

The FODO cell (cont.)

If we compare the previous matrix with the Twiss representation over one period,

$$M_{\text{FODO}} = \begin{pmatrix} 1 + \frac{L}{2f} & L + \frac{L^2}{4f} \\ -\frac{2L}{f^2} & 1 - \frac{L}{2f} - \frac{L^2}{4f^2} \end{pmatrix}$$

$$M_{\text{Twiss}} = \begin{pmatrix} \cos\mu + \alpha\sin\mu & \beta\sin\mu \\ -\gamma\sin\mu & \cos\mu - \alpha\sin\mu \end{pmatrix} = \cos\mu \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}_{\mathbf{I}} + \sin\mu \underbrace{\begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix}}_{\mathbf{J}}$$

we can derive interesting properties.

- Phase advance

$$\cos\mu = \frac{1}{2}\text{trace}\,(M) = 1 - \frac{L^2}{8f^2}$$

remembering that $\cos\mu = 1 - 2\sin^2\frac{\mu}{2}$

$$\left|\sin\frac{\mu}{2}\right| = \frac{L}{4f}$$

This equation allows to compute the phase advance per cell from the cell length and the focal length of the quadrupoles.

The FODO cell (cont.)

- Example: compute the focal length in order to have a phase advance of 90° per cell

$$f = \frac{1}{\sqrt{2}} \frac{L}{2}$$

  e.g. an emittance measurement station

- Stability requires that $|\cos\mu| < 1$, that is

$$\frac{L}{4f} < 1 \qquad \rightarrow \qquad \text{stability is for:} \quad f > L/4 \quad (\text{or } L < 4f)$$

- Compute the phase advance per cell from the transfer matrix: From $\cos\mu = \frac{1}{2}\text{trace}\,(M)$

$$\mu = \arccos\left(\frac{1}{2}\text{trace}\,(M)\right)$$

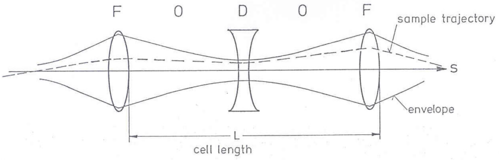
- Compute β-function and α parameter

$$\beta = \frac{M_{12}}{\sin\mu}$$

$$\alpha = \frac{M_{11} - \cos\mu}{\sin\mu}$$

The FODO cell: useful formulæ

For a FODO cell like in figure, with two thin quads separated by length $L/2$



one has:

$$f = \frac{L}{4\sin\frac{\mu}{2}}$$

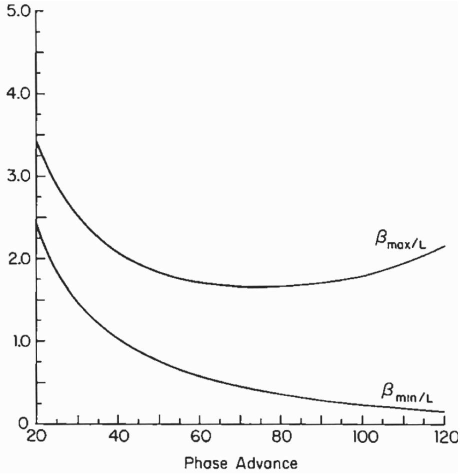
$$\beta^{\pm} = \frac{L\left(1 \pm \sin\frac{\mu}{2}\right)}{\sin\mu}$$

$$\alpha^{\pm} = \frac{\mp 1 - \sin\frac{\mu}{2}}{\cos\frac{\mu}{2}}$$

$$D^{\pm} = \frac{L\theta\left(1 \pm \frac{1}{2}\sin\frac{\mu}{2}\right)}{4\sin^2\frac{\mu}{2}}$$

θ is the total bending angle of the whole cell.

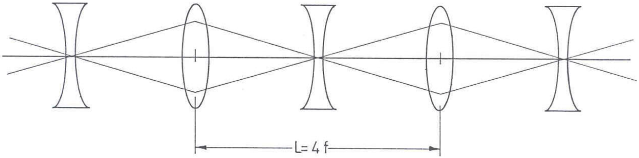
β_{max} and β_{min} as a function of μ



The FODO cell (example 1)

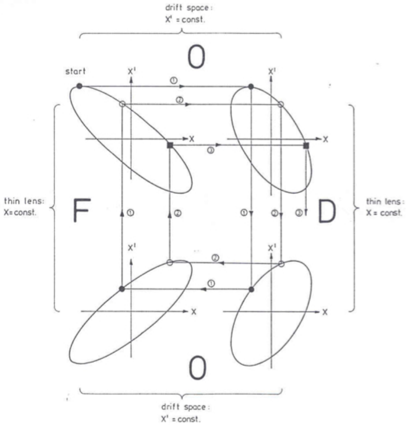
Stability condition $4f \geq L$, has a simple interpretation:

- It is well known from optics that an object at a distance $a = 2f$ from a focusing lens has its image at $b = 2f$



- The defocusing lenses have no effect if a point-like object is located exactly on the axis at distance $2f$ from a focusing lens, because they are traversed on the axis
- If however the lens system is moved further apart ($L > 4f$), this is no more true and the divergence of the light or particle beam is increased by every defocusing lens

The FODO cell (example 2)



- Phase space dynamics in a simple circular accelerator consisting of one FODO cell with two 180° bending magnets located in the drift spaces (the O's)
- The periodicity of α, β, and γ is reflected by the fact that the phase-space ellipse is transformed into itself after each turn
- An individual particle trajectory, however, which starts, for instance, somewhere on the ellipse at the exit of the focusing quadrupole (small circle), is seen to move on the ellipse from turn to turn as determined by the phase angle μ
- Thus, an individual particle trajectory is not periodic, while the envelope of a whole beam is

Exercise: phase-advance of a transfer line

We have seen that the phase advance of a periodic system is given by:

$$\mu = \arccos\left(\frac{1}{2}\text{trace}\,(M)\right)$$

<u>Question</u>: given the transfer matrix M of an arbitrary lattice, and knowing the initial Twiss parameters α_0 and β_0; compute the phase advance μ:

$$\mu = ?$$

Hint: M can be written as:

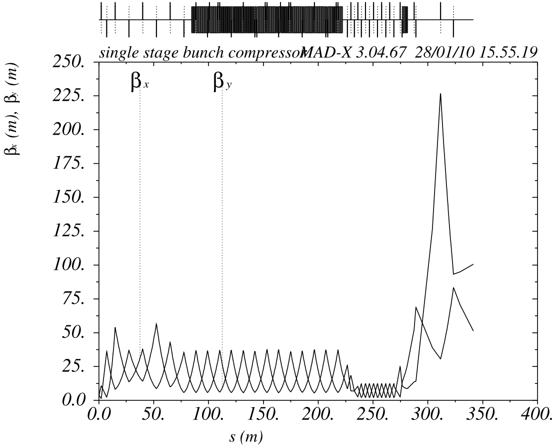
$$M(s) = \begin{pmatrix} \sqrt{\frac{\beta_s}{\beta_0}}\left(\cos\mu + \alpha_0 \sin\mu\right) & \sqrt{\beta_s \beta_0}\sin\mu \\ \frac{(\alpha_0 - \alpha_s)\cos\mu - (1 + \alpha_0 \alpha_s)\sin\mu}{\sqrt{\beta_s \beta_0}} & \sqrt{\frac{\beta_0}{\beta_s}}\left(\cos\mu - \alpha_s \sin\mu\right) \end{pmatrix}$$

Non-periodic beam optics

- In the previous sections the Twiss parameters α, β, γ, and μ have been derived for a periodic, circular accelerator. The condition of periodicity was essential for the definition of the beta function (Hill's equation)
- Often, however, a particle beam moves only **once** along a **beam transfer line**, but one is nonetheless interested in quantities like beam envelopes and beam divergence
- In a circular accelerator α, β, and γ are completely determined by the magnet optics and the condition of periodicity (beam properties are not involved - only the beam emittance is chosen to match the beam size)
- In a transfer line, α, β, and γ are no longer uniquely determined by the transfer matrix, but they also depend on initial conditions which have to be specified in an adequate way

Non-periodic optics: ILC bunch compressor (EX1)

Optics of a non-periodic system including non-periodic optics. "Matching" sections connect parts with different periodic conditions.



The matrix

$$\begin{pmatrix} \beta \\ \alpha \\ \gamma \end{pmatrix}_s = M_{3\times3} \begin{pmatrix} \beta \\ \alpha \\ \gamma \end{pmatrix}_0$$

with

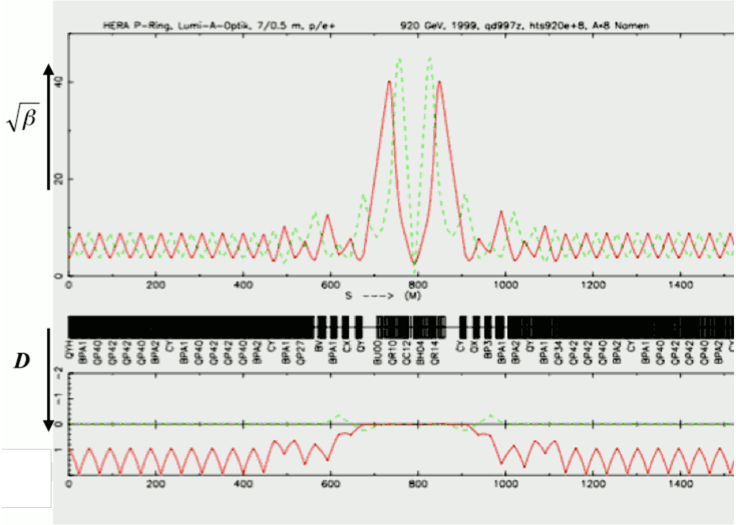
$$M_{3\times3} = \begin{pmatrix} C^2 & -2SC & S^2 \\ -CC' & SC' + S'C & -SS' \\ C'^2 & -2S'C' & S'^2 \end{pmatrix}$$

allows to compute the magnets parameters for the matching sections

Note: even if the β functions are very large, the beam size keeps small: $\sigma = \sqrt{\beta\epsilon}$, with

$$\epsilon_y = \frac{\epsilon_{y,N}}{\gamma_{\text{rel}}} = \frac{5 \times 10^{-9}\ \text{m}}{5\ \text{GeV} / 0.5\ \text{MeV}} = 10^{-13}\ \text{m}$$

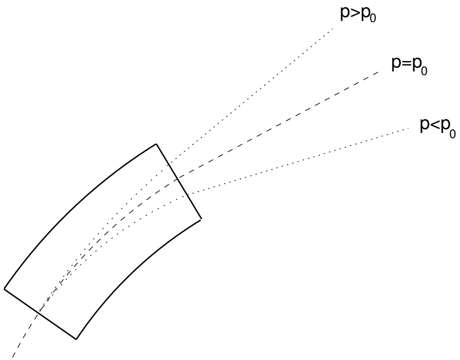
Non-periodic optics: final focus of a HEP experiment (EX2)



Introducing dispersion: $D(s)$

So far we have studied monochromatic beams of particles, but this is slightly unrealistic:
We always have some (small?) momentum spread among all particles:
$\Delta P = P - P_0 \neq 0$.

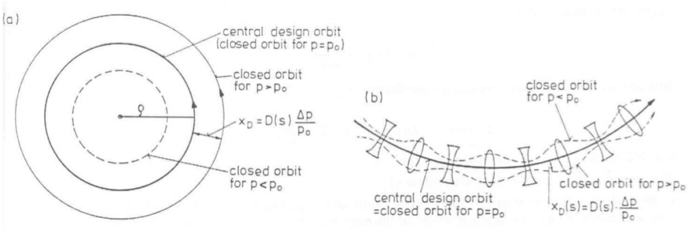
Consider three particles with P respectively: less than, greater than, and equal to P_0 , traveling through a dipole. Remembering $B\rho = \frac{P}{q}$:



The system introduces a correlation of momentum with transverse position. This correlation is known as **dispersion** (an intrinsic property of the dipole magnets).

Orbit of off-momentum particles

- in a circular particle accelerator, a particle with $P = P_0$ and $x = y = x' = y' = 0$ (i.e. zero displacement and zero slope) moves on the design orbit for an arbitrary number of revolutions
- particles with $P = P_0$ but non-zero displacement and slope perform betatron oscillations, with a certain tune Q
- what happens to particles with momentum $P \neq P_0$? they no longer move on the design orbit



Closed orbit for particles with momentum $P \neq P_0$ in a weakly (a) and strongly (b) focusing circular accelerator.

The Inhomogeneous Hill's equation

Let's go back to the magnetic rigidity. If $P \neq P_0$ (define $\delta = \frac{P-P_0}{P_0} = \frac{\Delta P}{P_0}$) we can work out how the bending radius ρ depends on the particle momentum, w.r.t. ρ_0:

$$\Rightarrow B\rho = \frac{P}{q} \quad = \frac{P_0\,(1+\delta)}{q} = B\rho_0\,(1+\delta) \qquad \Rightarrow \quad \rho = \rho_0\,(1+\delta)\,.$$

When we derived the equation of motion at some point we had (slide 15):

$$\underbrace{x''}_{\text{term 1}} - \frac{1}{\underbrace{\rho + x}_{\text{term 2}}} = -\frac{B_y}{P/q} \quad \text{that later became: } x'' + \left(\frac{1}{\rho^2} + k\right) x = 0$$

On the way we had "Taylor expanded" term 2: $\displaystyle \frac{1}{\rho + x} \approx \frac{1}{\rho}\left(1 - \frac{x}{\rho}\right).$

Now we need to redo it for ρ as $\rho_0\,(1+\delta)$: $\displaystyle \frac{1}{\rho + x} = \frac{1}{\rho_0\,(1+\delta) + x} \approx \frac{1}{\rho_0}\left(1 - \frac{x}{\rho_0} - \delta\right)$

and the equation of motion becomes:

$$x'' + \left(\frac{1}{\rho_0^2} + k\right) x - \frac{\delta}{\rho_0} = 0.$$

If we drop the suffix 0 and explicit δ, this is "the inhomogeneous Hill's equation":

$$\boxed{x'' + \left(\frac{1}{\rho^2} + k\right) x = \frac{1}{\rho}\frac{\Delta P}{P_0}}$$

Solution of the inhomogeneous Hill's equation

A particle with $\Delta P = P - P_0 \neq 0$ satisfies the inhomogeneous Hill equation for the horizontal motion:

$$x''(s) + K(s)\,x(s) = \frac{1}{\rho}\frac{\Delta P}{P_0}$$

the total deviation of the particle from the reference orbit can be written as

$$x(s) = x_\beta(s) + x_D(s)$$

where:

- $x_D(s)$ describes the deviation of the closed orbit for an off-momentum particle with $P = P_0 + \Delta P$. It is rewritten as $x_D(s) = D(s)\,\frac{\Delta P}{P_0}$, where $D(s)$ is the solution of the equation

$$D''(s) + K(s)\,D(s) = \frac{1}{\rho}$$

- $x_\beta(s)$ describes the betatron oscillation around the new closed orbit, and it's the solution of the homogeneous equation $x_\beta''(s) + K(s)\,x_\beta(s) = 0$

$D(s)$ is the *dispersion function*.

Dispersion function and orbit

The dispersion function $D(s)$ is the solution of the inhomogeneous Hill's equation:

$$D''(s) + K(s)\,D(s) = \frac{1}{\rho}$$

$D(s)$:

- is that special orbit that an ideal particle would have for $\Delta P/P_0 = 1$
- It can be proved that the solution is:

$$D(s) = S(s)\int_0^s \frac{1}{\rho(t)}C(t)\,\mathrm{d}t - C(s)\int_0^s \frac{1}{\rho(t)}S(t)\,\mathrm{d}t$$

Once one knows $D(s)$, the orbit $x(s) = x_\beta(s) + x_D(s)$, with $x_D(s) = D(s)\,\frac{\Delta P}{P_0}$, can be rewritten as

$$\begin{aligned} x(s) &= x_\beta(s) + x_D(s) \\ &= C(s)\,x_0 + S(s)\,x_0' + D(s)\,\frac{\Delta P}{P_0} \end{aligned}$$

Dispersion function and orbit

The equation of motion:

$$\begin{aligned} x(s) &= x_\beta(s) + x_D(s) \\ &= C(s)\,x_0 + S(s)\,x_0' + D(s)\,\frac{\Delta P}{P_0} \end{aligned}$$

can be written in matrix form:

$$\begin{pmatrix} x \\ x' \end{pmatrix}_s = \begin{pmatrix} C & S \\ C' & S' \end{pmatrix} \begin{pmatrix} x \\ x' \end{pmatrix}_0 + \frac{\Delta P}{P_0} \begin{pmatrix} D \\ D' \end{pmatrix}_0$$

Or, in a more compact way:

$$\begin{pmatrix} x \\ x' \\ \Delta P/P_0 \end{pmatrix}_s = \begin{pmatrix} C & S & D \\ C' & S' & D' \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ x' \\ \Delta P/P_0 \end{pmatrix}_0$$

Summary

integrated dipole field over a turn	$\int B dl \approx N L_{\text{Bend}} B = 2\pi \frac{P_0}{q}$
transfer matrix of a FODO cell	$M_{\text{FODO}} = \begin{pmatrix} 1 + \frac{L}{2f} & L + \frac{L^2}{4f} \\ -\frac{2L}{f^2} & 1 - \frac{L}{2f} - \frac{L^2}{4f^2} \end{pmatrix}$
stability in a FODO cell	$f > L/4$
phase advance in a FODO cell	$\mu = \arccos\left(\frac{1}{2}\text{trace}\left(M\right)\right)$
there exist matching sections	$\begin{pmatrix} \beta \\ \alpha \\ \gamma \end{pmatrix}_s = M_{3\times 3} \begin{pmatrix} \beta \\ \alpha \\ \gamma \end{pmatrix}_0$
inhomogeneous Hill's equation	$x'' + K\left(s\right)x = \frac{1}{\rho}\frac{\Delta P}{P_0}$
...and its solution	$x\left(s\right) = x_\beta\left(s\right) + D\left(s\right)\frac{\Delta P}{P_0}$
dispersion function	$D\left(s\right)$

Dispersion function and orbit

We need to study the motion for particles with $\Delta P = P - P_0 \neq 0$:

$$x''(s) + K(s)\,x(s) = \frac{1}{\rho}\frac{\Delta P}{P_0}$$

The general solution of this equation is:

$$x(s) = x_\beta(s) + x_D(s) \qquad \begin{cases} x_\beta''(s) + K(s)\,x_\beta(s) = 0 \\ D''(s) + K(s)\,D(s) = \dfrac{1}{\rho} \end{cases}$$

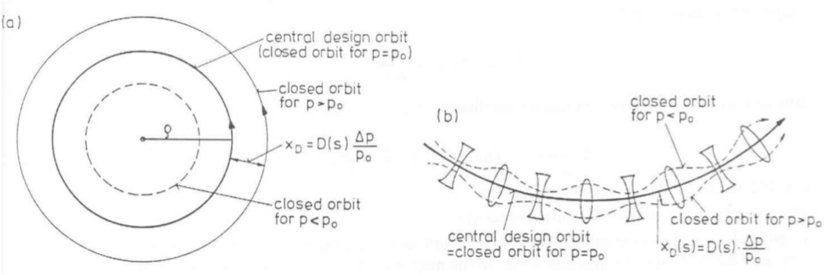
with $x_D(s) = D(s)\,\frac{\Delta P}{P_0}$.

Remarks

- $D(s)$ is that special orbit that a particle would have for $\Delta P / P_0 = 1$
- $x_D(s)$ describes the deviation from the new **closed orbit** for an off-momentum particle with a certain ΔP
- the orbit of a generic particle is the sum of the well known $x_\beta(s)$ and $x_D(s)$

Understanding the solution $x(s) = x_\beta(s) + x_D(s)$

with $x_D(s) = D(s)\,\frac{\Delta P}{P_0}$.



Closed orbit for particles with momentum $P \neq P_0$ in a weakly (a) and strongly (b) focusing circular accelerator.

- $x_D(s)$ describes the deviation from the reference orbit of an off-momentum particle with $P = P_0 + \Delta P$
- $x_\beta(s)$ describes the betatron oscillation around the orbit $x_D(s)$

Dispersion and orbit propagation

The dispersion orbit is solution of $D''(s) + K(s)\,D(s) = \frac{1}{\rho}$:

$$D(s) = S(s)\int_0^s \frac{1}{\rho(t)} C(t)\,\mathrm{d}t - C(s)\int_0^s \frac{1}{\rho(t)} S(t)\,\mathrm{d}t$$

Now the orbit:

$$x(s) = x_\beta(s) + x_D(s)$$
$$x(s) = C(s)\,x_0 + S(s)\,x'_0 + D(s)\,\frac{\Delta P}{P_0}$$

In matrix form

$$\begin{pmatrix} x \\ x' \end{pmatrix}_s = \begin{pmatrix} C & S \\ C' & S' \end{pmatrix}\begin{pmatrix} x \\ x' \end{pmatrix}_0 + \frac{\Delta P}{P_0}\begin{pmatrix} D \\ D' \end{pmatrix}_0$$

We can rewrite the solution in matrix form:

$$\begin{pmatrix} x \\ x' \\ \Delta P/P_0 \end{pmatrix}_s = \begin{pmatrix} C & S & D \\ C' & S' & D' \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ x' \\ \Delta P/P_0 \end{pmatrix}_0$$

Exercise: show that $D(s)$ is a solution for the equation of motion, with the initial conditions $D_0 = D'_0 = 0$.

Examples of dispersion function

Let's study, for different magnetic elements, the solution of:

$$D(s) = S(s)\int_0^s \frac{1}{\rho(t)} C(t)\,\mathrm{d}t - C(s)\int_0^s \frac{1}{\rho(t)} S(t)\,\mathrm{d}t$$

at the exit of the element: that is, $D(s)$ with $s = L_{\text{magnet}}$

- Drift space:

$$M_{\text{Drift}} = \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix}$$

$C(t) = 1,\ S(t) = L,\ \rho(t) = \infty \quad \Rightarrow$ the integrals cancel

$$M_{\text{Drift}} = \begin{pmatrix} 1 & L & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Dispersion function in a sector dipole

- Sector dipole:
  $K = \frac{1}{\rho^2}$:

$$M_{\text{Dipole}} = \begin{pmatrix} \cos\left(\sqrt{K}L\right) & \frac{1}{\sqrt{K}}\sin\left(\sqrt{K}L\right) \\ -\sqrt{K}\sin\left(\sqrt{K}L\right) & \cos\left(\sqrt{K}L\right) \end{pmatrix} = \begin{pmatrix} \cos\frac{L}{\rho} & \rho\sin\frac{L}{\rho} \\ -\frac{1}{\rho}\sin\frac{L}{\rho} & \cos\frac{L}{\rho} \end{pmatrix}$$

which gives

$$D(L) = \rho\left(1 - \cos\frac{L}{\rho}\right)$$

$$D'(L) = \sin\frac{L}{\rho}$$

therefore

$$M_{\text{Dipole}} = \begin{pmatrix} \cos\frac{L}{\rho} & \rho\sin\frac{L}{\rho} & \rho\left(1-\cos\frac{L}{\rho}\right) \\ -\frac{1}{\rho}\sin\frac{L}{\rho} & \cos\frac{L}{\rho} & \sin\frac{L}{\rho} \\ 0 & 0 & 1 \end{pmatrix}$$

Notice: $\frac{L}{\rho} = \phi$ is the beding angle.

Dispersion function in a quadrupole

- Focusing quadrupole, $K > 0$:

$$M_{\text{QF}} = \begin{pmatrix} \cos\left(\sqrt{K}L\right) & \frac{1}{\sqrt{K}}\sin\left(\sqrt{K}L\right) & 0 \\ -\sqrt{K}\sin\left(\sqrt{K}L\right) & \cos\left(\sqrt{K}L\right) & 0 \\ 0 & 0 & 1 \end{pmatrix};$$

- Defocusing quadrupole, $K < 0$:

$$M_{\text{QD}} = \begin{pmatrix} \cosh\left(\sqrt{|K|}L\right) & \frac{1}{\sqrt{|K|}}\sinh\left(\sqrt{|K|}L\right) & 0 \\ \sqrt{|K|}\sinh\left(\sqrt{|K|}L\right) & \cosh\left(\sqrt{|K|}L\right) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Dispersion propagation through the lattice

- The equation:

$$D(s) = S(s)\int_0^s \frac{1}{\rho(t)} C(t)\,\mathrm{d}t - C(s)\int_0^s \frac{1}{\rho(t)} S(t)\,\mathrm{d}t$$

allows to compute **the dispersion inside a magnet**, which does not depend on the dispersion that might have been generated by the upstreams magnets.

- At the exit of a magnet of length L_m the dispersion reaches the value $D(L_\mathrm{m})$
- The dispersion (also indicated as η, with its derivative η') propagates from there, through the rest of the machine, just like any other particle:

$$\begin{pmatrix} \eta \\ \eta' \\ 1 \end{pmatrix}_s = \begin{pmatrix} C & S & D \\ C' & S' & D' \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \eta \\ \eta' \\ 1 \end{pmatrix}_0$$

Periodic dispersion

In a periodic lattice, also the dispersion must be periodic.

That is, for $\begin{pmatrix} \eta \\ \eta' \\ 1 \end{pmatrix}$ we need to have:

$$\begin{pmatrix} \eta \\ \eta' \\ 1 \end{pmatrix} = \begin{pmatrix} C & S & D \\ C' & S' & D' \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \eta \\ \eta\prime \\ 1 \end{pmatrix}$$

Let's rewrite this in 2×2 form:

$$\begin{pmatrix} \eta \\ \eta' \end{pmatrix} = \begin{pmatrix} C & S \\ C' & S' \end{pmatrix} \begin{pmatrix} \eta \\ \eta' \end{pmatrix} + \begin{pmatrix} D \\ D' \end{pmatrix}$$

$$\begin{pmatrix} 1 - C & -S \\ -C' & 1 - S' \end{pmatrix} \begin{pmatrix} \eta \\ \eta' \end{pmatrix} = \begin{pmatrix} D \\ D' \end{pmatrix}$$

The solution is:

$$\begin{pmatrix} \eta \\ \eta' \end{pmatrix} = \frac{1}{(1 - C)(1 - S') - C'S} \begin{pmatrix} 1 - S' & S \\ C' & 1 - C \end{pmatrix} \begin{pmatrix} D \\ D' \end{pmatrix}$$

Dispersion function in a FODO lattice

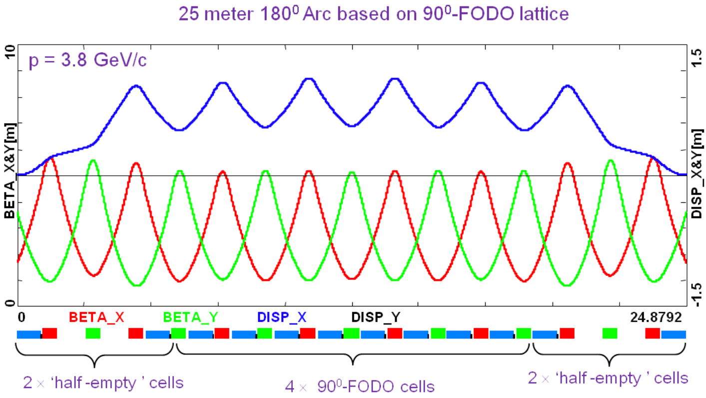
The dispersion function in a FODO cell is a periodic function with maxima at the focusing quadrupoles and minima at the defocusing quadrupoles:

$$D^{\pm} = \frac{L\phi\left(1 \pm \frac{1}{2}\sin\frac{\mu}{2}\right)}{4\sin^2\frac{\mu}{2}}$$

where:

- L is the total length of the cell
- ϕ is the total bending angle of the cell
- μ is the phase advance of the cell

Example of dispersion function in a FODO lattice



Aperture radius: r = 15 cm

12 × Dipoles:	field: 3.9 Tesla	length: 85 cm
15 × Quads:	gradient : 25 Tesla/m (3.8 Tesla at the pole)	length: 50 cm

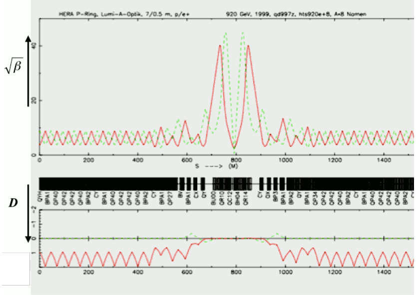
Impact of dispersion on the beam size

In this example from the HERA storage ring (DESY) we see the Twiss parameters and the dispersion near the interaction point. In the periodic region,

$$x_\beta(s) = 1 \ldots 2 \text{ mm}$$
$$D(s) = 1 \ldots 2 \text{ m}$$
$$\Delta P / P_0 \approx 1 \cdot 10^{-3}$$

Remember:

$$x(s) = x_\beta(s) + D(s)\,\frac{\Delta P}{P_0}$$



Beware: the dispersion contributes to the beam size:

$$\sigma_x = \sqrt{\sigma_{x_\beta}^2 + \text{std}\left(D \cdot \frac{\Delta P}{P_0}\right)^2} = \sqrt{\epsilon_{\text{geometric}} \cdot \beta + D^2 \cdot \frac{\sigma_P^2}{P_0^2}}$$

- We need to suppress the dispersion at the IP !
- We need a special insertion section: a *dispersion suppressor*
- Remember: $\epsilon_{\text{geometric}} = \dfrac{\epsilon_{\text{normalised}}}{\beta_{\text{rel}}\gamma_{\text{rel}}}$

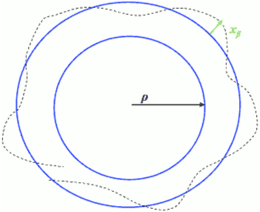
The momentum compaction factor

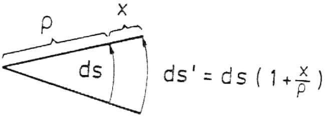
The dispersion function relates the momentum error of a particle to the horizontal orbit coordinate

The general solution of the equation of motion is

$$x(s) = x_\beta(s) + D(s)\,\frac{\Delta P}{P_0}$$

The dispersion changes also the length of the off-energy orbit.





particle with offset x w.r.t. the design orbit:

$$\frac{\mathrm{d}s'}{\mathrm{d}s} = \frac{\rho + x}{\rho} \quad \rightarrow \quad \mathrm{d}s' = \left(1 + \frac{x}{\rho}\right)\mathrm{d}s$$

The circumference change is ΔC, that is $C' = \oint \left(1 + \frac{x}{\rho}\right)\mathrm{d}s = C + \Delta C$

We define the "momentum compaction factor" α_P, such that:

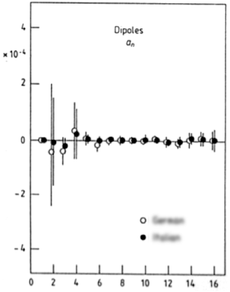
$$\frac{\Delta C}{C} = \alpha_P \frac{\Delta P}{P_0} \quad \rightarrow \text{to the lowest order in } \Delta P/P_0 : \quad \alpha_P = \frac{1}{C}\oint \frac{D(s)}{\rho}\mathrm{d}s \approx \frac{1}{Q_x^2}$$

Magnetic imperfections

High-order multipolar components and misalignments

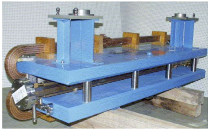
Taylor expansion of the B field:

$$B_y(x) = \underbrace{B_{y0}}_{\text{dipole}} + \underbrace{\frac{\partial B_y}{\partial x}}_{\text{quad}} x + \frac{1}{2}\underbrace{\frac{\partial^2 B_y}{\partial x^2}}_{\text{sextupole}} x^2 + \frac{1}{3!}\underbrace{\frac{\partial^3 B_y}{\partial x^3}}_{\text{octupole}} x^3 + \dots \qquad \text{divide by } B_{y0}$$



There can be undesired multipolar components, due to small fabrication defects

Or also errors in the windings, in the gap h, ... remember: $B = \dfrac{\mu_0 n I}{h}$



Moreover: **"feed-down"** effect $\Rightarrow$ a misalign magnet of order n, behaves like a magnet of order n, plus a magnet of order $n-1$ overlapped

Dipole magnet errors

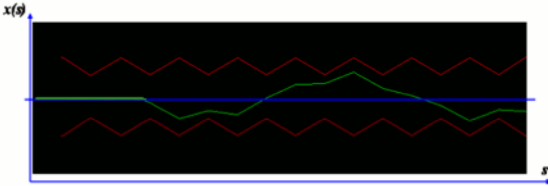
Let's imagine to have a magnet with $B = B_0 + \Delta B$. This will give an additional kick to each particle, and will distort the ideal design orbit

$$F_x = ev\left(B_0 + \Delta B\right); \qquad \Delta x' = \Delta B \mathrm{d}s / B\rho$$

A dipole error will cause a distortion of the closed orbit, that will „run around" the storage ring, being observable everywhere. If the distortion is small enough, it will still lead to a closed orbit.

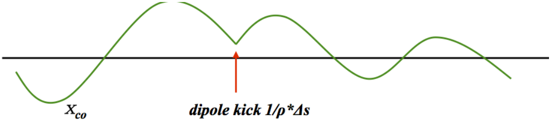
Example: 1 single dipole error

$$\begin{pmatrix} x \\ x' \end{pmatrix}_s = M_{\text{lattice}} \begin{pmatrix} 0 \\ \Delta x' \end{pmatrix}_0$$



In order to have bounded motion the tune Q must be non-integer, $Q \neq 1$. We see that even for particles with reference momentum P_0 an integer Q value is forbidden, since small field errors are always present.

Orbit distortion for a single dipole field error



We consider a single thin dipole field error at the location $s = s_0$, with a kick angle $\Delta x'$.

$$X_- = \begin{pmatrix} x_0 \\ x_0' + \Delta x' \end{pmatrix}, \quad X_+ = \begin{pmatrix} x_0 \\ x_0' \end{pmatrix}$$

are the phase space coordinates before and after the kick located at s_0. The closed-orbit condition becomes

$$M_{\text{Lattice}} \begin{pmatrix} x_0 \\ x_0' \end{pmatrix} = \begin{pmatrix} x_0 \\ x_0' + \Delta x' \end{pmatrix}$$

The resulting closed orbit at s_0 is

$$x_0 = \frac{\beta_0 \Delta x'}{2 \sin \pi Q} \cos \pi Q; \quad x_0' = \frac{\Delta x'}{2 \sin \pi Q} \left(\sin \pi Q - \alpha_0 \cos \pi Q \right)$$

where Q is the tune. The orbit at any other location s is

$$x(s) = \frac{\sqrt{\beta_s \beta_0} \Delta x'}{2 \sin \pi Q} \cos \left(\pi Q - |\mu_s - \mu_0| \right)$$

(see the references for a demonstration)

Orbit distortion for distributed dipole field errors

One single dipole field error

$$x(s) = \frac{\sqrt{\beta_s \beta_0} \Delta x'}{2 \sin \pi Q} \cos\left(\pi Q - |\mu_s - \mu_0|\right)$$

Distributed dipole field errors

$$x(s) = \frac{\sqrt{\beta_s}}{2 \sin \pi Q} \sum_i \sqrt{\beta_i} \Delta x'_i \cos\left(\pi Q - |\mu_s - \mu_i|\right)$$

- orbit distortion is visible at any position s in the ring, even if the dipole error is located at one single point s_0
- the β function describes the sensitivity of the beam to external fields
- the β function acts as amplification factor for the orbit amplitude at the given observation point
- there is a singularity at the denominator when Q integer $\Rightarrow$ it's called resonance

Quadrupole errors: tune shift

Orbit perturbation described by a thin lens quadrupole:

$$M_{\text{Perturbed}} = \underbrace{\begin{pmatrix} 1 & 0 \\ \Delta k \mathrm{d}s & 1 \end{pmatrix}}_{\text{perturbation}} \underbrace{\begin{pmatrix} \cos\mu_0 + \alpha\sin\mu_0 & \beta\sin\mu_0 \\ -\gamma\sin\mu_0 & \cos\mu_0 - \alpha\sin\mu_0 \end{pmatrix}}_{\text{ideal ring}}$$

Let's see how the tunes changes: one-turn map

$$M_{\text{Perturbed}} = \begin{pmatrix} \cos\mu_0 + \alpha\sin\mu_0 & \beta\sin\mu_0 \\ \Delta k \mathrm{d}s \left(\cos\mu_0 + \alpha\sin\mu_0\right) - \gamma\sin\mu_0 & \Delta k \mathrm{d}s \beta\sin\mu_0 + \cos\mu_0 - \alpha\sin\mu_0 \end{pmatrix}$$

Remember the rule for computing the tune:

$$2\cos\mu = \mathrm{trace}\left(M\right) = 2\cos\mu_0 + \Delta k \mathrm{d}s \beta\sin\mu_0$$

Quadrupole errors: tune shift (cont.)

We rewrite $\cos\mu = \cos(\mu_0 + \Delta\mu)$

$$\cos(\mu_0 + \Delta\mu) = \cos\mu_0 + \frac{1}{2}\Delta k \mathrm{d}s \beta \sin\mu_0$$

from which we can compute that

$$\Delta\mu = \frac{\Delta k \, \mathrm{d}s \, \beta}{2} \quad \text{shift in the phase advance}$$

$$\Delta Q = \oint_{\text{quads}} \frac{\Delta k(s) \, \beta(s) \, \mathrm{d}s}{4\pi} \quad \text{tune shift}$$

Important remarks:

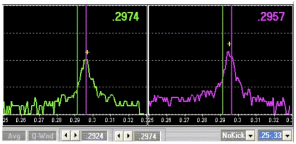
- the tune shift if proportional to the β-function at the location of the quadrupole
  - field quality, power supply tolerances etc. are much tighter at places where β is large
- β is a measurement of the sensitivity of the beam

Quadrupole errors: tune shift example

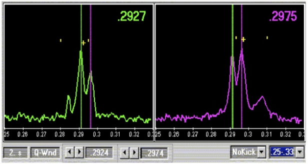
Deliberate change of a quadrupole strength in a synchrotron:

$$\Delta Q = \oint_{\text{quads}} \frac{\Delta K(s)\,\beta(s)\,\mathrm{d}s}{4\pi} \approx \frac{\Delta K(s)\; L_{\text{quad}}\,\overline{\beta}}{4\pi}$$

$\Rightarrow$



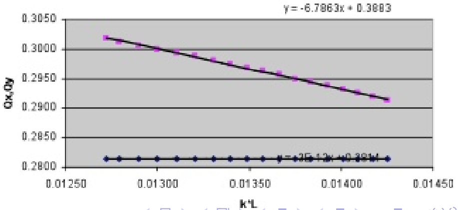
The tune is measured permanently



We change the strength of "trim" quads to fix Q

Horizontal axis is a scan of K_1 (quad integrated focusing strength):

- tune shift is proportional to β through $\Delta Q \propto \Delta K \cdot \beta$
- En passant, we use this to measure β.



Tune shift correction

Errors in the quadrupole fields induce tune shift:

$$\Delta Q = \oint_{\text{quads}} \frac{\Delta k(s)\,\beta(s)\,ds}{4\pi}$$

Cure: we compensate the quad errors using other (correcting) quadrupoles

- If you use only one correcting quadrupole, with $1/f = \Delta k_1 L$
  - it changes both Q_x and Q_y:

$$\Delta Q_x = \frac{\beta_{1x}}{4\pi f_1} \quad \text{and} \quad \Delta Q_y = -\frac{\beta_{1y}}{4\pi f_1}$$

- We need to use two independent correcting quadrupoles:

$$\Delta Q_x = \frac{\beta_{1x}}{4\pi f_1} + \frac{\beta_{2x}}{4\pi f_2}$$
$$\Delta Q_y = -\frac{\beta_{1y}}{4\pi f_1} - \frac{\beta_{2y}}{4\pi f_2}$$

$$\begin{pmatrix} \Delta Q_x \\ \Delta Q_y \end{pmatrix} = \frac{1}{4\pi} \begin{pmatrix} \beta_{1x} & \beta_{2x} \\ \beta_{1y} & \beta_{2y} \end{pmatrix} \begin{pmatrix} 1/f_1 \\ 1/f_2 \end{pmatrix}$$

- Solve by inversion:

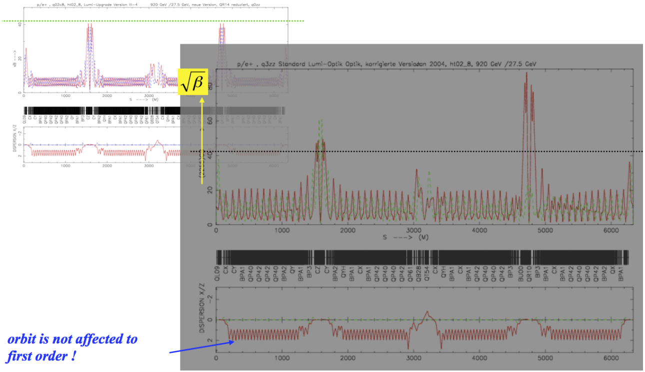
$$\begin{pmatrix} 1/f_1 \\ 1/f_2 \end{pmatrix} = \frac{4\pi}{\beta_{1x}\beta_{2y} - \beta_{2x}\beta_{1y}} \begin{pmatrix} \beta_{2y} & -\beta_{2x} \\ -\beta_{1y} & \beta_{1x} \end{pmatrix} \begin{pmatrix} \Delta Q_x \\ \Delta Q_y \end{pmatrix}$$

Quadrupole errors: beta beat

A quadrupole error at s_0 causes distortion of β-function at s: $\Delta\beta(s)$ due to the errors of all quadrupoles:

$$\frac{\Delta\beta_s}{\beta_s} = \frac{1}{2\sin 2\pi Q} \sum_i \beta_i \Delta k_i \cos\left(2\pi Q - 2\left(\mu_i - \mu_s\right)\right)$$

Note: Unstable betatron motion if tune is half integer!

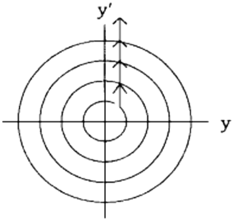


This imperfection can be corrected with an appropriate distribution of tuneable sextupoles.

Tunes and resonances

The particles – oscillating under the influence of the external magnetic fields – can be excited in case of resonant tunes to infinite high amplitudes.

There is particle loss within a short number of turns.



ν_y = integer

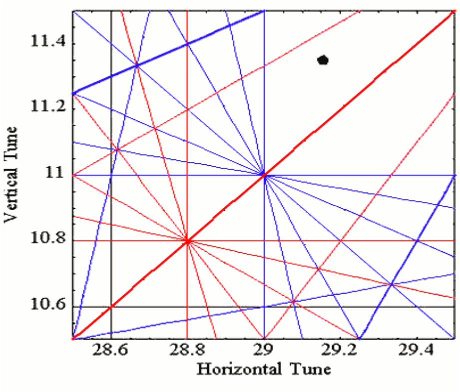
The cure:

1. **avoid** large magnet errors
2. **avoid forbidden tune values** in both planes

$$\mathrm{m}\cdot Q_x + \mathrm{n}\cdot Q_y \neq \mathrm{p}$$

with m, n, p integer numbers

Resonance diagram

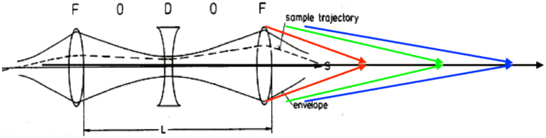


$m \cdot Q_x + n \cdot Q_y \neq p$ where $|m| + |n|$ is the order of the resonance

A resonance diagram for the Diamond light source. The lines shown are the resonances and the black dot shows a suitable place where the machine could be operated.

Quadrupole errors: chromaticity, ξ

Is an error (optical aberration) that happens in quadrupoles when $\Delta P/P_0 \neq 0$:



The chromaticity ξ is the variation of tune ΔQ with the relative momentum error:

$$\Delta Q = \xi \frac{\Delta P}{P_0} \quad \Rightarrow \quad \xi = \frac{\Delta Q}{\Delta P/P_0}$$

Remember the quadrupole strength:

$$k = \frac{g}{P/q} \qquad \text{with } P = P_0 + \Delta P = P_0\,(1+\delta)$$

then

$$k = \frac{qg}{P_0 + \Delta P} = \frac{k_0}{1+\delta} \approx \frac{q}{P_0}\left(1 - \frac{\Delta P}{P_0}\right) g = k_0 + \Delta k$$

$$\Delta k = -\frac{\Delta P}{P_0} k_0$$

Quadrupole errors: chromaticity (cont.)

$$\Delta k = -\frac{\Delta P}{P_0} k_0$$

$\Rightarrow$ Chromaticity acts like a quadrupole error and leads to a *tune spread:*

$$\Delta Q_{\text{one quad}} = -\frac{1}{4\pi}\frac{\Delta P}{P_0} k_0 \beta(s)\,\mathrm{d}s \qquad \Rightarrow \Delta Q_{\text{all quads}} = -\frac{1}{4\pi}\frac{\Delta P}{P_0}\oint k(s)\,\beta(s)\,\mathrm{d}s$$

Therefore the definition of chromaticity ξ is

$$\xi = -\frac{1}{4\pi}\oint_{\text{quads}} k(s)\,\beta(s)\,\mathrm{d}s$$

The peculiarity of chromaticity is that it isn't due to external agents, it is generated by the lattice itself!

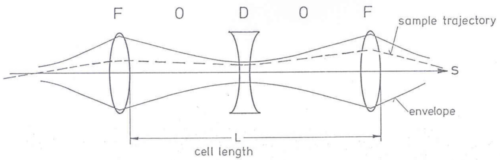
Remarks:

- ξ is a number indicating the size of the tune spot in the working diagram
- ξ is always created by the focusing strength k of **all** quadrupoles
- **natural chromaticity is always negative**

In other words, because of chromaticity the tune is not a sharp point, but is a **spot**

Example: Chromaticity of the FODO cell

Consider a FODO cells like in figure, with two thin quads, each with focal length f, separated by length $L/2$, and total phase advance μ:



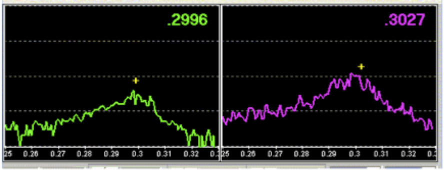
The natural chromaticity ξ_N of the cell is:

$$\xi_N = -\frac{1}{4\pi}\oint \beta(s)k(s)ds = -\frac{1}{4\pi}\int_{\text{cell}} \beta(s)\underbrace{k(s)ds}_{\frac{1}{f}} = -\frac{1}{4\pi}\left[\frac{\beta^+}{f} - \frac{\beta^-}{f}\right]$$

$$= -\frac{1}{4\pi\sin\mu}\left[\left(L+\frac{L^2}{4f}\right)\frac{1}{f} - \left(L-\frac{L^2}{4f}\right)\frac{1}{f}\right] = -\frac{1}{4\pi\sin\mu}\left[\frac{L}{f} - \frac{L}{f} + \frac{L^2}{2f^2}\right] = -\frac{1}{8\pi\sin\mu}\frac{L^2}{f^2} \simeq -\frac{1}{\pi}\tan\frac{\mu}{2}$$

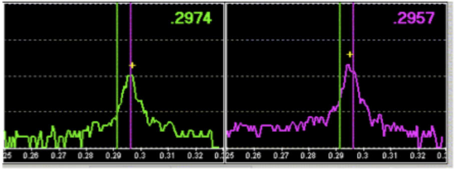
For N_{cell} cells, the total chromaticity is N_{cell} times the chromaticity of each cell $\xi_{N,N_{\text{cell}}} = -\frac{N_{\text{cell}}}{\pi}\tan\frac{\mu}{2}$.

Quadrupole errors: chromaticity



Tune signal for a nearly uncompensated cromaticity (Q' ≈ 20)

Ideal situation: cromaticity well corrected, (Q' ≈ 1)

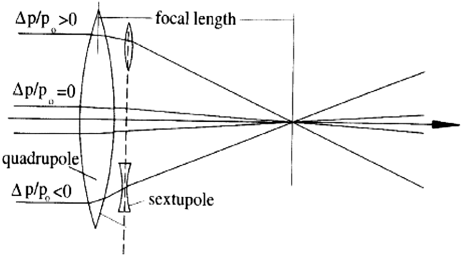


Chromaticity correction

Remember what is chromaticity: the quadrupole focusing experienced by particles changes with energy

- it induces tune shift, which can cause beam lifetime reduction due to resonances

Cure: we need additional, energy-dependent, focusing. This is given by sextupoles



- The sextupole magnetic field rises quadratically:

$$B_x = \tilde{g}xy$$
$$B_y = \frac{1}{2}\tilde{g}\left(x^2 - y^2\right) \qquad \Rightarrow \frac{\partial B_x}{\partial y} = \frac{\partial B_y}{\partial x} = \tilde{g}x \quad \text{a "gradient"}$$

it provides a linearly increasing quadrupole gradient

Chromaticity correction (cont.)

Now remember:

- Normalised quadrupole strength is

$$k = \frac{g}{P/q} \; [\mathrm{m}^{-2}]$$

- Sextupoles are characterised by a normalised sextupole strength k_2, which carries a focusing quadrupolar component k_1:

$$k_2 = \frac{\tilde{g}}{P/q} \; [\mathrm{m}^{-3}]; \qquad \tilde{k}_1 = \frac{\tilde{g}x}{P/q} \; [\mathrm{m}^{-2}]$$

Cure for chromaticity: we need sextupole magnets installed in the storage ring in order to increase the focusing strength for particles with larger energy

- A sextupole at a location with dispersion does the trick: $x = D \cdot \frac{\Delta P}{P_0}$

$$\tilde{k}_1 = \frac{\tilde{g}\left(D\frac{\Delta P}{P_0}\right)}{P/q}\ [\mathrm{m}^{-2}]$$

- for $x = 0$ it corresponds to an energy-dependent focal length

$$\frac{1}{f_{\mathrm{sext}}} = \tilde{k}_1 L_{\mathrm{sext}} = \overbrace{\underbrace{\frac{\tilde{g}}{P/q}}_{k_2} \underbrace{D\frac{\Delta P}{P_0}}_{[\mathrm{m}]}}^{\tilde{k}_1} \cdot L_{\mathrm{sext}} = k_2 D \cdot \frac{\Delta P}{P_0} \cdot L_{\mathrm{sext}}$$

Now the formula for the chromaticity rewrites:

$$\xi = \underbrace{-\frac{1}{4\pi}\oint k(s)\,\beta(s)\,\mathrm{d}s}_{\text{chromaticity due to quadrupoles}} + \underbrace{\frac{1}{4\pi}\oint k_2(s)\,D\beta(s)\,\mathrm{d}s}_{\text{chromaticity due to sextupoles}}$$

Design rules for sextupole scheme

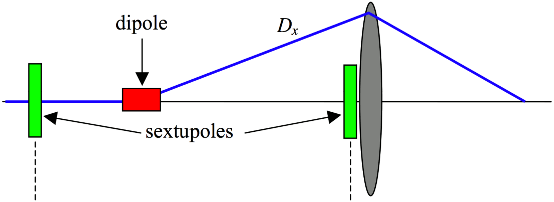
- Chromatic aberrations must be corrected in both planes ⇒ you need at least two sextupoles, S_F and S_D (sextupole strengths)
- In each plane the sextupole fields contribute with different signs to the chromaticity ξ_x and ξ_y:

$$\xi_x = -\frac{1}{4\pi}\oint \beta_x(s)\left[\ \ k(s) - S_F D_x(s) + S_D D_x(s)\right] \mathrm{d}s$$
$$\xi_y = -\frac{1}{4\pi}\oint \beta_y(s)\left[-k(s) + S_F D_x(s) - S_D D_x(s)\right] \mathrm{d}s$$

- To minimise chromatic sextupoles strengths, sextupoles should be located near quadrupoles where $\beta_x D_x$ and $\beta_y D_x$ are large
- For optimal independent chromatic correction S_F should be located where the ratio β_x/β_y is large, S_D where β_y/β_x is large.

Example of chromaticity correction scheme

- *Chromatic aberrations* introduced by a quadrupole are locally cancelled by a sextupole, placed near the quadrupole itself in a dispersive region (in straght sections dispersion is generated using an upstream bending magnet)
- Notice that the sextupoles affect also the on-momentum particles: they intriduce *geometric aberrations*. These can be cancelled by adding one additional sextupole at $\Delta\mu = \pi$



The phase advance between the two sextupoles S_1 and S_2 must be π, so that:

$$\begin{pmatrix} x \\ x' \end{pmatrix}_{s_1} \rightarrow \underbrace{M = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}}_{s_1 \rightarrow s_2} \rightarrow \begin{pmatrix} -x \\ -x' \end{pmatrix}_{s_2}$$

with $\Delta\mu = \pi \Updownarrow M$

Summary of imperfections

Error	Effect	Cure
fabrication imperfections	unwanted multipolar components	better fabrication / multipolar corrector coils
transverse offsets	"feed-down" effect	better alignment / corrector kickers
roll effects	*couplings* $x - y$	*skew quads*
dipole kicks along the ring	orbit distortion $\propto \beta_{\text{kick location}}$, residual dispersion	corrector kickers
quad field errors	tune shift	trim special quadrupoles
chromaticity	tune spread	design / sextupoles
power supplies	closed orbit distortion tune shift / spread	try to correct / improve power supplies

Summary

orbit for an off-momentum particle $x(s) = x_\beta(s) + D(s)\frac{\Delta P}{P_0}$

dispersion trajectory $D(s) = S(s)\int_0^s \frac{1}{\rho(t)} C(t)\,\mathrm{d}t - C(s)\int_0^s \frac{1}{\rho(t)} S(t)\,\mathrm{d}t$

equations of motion with dispersion

$$\begin{pmatrix} x \\ x' \\ \Delta P/P_0 \end{pmatrix}_s = \begin{pmatrix} C & S & D \\ C' & S' & D' \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ x' \\ \Delta P/P_0 \end{pmatrix}_0$$

definition of momentum compaction, α_P $\frac{\Delta C}{C} = \alpha_P \frac{\Delta P}{P_0}$

stability condition $m \cdot Q_x + n \cdot Q_y \neq p$ with n, m, p integers

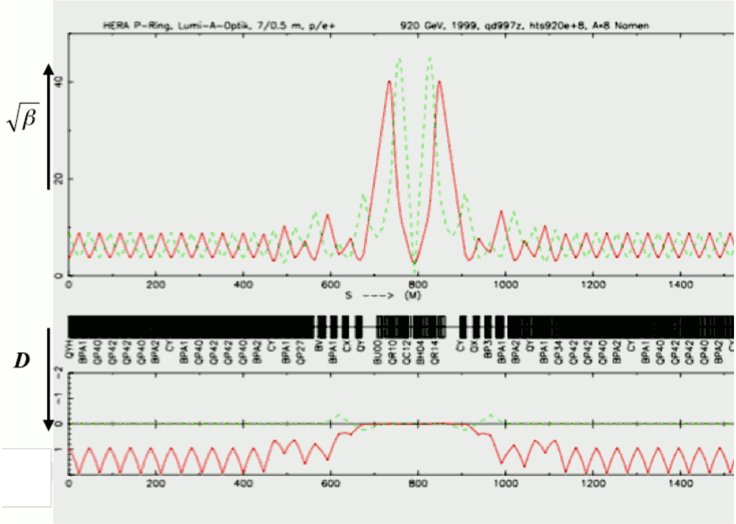
tune shift $\Delta Q = \frac{1}{4\pi}\oint_{\text{quads}} \Delta k(s)\,\beta(s)\,\mathrm{d}s$

beta beat

$$\frac{\Delta\beta(s)}{\beta(s)} = \frac{1}{2\sin 2\pi Q} \cdot \oint \beta(t)\,\Delta k(t)\cos(2\pi Q - 2(\mu(t) - \mu(s)))$$

chromaticity $\xi = \frac{\Delta Q}{\Delta P/P_0} = -\frac{1}{4\pi}\oint_{\text{quads}} k(s)\,\beta(s)\,\mathrm{d}s$

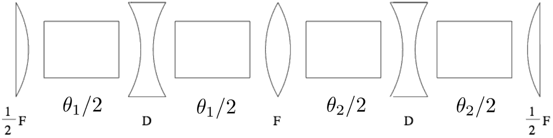
Insertions


HERA P-Ring, Lumi-A-Optik, 7/0.5 m, p/e+
920 GeV, 1999, qd997z, hts920e+8, A+8 Nomen
$\sqrt{\beta}$
S ---> (M)
D

Dispersion suppressor

In an arc, the FODO dispersion is non-zero everywhere. However, in straight sections, we often want to have $\eta = \eta' = 0$. $\Rightarrow$ for instance to keep small the beam size at the interaction point.

We can "match" between these two conditions with a "dispersion suppressor": a non-periodic set of magnets that transforms FODO η, η' to zero



Consider two FODO cells with length L and different total bend angles: θ_1, θ_2: we want to have

$$\begin{pmatrix} \eta \\ \eta' \end{pmatrix}_{\text{entrance}} \equiv \begin{pmatrix} \eta_0 \\ 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \eta \\ \eta' \end{pmatrix}_{\text{exit}} \equiv \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Note:

- the two cells have the same quadrupole strengths, so that they have also the same β, and μ (phase advance per cell)
- remember that $\alpha = 0$ at both ends, and that, if the incoming beam comes from a FODO cell with the same length L, phase advance μ, and with a total bending angle θ, then the initial dispersion is

$$\eta_0 = \eta^+_{\text{FODO}}$$

$\eta^+_{\text{FODO}} \approx \frac{4f^2}{L}\left(1 + \frac{L}{8f}\right)\theta$, in thin-lens approximation

Dispersion suppressor (cont.)

Transport for the dispersion:

$$\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} C & S & D \\ C' & S' & D' \\ 0 & 0 & 1 \end{pmatrix}_{suppressor} \begin{pmatrix} \eta_0 \\ 0 \\ 1 \end{pmatrix}$$

In 2×2 form reads

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} C & S \\ C' & S' \end{pmatrix} \begin{pmatrix} \eta_0 \\ 0 \end{pmatrix} + \begin{pmatrix} D \\ D' \end{pmatrix}$$

which has solution

$$\begin{pmatrix} D \\ D' \end{pmatrix} = - \begin{pmatrix} C & S \\ C' & S' \end{pmatrix} \begin{pmatrix} \eta_0 \\ 0 \end{pmatrix}$$

The transfer matrix for the suppressor is

$$M_{\text{suppressor}} = M_{\text{FODO 2}} \cdot M_{\text{FODO 1}}$$

For each FODO cell, $M_{\text{FODO}} = M_{1/2\text{F}} \cdot M_{\text{dipole}} \cdot M_{\text{D}} \cdot M_{\text{dipole}} \cdot M_{1/2\text{F}}$, in thin-lens approximation:

$$M_{FODO\ j} = \begin{pmatrix} 1 - \frac{L^2}{8f^2} & L\left(1 + \frac{l}{4f}\right) & \frac{L}{2}\left(1 + \frac{L}{8f}\right)\theta_j \\ -\frac{L}{4f^2}\left(1 - \frac{L}{4f}\right) & 1 - \frac{L^2}{8f^2} & \left(1 - \frac{L}{8f} - \frac{L^2}{32f^2}\right)\theta_j \\ 0 & 0 & 1 \end{pmatrix}$$

where $j = 1, 2$ (1=first cell, 2=second cell)

Dispersion suppressor (cont.)

If we do the math, we find the expressions that we have to set to zero:

$$\begin{cases} D(s) = \frac{L}{2}\left(1+\frac{L}{8f}\right)\left[\left(3-\frac{L^2}{4f^2}\right)\theta_1+\theta_2\right] \\ D'(s) = \left(1-\frac{L}{8f}-\frac{L^2}{32f^2}\right)\left[\left(1-\frac{L^2}{4f^2}\right)\theta_1+\theta_2\right] \end{cases}$$

From lecture 3, we remember that the phase advance μ for a FODO cell, in terms of the length L and the focal length f, is

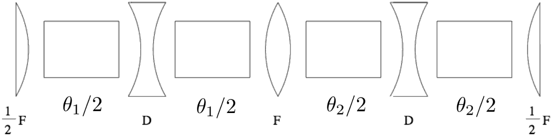
$$\left|\sin\frac{\mu}{2}\right| = \frac{L}{4f}$$

Thus, one can write the solution as a function of the phase advance μ, and of $\theta = \theta_1 + \theta_2$:

$$\begin{cases} \theta_1 = \left(1-\frac{1}{4\sin^2\frac{\mu}{2}}\right)\theta \\ \theta_2 = \frac{1}{4\sin^2\frac{\mu}{2}}\theta \end{cases}$$

Dispersion suppressor (summary)

Dispersion suppressor, a non-periodic set of magnets that transforms FODO η, η' to zero:



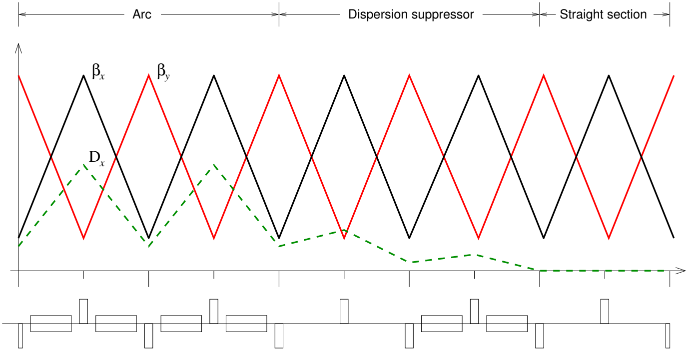
One possibility: two FODO cells with length L, phase advance μ, and different total bend angles: θ_1, θ_2:

$$\begin{cases} \theta_1 = \left(1 - \dfrac{1}{4\sin^2\frac{\mu}{2}}\right)\theta \\ \theta_2 = \dfrac{1}{4\sin^2\frac{\mu}{2}}\theta \end{cases}$$

An interesting solution is for $\mu = 60^\circ$: in this case

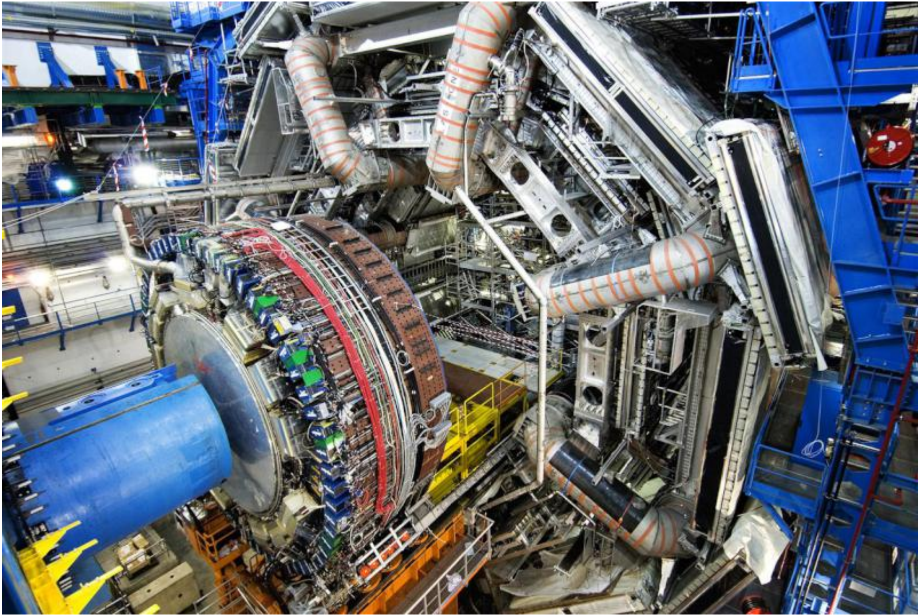
- then $\theta_1 = 0$, and $\theta_2 = \theta \Rightarrow$ we just leave out two dipole magnets in the first FODO cell insertion
- this is called the "missing-magnet" scheme

Optics functions in the dispersion suppressor, with $\mu = 60^\circ$



This is the "missing-magnet" scheme.

Often the insertions are **bigger** than few meters...



The most problematic insertion: the drift space

The most problematic insertion is the drift space !

Let's see what happens to the Twiss parameters α, β, and γ if we stop focusing for a while

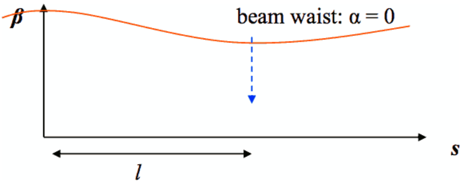
$$\left(\begin{array}{c}\beta\\ \alpha\\ \gamma\end{array}\right)_s = \left(\begin{array}{ccc} C^2 & -2SC & S^2 \\ -CC' & SC'+S'C & -SS' \\ C'^2 & -2S'C' & S'^2 \end{array}\right)\left(\begin{array}{c}\beta\\ \alpha\\ \gamma\end{array}\right)_0$$

for a drift:

$$M_{\text{drift}} = \left(\begin{array}{cc} C & S \\ C' & S' \end{array}\right) = \left(\begin{array}{cc} 1 & s \\ 0 & 1 \end{array}\right) \quad \Rightarrow \begin{cases} \beta(s) = \beta_0 - 2\alpha_0 s + \gamma_0 s^2 \\ \alpha(s) = \alpha_0 - \gamma_0 s \\ \gamma(s) = \gamma_0 \end{cases}$$

Let's find the location of the waist: $\alpha = 0$

- the location of the point of smallest beam size, $\beta^\star$



Beam waist:

$$\alpha(s) = \alpha_0 - \gamma_0 s = 0 \qquad \rightarrow \qquad s = \frac{\alpha_0}{\gamma_0} = l_{\text{waist}}$$

Beam size at that point

$$\left.\begin{array}{l} \gamma(l) = \gamma_0 \\ \alpha(l) = 0 \end{array}\right\} \qquad \rightarrow \gamma(l) = \frac{1 + \alpha^2(l)}{\beta(l)} = \frac{1}{\beta(l)} \qquad \rightarrow \beta_{\text{min}} = \frac{1}{\gamma_0}$$

This beta, at $l = l_{\text{waist}}$, is also called "beta star":

$$\Rightarrow \beta^\star = \beta_{\text{min}}$$

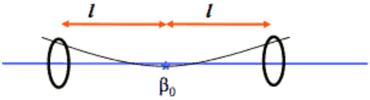
It's at $l = l_{\text{waist}}$ that the interaction point (IP) is located.

A drift space with $L = l_{\text{waist}}$: the Low β-insertion

We can assume we have a symmetry point at a distance l_{waist}:

$$\beta(s) = \beta_0 - 2\alpha_0 s + \gamma_0 s^2, \text{ at } \alpha(s) = 0 \quad \rightarrow \beta^\star = \frac{1}{\gamma_0}$$

On each side of the symmetry point

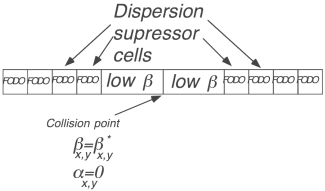


we have

$$\beta(s) = \beta^\star + \frac{s^2}{\beta^\star}$$

$\Rightarrow \beta$ grows quadratically with s.

A drift space at the interaction point, with length $L = l_{\text{waist}}$, is called "low-β insertion":



Phase advance in a low-β insertion

We have:

$$\beta(s) = \beta^{\star} + \frac{s^2}{\beta^{\star}}$$

The phase advance across the straight section is:

$$\Delta\mu = \int_{-L_{\text{waist}}}^{L_{\text{waist}}} \frac{\mathrm{d}s}{\beta^{\star} + \frac{s^2}{\beta^{\star}}} = 2\arctan\frac{L_{\text{waist}}}{\beta^{\star}}$$

which is close to $\Delta\mu = \pi$ for $L_{\text{waist}} \gg \beta^{\star}$.

In other words: in the interaction region the tune increases by half an integer!

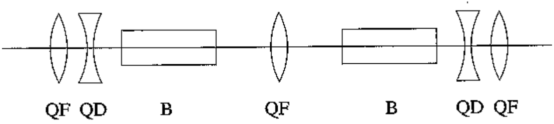
Achromatic insertions

There exist insertions (arcs) that don't introduce dispersion: they are called *achromatic arcs*

- In principle, dispersion can be suppressed by one focusing quadrupole and one bending magnet
- With one focusing quad in between two dipoles, one can get achromat condition: In between two bends, we call it arc section. Outside the arc section, we can match dispersion to zero. This is called "Double Bend Achromat" (DBA) structure
- We need quads outside the arc section to match the betatron functions, tunes, etc.
- Similarly, one can design "Triple Bend Achromat" (TBA), "Quadruple Bend Achromat" (QBA), and "Multi Bend Achromat" (MBA or nBA) structure
- For FODO cells structure, dispersion suppression section at both ends of the standard cells (see previous slides)

The Double Bend Achromat lattice (DBA)

Consider a simple DBA cell with a single quadrupole in the middle (plus external quadrupoles for matching).



$$M_{\mathrm{DBA}} = M_{\mathrm{B}} \cdot M_{\mathrm{drift}} \cdot \underbrace{M_{1/2\mathrm{F}} \cdot M_{1/2\mathrm{F}}}_{M_{\mathrm{F}}} \cdot M_{\mathrm{drift}} \cdot M_{\mathrm{B}}$$

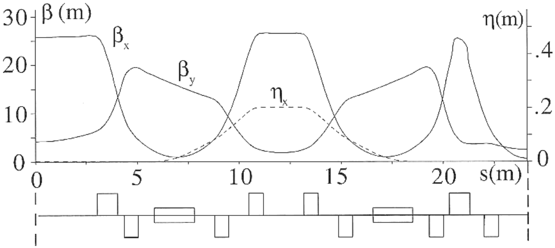
In thin-lens approximation, the dispersion matching condition:

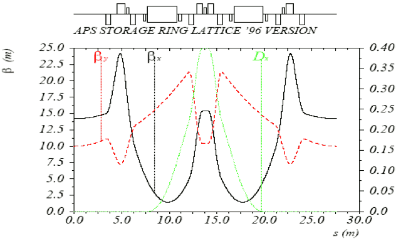
$$\begin{pmatrix} D_{\mathrm{center}} \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ -\frac{1}{2f} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & L_1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & L & L\theta/2 \\ 0 & 1 & \theta \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

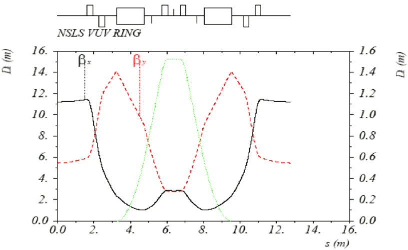
where f is the focal length of the quad, θ and L are the bend angle and the length of the dipole, and L_1 is the distance between the dipole and the centre of the quad.

$$f = \frac{1}{2}\left(L_1 + \frac{1}{2}L\right); \qquad D_{\mathrm{center}} = \left(L_1 + \frac{1}{2}L\right)\theta$$

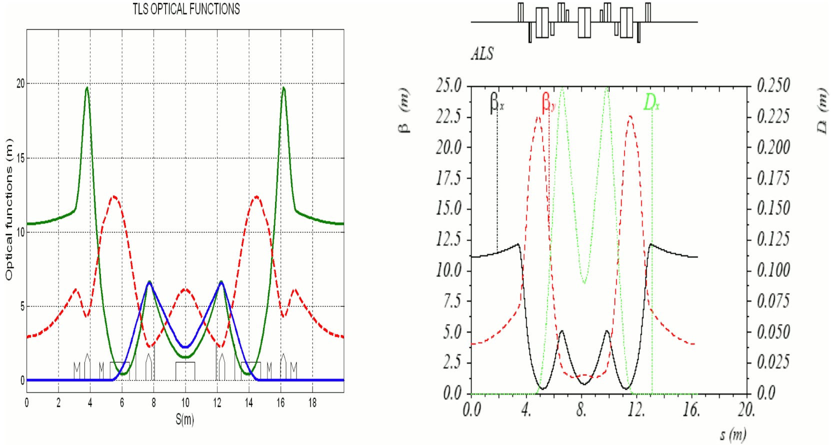
DBA optical functions





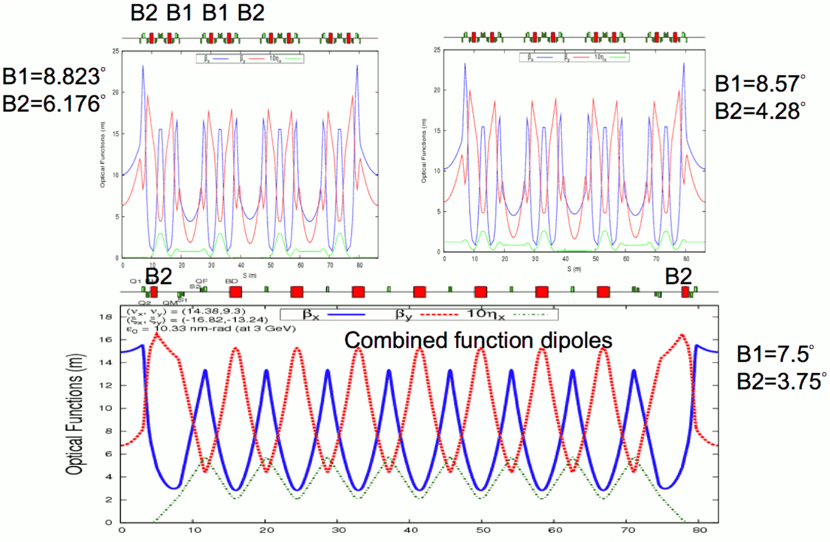


Triple Bend Achromat (TBA)



Combined function dipoles

QBA, OBA, and nBA



Completing the picture: 6-D phase space

In the real life the state vector is six-dimensional:

$$\begin{pmatrix} x & x' & y & y' & z & \Delta P/P_0 \end{pmatrix}^T$$

and the transfer matrix is typically

$$\begin{pmatrix} x \\ x' \\ y \\ y' \\ z \\ \frac{\Delta P}{P_0} \end{pmatrix}_s = \begin{pmatrix} R_{11} & R_{12} & \mathbf{0} & \mathbf{0} & 0 & R_{16} \\ R_{21} & R_{22} & \mathbf{0} & \mathbf{0} & 0 & R_{26} \\ \mathbf{0} & \mathbf{0} & R_{33} & R_{34} & 0 & 0 \\ \mathbf{0} & \mathbf{0} & R_{43} & R_{44} & 0 & 0 \\ R_{51} & R_{52} & 0 & 0 & 1 & R_{56} \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ x' \\ y \\ y' \\ z \\ \frac{\Delta P}{P_0} \end{pmatrix}_0$$

in bold the elements that would couple the $x - y$ motion.

Nota bene: this matrix can still represent **only** linear elements.

- if we want to consider high-order elements: e.g. sextupoles, octupoles, etc. $\Rightarrow$ we need computer simulations ! "particle tracking" or "maps" (MAD-X, for instance)
- because such elements introduce non-linear motion, which is too difficult to treat analytically

Coupled motion

Certain elements might be used to intentionally couple horizontal and vertical motions, for example: skew quadrupoles, and solenoids:

$$
\begin{aligned}
M_{\text{skew quad}} &= R_{\text{rot}}(\phi) \times M_{\text{quad}} \times R_{\text{rot}}(-\phi) = \\
&= \begin{pmatrix} \cos\phi & 0 & \sin\phi & 0 \\ 0 & \cos\phi & 0 & \sin\phi \\ -\sin\phi & 0 & \cos\phi & 0 \\ 0 & -\sin\phi & 0 & \cos\phi \end{pmatrix} \times \\
&\times \begin{pmatrix} \cos\sqrt{K}L & \frac{1}{\sqrt{K}}\sin\sqrt{K}L & 0 & 0 \\ -\sqrt{K}\sin\sqrt{K}L & \cos\sqrt{K}L & 0 & 0 \\ 0 & 0 & \cosh\sqrt{|K|}L & \frac{1}{\sqrt{|K|}}\sinh\sqrt{|K|}L \\ 0 & 0 & \sqrt{|K|}\sinh\sqrt{|K|}L & \cosh\sqrt{|K|}L \end{pmatrix} \times \\
&\times \begin{pmatrix} \cos\phi & 0 & -\sin\phi & 0 \\ 0 & \cos\phi & 0 & -\sin\phi \\ \sin\phi & 0 & \cos\phi & 0 \\ 0 & \sin\phi & 0 & \cos\phi \end{pmatrix}
\end{aligned}
$$

(typically $\phi = 45^\circ$)

Notice: coupling can be induced even by normal elements, including quadrupoles and dipoles, just because of alignment errors ("roll error", i.e. small angles about the optical axis).

Coupled motion: solenoid magnets

Solenoids are magnets with only $B_z \neq 0$. Their transfer matrix reads

$$M_{\text{solenoid}} = \begin{pmatrix} C^2 & \frac{SC}{K} & SC & \frac{S^2}{K} \\ -KSC & C^2 & -KS^2 & SC \\ -SC & -\frac{S^2}{K} & C^2 & \frac{SC}{K} \\ KS^2 & -SC & -KSC & C^2 \end{pmatrix}$$

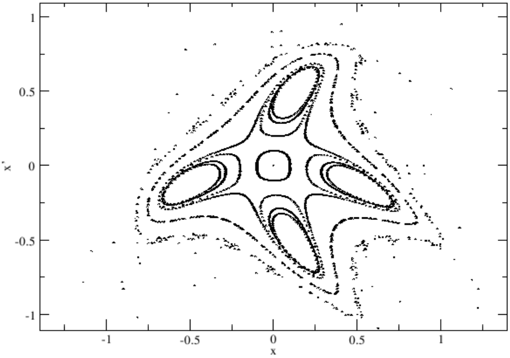
with: $L =$ effective length of the solenoid, $K = B_z/(2B\rho) = B_z/(2P/q)$, $C = \cos KL$, $S = \sin KL$.

Notice: a rotation of the transverse coordinates x and y about the optical axis at the exit of the solenoid, by an angle $-KL$, decouples the x and y first order terms, and allows to write,

$$M_{\text{solenoid}} = R_{\text{rot}}(-KL) \times \begin{pmatrix} C & \frac{S}{K} & 0 & 0 \\ -KS & C & 0 & 0 \\ 0 & 0 & C & \frac{S}{K} \\ 0 & 0 & -KS & C \end{pmatrix}$$

$\Rightarrow$ a solenoid behaves like a rotating quadrupole that focuses in both $x$and y.

Non-linear dynamics



$$\begin{pmatrix} x_{n+1} \\ x'_{n+1} \end{pmatrix} = \begin{pmatrix} \cos(2\pi Q) & \sin(2\pi Q) \\ -\sin(2\pi Q) & \cos(2\pi Q) \end{pmatrix} \begin{pmatrix} x_n \\ x'_n + x_n^2 \end{pmatrix}$$

- Q=0.2516
- linear motion near center (circles)
- More and more square
- Non-linear tuneshift
- Islands
- Limit of stability
- Dynamic Aperture
- Crucial if strong quads and chromaticity correction in s.r. light sources
- many non-linearities in LHC due to s.c. magnet and finite manufacturing tolerances

[NEW VIDEO!]

Particle tracking and dynamic aperture

Dynamic aperture: *is a method used to calculate the amplitude threshold of stable motion of particles. Numerical simulations of particle tracking aim at determining the "dynamic aperture".*

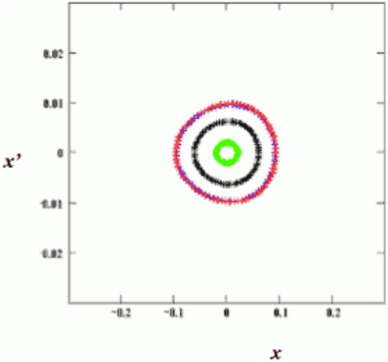
Dynamic aperture for hadrons

- in the case of protons or heavy ion accelerators, (or synchrotrons, or storage rings), there is minimal radiation, and hence the dynamics is symplectic
- for long term stability, a tiny dynamical diffusion can lead an initially stable orbit slowly into an unstable region
- this makes the dynamic aperture problem particularly challenging: One may need to consider the stability over billions of turns

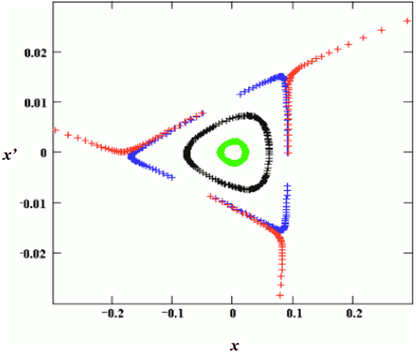
For the case of electrons

- in bending magnetic fields, the electrons radiate which causes a damping effect.
- this means that one typically only cares about stability over few (˜thousands) of turns

Dynamic Aperture and tracking simulations



a beam of four particles in a storage ring composed by only linear elements



a beam of four particles in a storage ring where there is a strong sextupole: it's a catastrophe!

The end!

I'd like to thank you all

for your attention!

Some Excellent References

1. The CERN Accelerator School (CAS) Proceedings: e.g. 1992, Jyväskylä, Finland; or 2013, Trondheim, Norway
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3. Mario Conte, William W. MacKay, An Introduction to the Physics of Particle Accelerators, Second Edition, World Scientific, 2008
4. Andrzej Wokski, Beam Dynamics in High Energy Particle Accelerators, Imperial College Press, 2014