Nonlinear dynamics of the Boltzmann equation in the FRW universe

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The big bang and little bangs
Non equilibrium phenomena in cosmology and heavy ion collisions

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THE OHIO STATE UNIVERSITY

Motivation

- Exact solutions to the relativistic Boltzmann equation have been obtained within the relaxation time approximation (RTA) for rapidly expanding systems
  - Bjorken flow: Baym; PLB 138 (1984) 18
  - Gubser flow: Denicol,Heinz, Martinez, Noronha and Strickland; PRL 113 (2014) 202301, PRD 90 (2014) 125026
- Those solutions have been useful to determine the accuracy and validity of hydrodynamical approximations for weakly coupled systems.
- In this work we make a step forward by solving exactly the non-linear relativistic Boltzmann equation for an expanding gas with constant cross section in a spatially flat FRW spacetime.

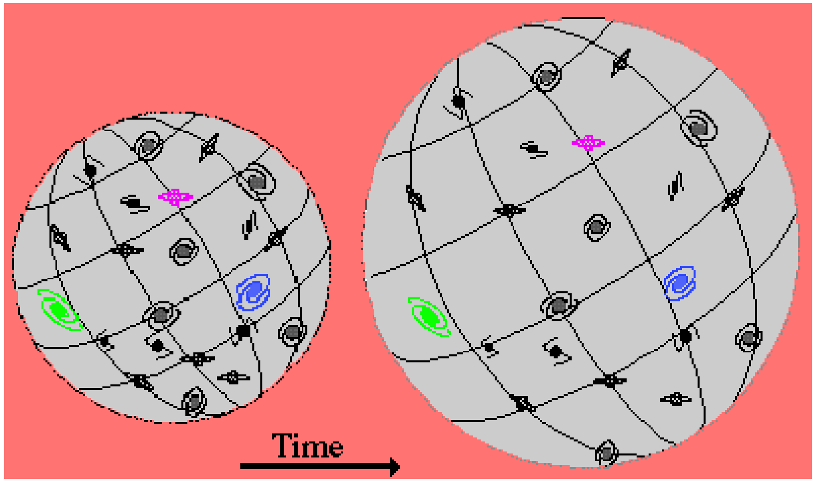
Outline

- Basics of the FRW universe
- Nonlinear Boltzmann equation in the FRW universe
  - Normalized energy moments
  - Laguerre moments
  - Reconstructing the distribution function from its moments
- Linear Boltzmann equation
- Relaxation time approximation (RTA) Boltzmann equation
- Comparing the predictions of the nonlinear, linear and RTA Boltzmann equation
- Entropy production by non-hydrodynamic modes
- Conclusions and outlook

Basics of the FRW universe

Metric of the (flat) FRW spacetime

For a spatially flat universe that expands homogeneously and isotropically



$$ds^2 = dt^2 - a^2(t)\left(dx^2 + dy^2 + dz^2\right)$$

Some important properties of the FRW metric are

- Scale factor depends on time (i. e. a=a(t)) and the equation of state
- Invariant under the rescaling symmetry

$$x^i \to x^i/\Lambda\,, \qquad a(t) \to \Lambda\, a(t)$$

$$\Rightarrow a(t_0) = 1$$

- Invariant under spatial rotations & translations
- We will focus on the case of massless particles

Conservation laws in the FRW universe

The most generic forms for the particle current and the energy momentum tensor in the FRW spacetime are

$$N^\mu = \rho_0(t)\, u^\mu$$
$$T^{\mu\nu} = (\rho_1(t) + P) u^\mu u^\nu - P g^{\mu\nu}$$

In the LRF
$u^\mu = (1, 0, 0, 0)$

For massless particles ($P = 3\rho_1$) the conservation laws read as

$$D_\mu N^\mu = 0 \Rightarrow \partial_t \rho_0(t) + 3\rho_0(t) H(t) = 0$$
$$D_\mu T^{\mu\nu} = 0 \Rightarrow \partial_t \rho_1(t) + 4\rho_1(t) H(t) = 0$$

$$\underbrace{H(t) = \frac{\dot{a}}{a}}_{Hubble\, parameter}$$

whose solutions are

$$\rho_0(t) = \frac{\lambda(t)\, T^3(t)}{\pi^2},$$
$$\rho_1(t) = \frac{3\lambda(t)\, T^4(t)}{\pi^2},$$

$$T(t) = \frac{T_0}{a(t)}$$
$$\lambda(t) = \lambda(t_0)$$

Nonlinear Boltzmann equation in the FRW universe

The Boltzmann equation in the FRW universe

Spatial rotations + translations of the FRW metric

$$f(t,x,y,z,p_x,p_y,p_z) \Longrightarrow f\left(t,E_{\mathbf{p}}\right) \equiv f_{\mathbf{p}}$$

The Boltzmann equation in the FRW universe

Spatial rotations + translations of the FRW metric

$$f(t, x, y, z, p_x, p_y, p_z) \Longrightarrow f\left(t, E_{\mathbf{p}}\right) \equiv f_{\mathbf{p}}$$

Thus, the Boltzmann equation in the FRW spacetime is

$$E_{\mathbf{p}} \frac{\partial f_{\mathbf{p}}}{\partial t} = \mathcal{C}[f]$$

The Boltzmann equation in the FRW universe

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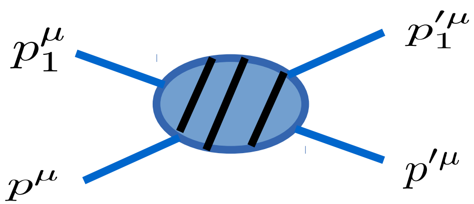
$$E_{\mathbf{p}}\frac{\partial f_{\mathbf{p}}}{\partial t} = \mathcal{C}[f] \qquad E_{\mathbf{p}} = \frac{|\mathbf{p}|}{a(t)}$$

The collisional kernel is

$$\mathcal{C}[f] =$$



$$= \int_{p',p_1',p_1} W(p,p_1|p',p_1')\left(f(x^\mu,p')\,f(x^\mu,p_1') - f(x^\mu,p_1)\,f(x^\mu,p)\right)$$

Nonlinear terms ~ f²

The Boltzmann equation in the FRW universe

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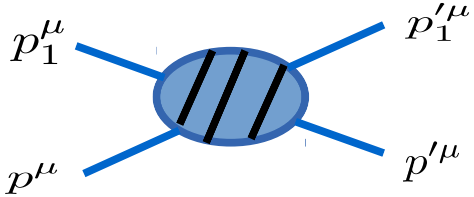
$\mathcal{C}[f] =$ 

$$= \int_{p',p_1',p_1} W(p,p_1|p',p_1')\left(f(x^\mu,p')\,f(x^\mu,p_1') - f(x^\mu,p_1)\,f(x^\mu,p)\right)$$

Transition rate. For a constant cross section σ

$$W(p,p_1|p',p_1') = (2\pi)^5\,\sqrt{-g}\,\sigma\,s\,\delta^{(4)}\left(p'^{\mu} + p_1'^{\mu} - p^{\mu} - p^{\mu}\right)$$

Non-linear evolution of the moments

The n-th normalized moment of the distribution function read as

$$M_n(t) = \frac{\rho_n(t)}{\rho_n^{\mathrm{eq}}(t)}$$

$$\rho_n(t) = \int_{\mathbf{p}} (u \cdot p)^n \, f_{\mathbf{p}}$$

$$\rho_n(t) = \int_{\mathbf{p}} (u \cdot p)^n \, f_{\mathbf{p}}^{eq} = (n+2)! \, \frac{\lambda}{2\pi^2} \, T^{n+3}(t)$$

- If f is the equilibrium distribution function $\Rightarrow$ $M_n = 1$
- Deviations from equilibrium corresponds to $M_n \neq 1$

  Different moments probe the distribution function in different kinematic regions of momentum space

Non-linear evolution of the moments

The evolution equation for the normalized energy moments

$$D_\tau M_n(\tau) + M_n(\tau) = \frac{1}{n+1}\sum_{m=0}^{n} M_m(\tau) M_{n-m}(\tau)$$

$$\tau = \int_{\hat{t}_0}^{\hat{t}} \frac{d\hat{t}'}{a^3(\hat{t}')}$$

Absorbs the information about the expansion of the universe

- This evolution equation was found 40 years ago by Krook, Wu and Bobylev (BKW) for a non-relativistic, spatially homogeneous and isotropic gas (static box).
- Positivity condition: $M_n(\tau)>0$ iff $M_n(0)>0$
- The equation of the n-th moment only couples to moments of the same or lower order. Thus, one can solve order by order!!!
- Conservation laws imply that $M_0(\tau)=M_1(\tau)=1$
- The fixed point of this evolution equation is $M_n \to 1$ for all n.

Laguerre moments

It is convenient to introduce the Laguerre moments

$$c_n(\tau) = \frac{2}{(n+1)(n+2)} \frac{1}{n(\tau)} \int_k (u\cdot k) \, \mathcal{L}_n^{(2)}\left(\frac{u\cdot k}{T(\tau)}\right) f_{\mathbf{k}}$$

Assoc. Laguerre polynomial

$$= \sum_{r=0}^{n} (-1)^r \binom{n}{r} M_r(\tau) \, ,$$

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$$= \sum_{r=0}^{n} (-1)^r \binom{n}{r} M_r(\tau)\,,$$

The Laguerre moments c_n satisfy the differential equation

$$D_\tau c_n(\tau) + c_n(\tau) = \frac{1}{n+1}\underbrace{\sum_{m=0}^{n} c_m(\tau)c_{n-m}(\tau)}_{\text{Non-linear mode by mode coupling}}$$

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- Conservation laws imply that $c_0(\tau)=1$ and $c_1(\tau)=0$
- The fixed point of this evolution equation is

$$c_n \to \delta_{n0} \text{ for } n\geq 2$$

Asymptotic behavior

The general solution of the Laguerre moments is

$$c_n(\tau) = \underbrace{e^{-\omega_n \tau} c_n(0)}_{\text{Dominant at late times}} + \frac{1}{n+1} \underbrace{\sum_{m=1}^{n-1} \int_0^{\tau} d\tau' \, e^{\omega_n(\tau'-\tau)} \, c_m(\tau') c_{n-m}(\tau')}_{\text{Sizable contributions at early times}}$$

$$\omega_n = \frac{n-1}{n+1}$$

For instance, up to n=5, the solutions of the Laguerre moments look like

$$c_2(\tau) = c_2(0) e^{-\omega_2 \tau}$$

$$c_3(\tau) = c_3(0) e^{-\omega_3 \tau},$$

$$c_4(\tau) = c_4(0) e^{-\omega_4 \tau} + 15\, c_2^2(\tau) \left[e^{-(\omega_4 - 2\omega_2)\tau} - 1 \right],$$

$$c_5(\tau) = c_5(0) e^{-\omega_5 \tau} + 6\, c_2(\tau) c_3(\tau) \left[e^{-(\omega_5 - \omega_2 - \omega_3)\tau} - 1 \right]$$

Reconstructing the distr. function

The distribution function can be reconstructed from its moments as follows

$$f_{\mathbf{k}} = \lambda\, e^{-(u\cdot p)/T(t)} \sum_{n=0}^{\infty} c_n(\tau)\, \mathcal{L}_n^{(2)}\left(\frac{u\cdot k}{T(\tau)}\right)$$

$$= \lambda\, e^{-u\cdot k/T(\tau)} \times \sum_{n=0}^{\infty} \left[\sum_{r=0}^{n} (-1)^r \binom{n}{r} M_r(\tau)\right] \mathcal{L}_n^{(2)}\left(\frac{u\cdot k}{T(\tau)}\right) .$$

- The distribution function can be entirely reconstructed from either the Laguerre moments or the normalized moments
- The thermal equilibrium is characterized by is $M_n \to 1$ or $c_n \to \delta_{n0}$
- The distribution function can be fully reconstructed for arbitrarily initial conditions once the moments' equations are solved.

Reconstructing the distr. function

The generic solution of the Laguerre moments implies

$$f_{\mathbf{k}} = f_{\mathbf{k}}^{lin} + f_{\mathbf{k}}^{non-lin.}$$

$$f_{\mathbf{k}}^{\mathrm{lin}}(\tau) = \lambda\, e^{-k/T_0}\left[1 + \sum_{m=2}^{\infty} c_m(0)\, e^{-\omega_m \tau} \mathcal{L}_r^{(2)}\left(\frac{k}{T_0}\right)\right]$$

$$f_{\mathbf{k}}^{\mathrm{non-lin}}(\tau) = \lambda\, e^{-k/T_0}\left[\sum_{m=4}^{\infty} \underbrace{h_m(\tau)}\, \mathcal{L}_r^{(2)}\left(\frac{k}{T_0}\right)\right]$$

$$h_m(\tau) = \frac{1}{m+1}\sum_{l=1}^{m-1}\int_0^{\tau} d\tau'\, e^{\omega_m(\tau'-\tau)}\, c_l(\tau')c_{m-l}(\tau')$$

- The distribution function is dominated by its linear part at late times. This component describes the dynamics of the slowest decaying modes.
- The non-linear component is important at early times. It carries the information of the thermalization of the high energy modes

An exact analytical solution to the nonlinear Boltzmann equation in the FRW spacetime

Exact analytical solution

An exact analytical solution to the moments' equation can be found for a particular non-equilibrated initial condition (Bobylev-Krook-Wu)

$$M_n(\tau) = \mathcal{K}(\tau)^{n-1}\left[n - (n-1)\mathcal{K}(\tau)\right]$$

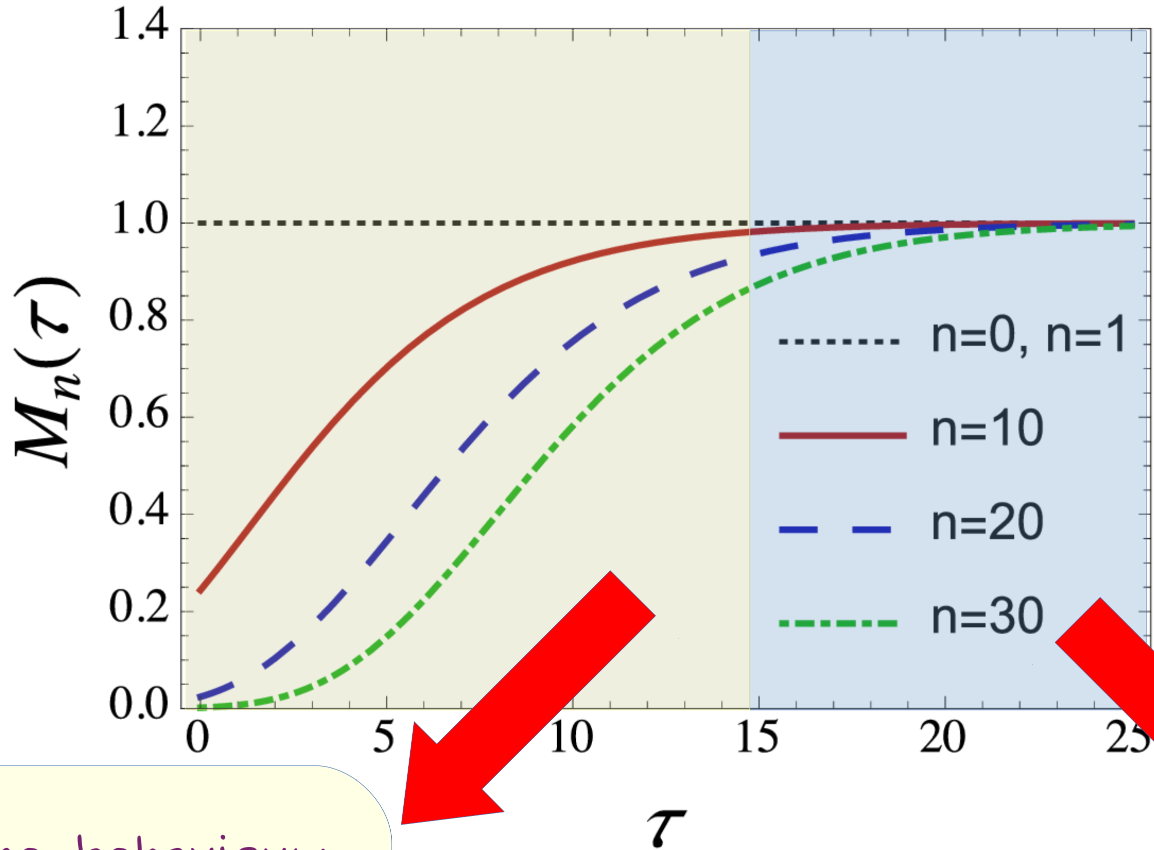
$$\mathcal{K}(\tau) = 1 - \frac{1}{4}e^{-\tau/6}$$

Thus, we get

$$f(\tau, |\mathbf{p}|) = \exp\left(-\frac{|\mathbf{p}|}{T_0\mathcal{K}(\tau)}\right)\left[\frac{4\mathcal{K}(\tau)-3}{\mathcal{K}^4(\tau)} + \frac{|\mathbf{p}|}{T_0}\left(\frac{1-\mathcal{K}(\tau)}{\mathcal{K}^5(\tau)}\right)\right]$$

Exact sol. of the energy moments

$$M_n(\tau) = \mathcal{K}(\tau)^{n-1}\left[n - (n-1)\mathcal{K}(\tau)\right]$$



$$\mathcal{K}(\tau) = 1 - \frac{1}{4}e^{-\tau/6}$$

$$\mathcal{K}(\tau_0 = 0) = \frac{3}{4}$$

Early time behaviour:

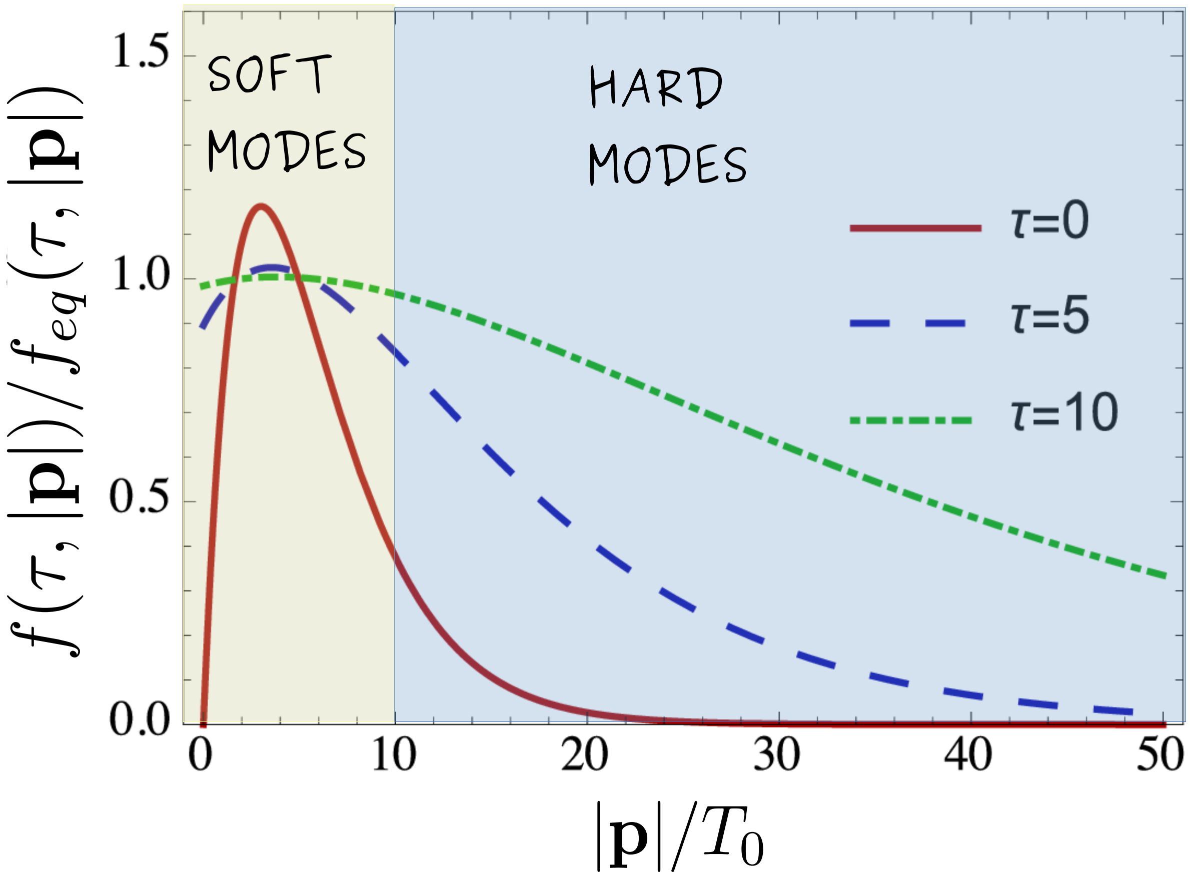
- Mn $\ll$ 1 when $\tau \gtrsim 0$
- System is far from equilibrium

Late time behaviour:

- Mn $\to$ 1 when $\tau \to \infty$
- Moments with large n thermalize later.

Evolution of the distribution function

$$f(\tau,|\mathbf{p}|) = \exp\left(-\frac{|\mathbf{p}|}{T_0\mathcal{K}(\tau)}\right)\left[\frac{4\mathcal{K}(\tau)-3}{\mathcal{K}^4(\tau)} + \frac{|\mathbf{p}|}{T_0}\left(\frac{1-\mathcal{K}(\tau)}{\mathcal{K}^5(\tau)}\right)\right]$$



$$f(\tau=0,|\mathbf{p}|) = \frac{256}{243}\frac{|\mathbf{p}|}{T_0}\exp\left(-\frac{4\,|\mathbf{p}|}{3\,T_0}\right)$$

- Soft modes thermalize first than the hard ones ("bottom-up")
- Formation of transient high energy tails
- When $\tau \to \infty$ the system reaches equilibrium

Exact analytical solution

Warning:

- the τ variable hides the information about the expansion of the system.
- For massless particles, the scaling factor is

$$a(t) = \sqrt{1 + b_r\, t}\ , \qquad b_r = 2\, H_0\, \sqrt{\Omega_r}$$

and thus

$$\tau = \int_{t_0}^{t} \frac{dt}{a^3(t)} = \frac{2}{b_r\, l_0}\, \left(1 - (b_r\, t + 1)^{-\frac{1}{2}}\right)$$

- Notice that

$$\tau_{\max} = \lim_{t\to\infty} \tau(t) = \frac{2}{b_r\, l_0}$$

$$\Rightarrow \quad \lim_{t\to\infty} f_{\mathbf{k}}(t) = \frac{\lambda\, e^{-k/(\mathcal{K}_{max}\, T_0)}}{\mathcal{K}_{max}^4} \left[4\, \mathcal{K}_{max} - 3 + \frac{k}{\mathcal{K}_{max} T_0} (1 - \mathcal{K}_{max})\right].$$

$$\mathcal{K}_{max} = 1 - \tfrac{1}{4} \exp\left[-\tau_{max}/6\right]$$

Non thermal energy tail

Linearized Boltzmann equation in the FRW universe

Linearized Boltzmann equation

It is common to expand around equilibrium f= feq +δf

$$\left.\begin{aligned} M_n \approx M_n^{\rm lin} = 1 + \delta M_n \\ c_n \approx c_n^{\rm lin} = \delta_{n0} + \delta c_n \end{aligned}\right\}$$

Conservation laws imply

$\delta M_0 = \delta M_1 = \delta c_0 = \delta c_1 = 0$

$$D_\tau M_n(\tau) + M_n(\tau) = \frac{1}{n+1}\sum_{m=0}^{n} M_m(\tau) M_{n-m}(\tau) \Rightarrow D_\tau \delta M_n(\tau) + \omega_n \delta M_n(\tau) = \frac{2}{n+1}\sum_{m=2}^{n-1} \delta M_m(\tau)$$

$$\omega_n = \frac{n-1}{n+1}$$

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There is no mode by mode coupling
in the linear approximation

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There is no mode by mode coupling in the linear approximation

$$D_\tau c_n(\tau) + c_n(\tau) = \frac{1}{n+1}\sum_{m=0}^{n} c_m(\tau) c_{n-m}(\tau) \Rightarrow D_\tau \delta c_n(\tau) + \omega_n \delta c_n(\tau) = 0$$

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There is no mode by mode coupling in the linear approximation

$$D_\tau c_n(\tau) + c_n(\tau) = \frac{1}{n+1}\sum_{m=0}^{n} c_m(\tau)c_{n-m}(\tau) \Rightarrow D_\tau \delta c_n(\tau) + \omega_n \delta c_n(\tau) = 0$$

No mode by mode coupling + full decoupling among different modes

$$\delta c_n = e^{-\omega_n \tau} c_n(0)$$

Linearized Boltzmann equation

The decoupling among different moments means

- δc_n are eigenfunction of the linearized collisional kernel with eigenvalues $w_n = (n-1)/(n+1)$
- The distribution function can be entirely reconstructed from the linearized Laguerre moments

$$f_{\mathbf{k}}^{\mathrm{lin}}(\tau) = \lambda\, e^{-k/T_0} \left[1 + \sum_{m=2}^{\infty} c_m(0)\, e^{-\omega_m \tau} \mathcal{L}_r^{(2)}\left(\frac{k}{T_0}\right)\right]$$

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- Depends only on the initial values of the Laguerre moments
- Asymptotic behavior depends on the slowest initially occupied Laguerre mode

Each mode decays at a different decay rate w_n

RTA Boltzmann equation in the FRW universe

RTA Boltzmann equation

The relaxation time approximation reads as

$$\mathcal{C}[f] = -\frac{(u \cdot p)}{\tau_{rel}(t)} \left(f(x,p) - f_{eq}(x,p)\right)$$

relaxation time is given by

$$\tau_{rel}(t) = \alpha \lambda_{mfp}\, a^3(t)$$

Denicol et. al. : $\alpha = 1.58375$ $\lambda_{mfp} = (\rho_0 \sigma_T)^{-1}$

- The general solution of the RTA Boltzmann is

$$f_{\mathbf{k}}^{\mathrm{RTA}}(\tau) = \lambda\, e^{-k/T_0} \left(1 + e^{-\tau/\alpha} \sum_{n=2}^{\infty} c_n(0)\, \mathcal{L}_n^{(2)}\left(\frac{k}{T_0}\right)\right).$$

- The moments of the RTA distribution function read

$$c_n^{\mathrm{RTA}}(\tau) = e^{-\tau/\alpha} c_n(0)\,,$$

$$M_n^{\mathrm{RTA}}(\tau) = 1 + e^{-\tau/\alpha}\left(M_n(0) - 1\right).$$

Comparing nonlinear, linear and RTA predictions

Numerical results

We compare the evolution of the distribution function obtained from the nonlinear, linear and RTA Boltzmann equation for the following initial conditions

- ES-IC

$$f_{\mathbf{k}}(0) = \lambda \frac{256}{243} \left(\frac{k}{T_0}\right) e^{-\frac{4}{3}\frac{k}{T_0}}$$

$$c_n(0) = \frac{1-n}{4^n}$$

- 1M-IC

$$f_{\mathbf{k}}(0) = \lambda\, e^{-\frac{k}{T_0}} \left[1 + \frac{3}{10} \mathcal{L}_2^{(2)} \left(\frac{k}{T_0}\right)\right]$$

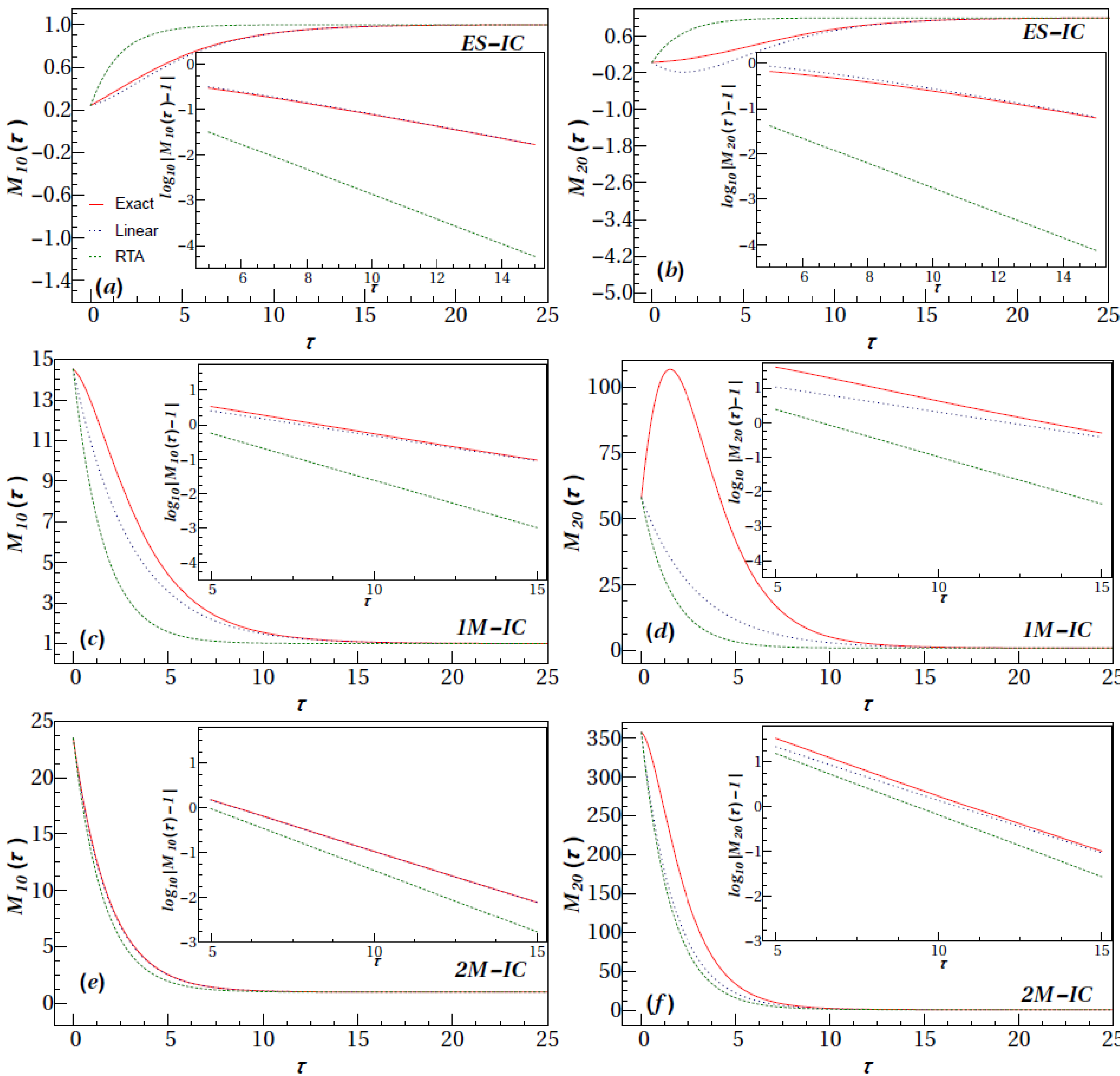
$$c_n(0) = \delta_{n0} + \frac{3}{10}\delta_{n2}$$

- 2M-IC

$$f_{\mathbf{k}}(0) = \lambda\, e^{-\frac{k}{T_0}} \left[1 - \frac{1}{10} \mathcal{L}_3^{(2)} \left(\frac{k}{T_0}\right) + \frac{1}{20} \mathcal{L}_4^{(2)} \left(\frac{k}{T_0}\right)\right]$$

$$c_n(0) = \delta_{n0} - \frac{1}{10}\delta_{n3} + \frac{1}{20}\delta_{n4}$$

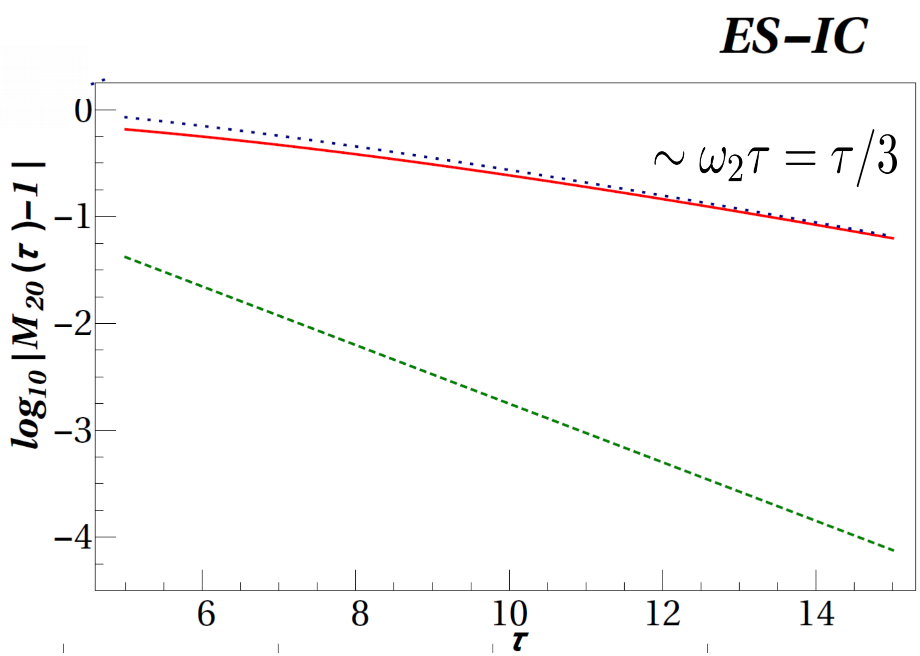
Comparing energy moments

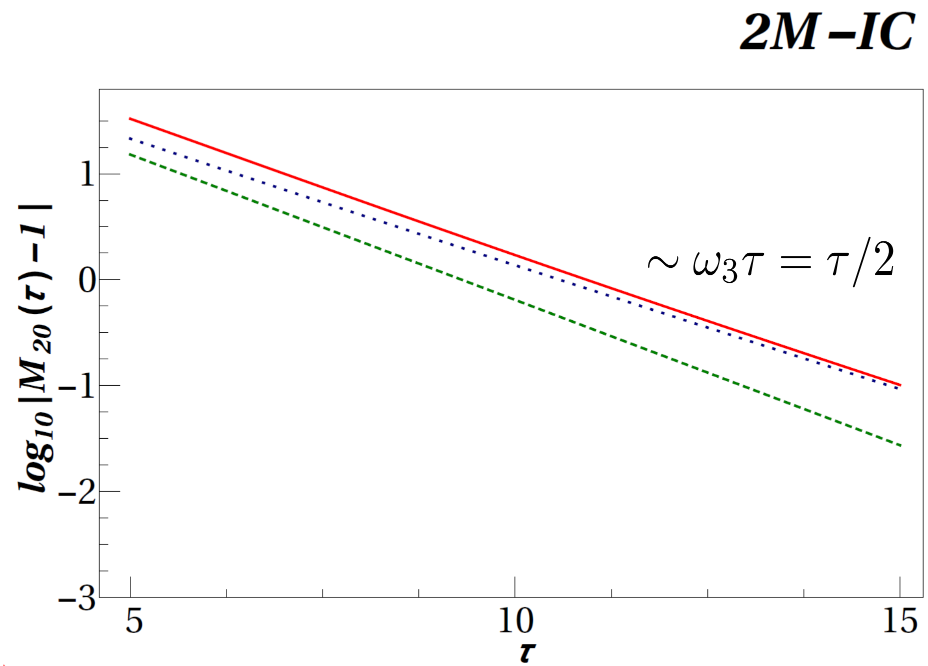


- Differences increase as the order n of the moment increases
- All moments relax to the unity asymptotically
- RTA energy moments relax faster than the linear and nonlinear

Comparing energy moments

$$\mathrm{M}_n(\tau) = \sum_{r=0}^{n} (-1)^r \binom{n}{r} c_r(\tau)\,,$$





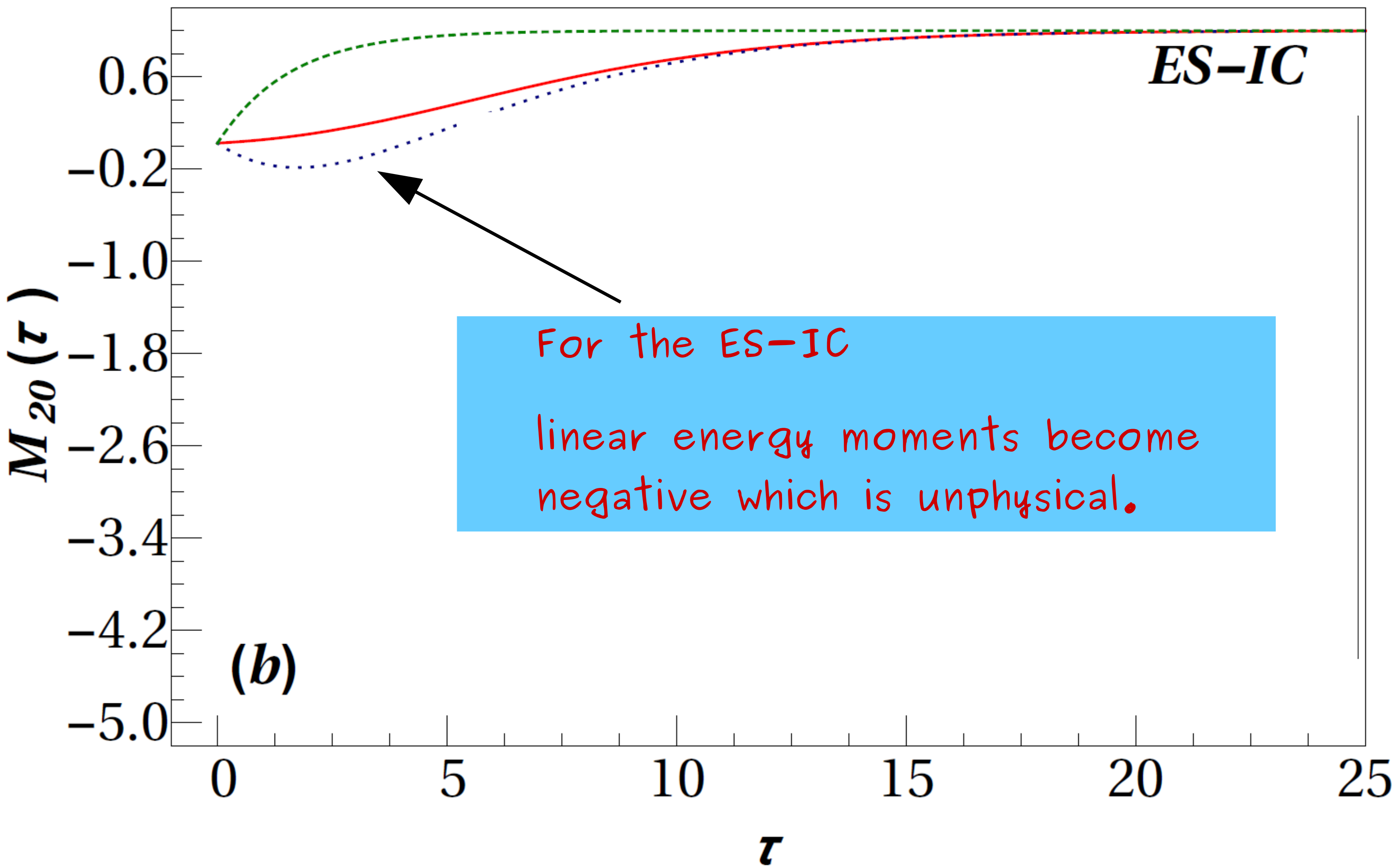
- In the linear and nonlinear cases, energy moments mix Laguerre moments of different orders that decay with different rates.

  ⇒ the asymptotic behavior of the energy moments in the linear and nonlinear case is determined by the first non-vanishing non-hydrodynamical mode

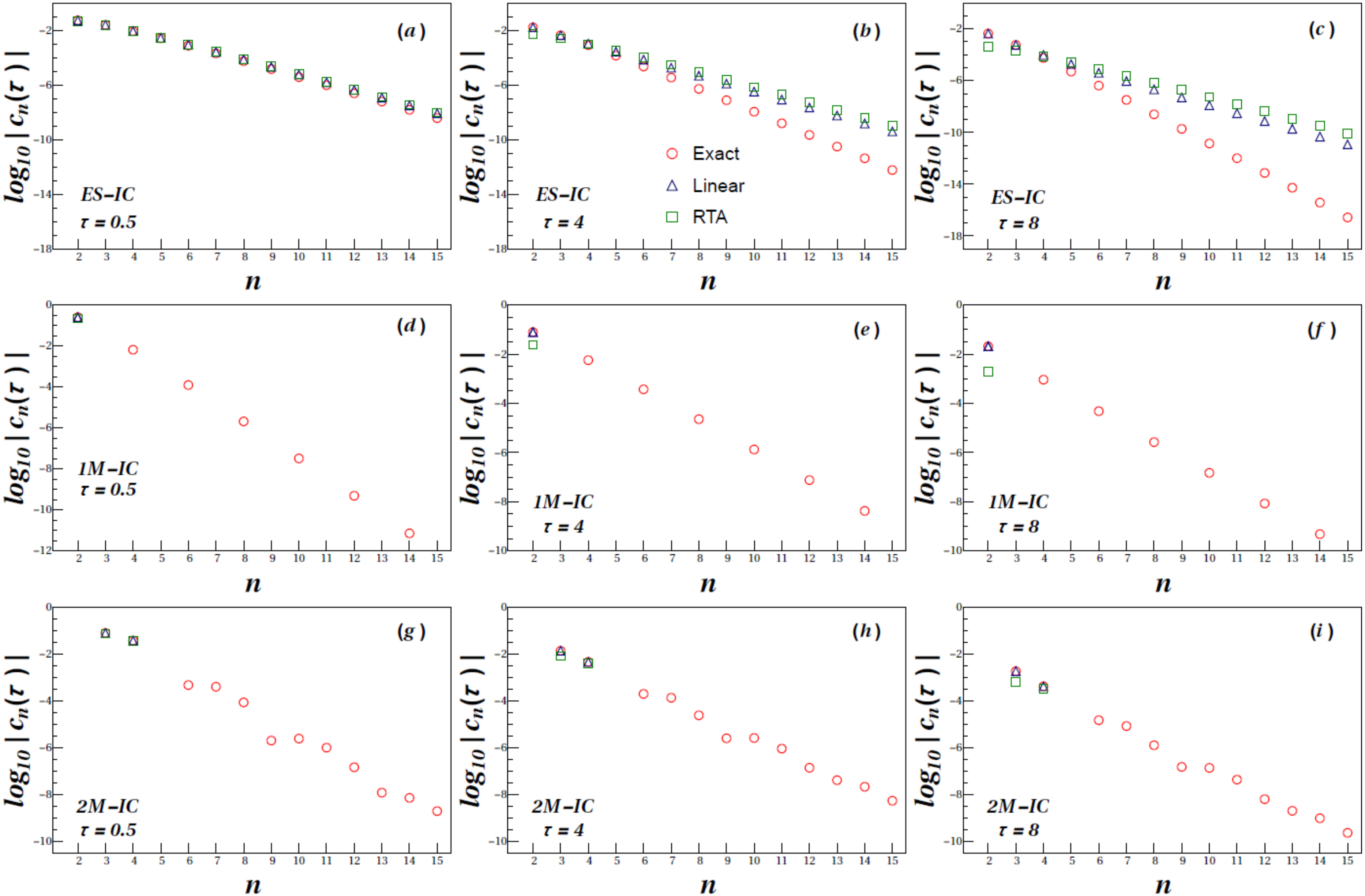
  For the ES−IC and 1M−IC the first non-vanishing non-hydro mode is c_2 while for the 2M−IC is c_3

- At late times the linear and nonlinear evolution converge

Comparing energy moments

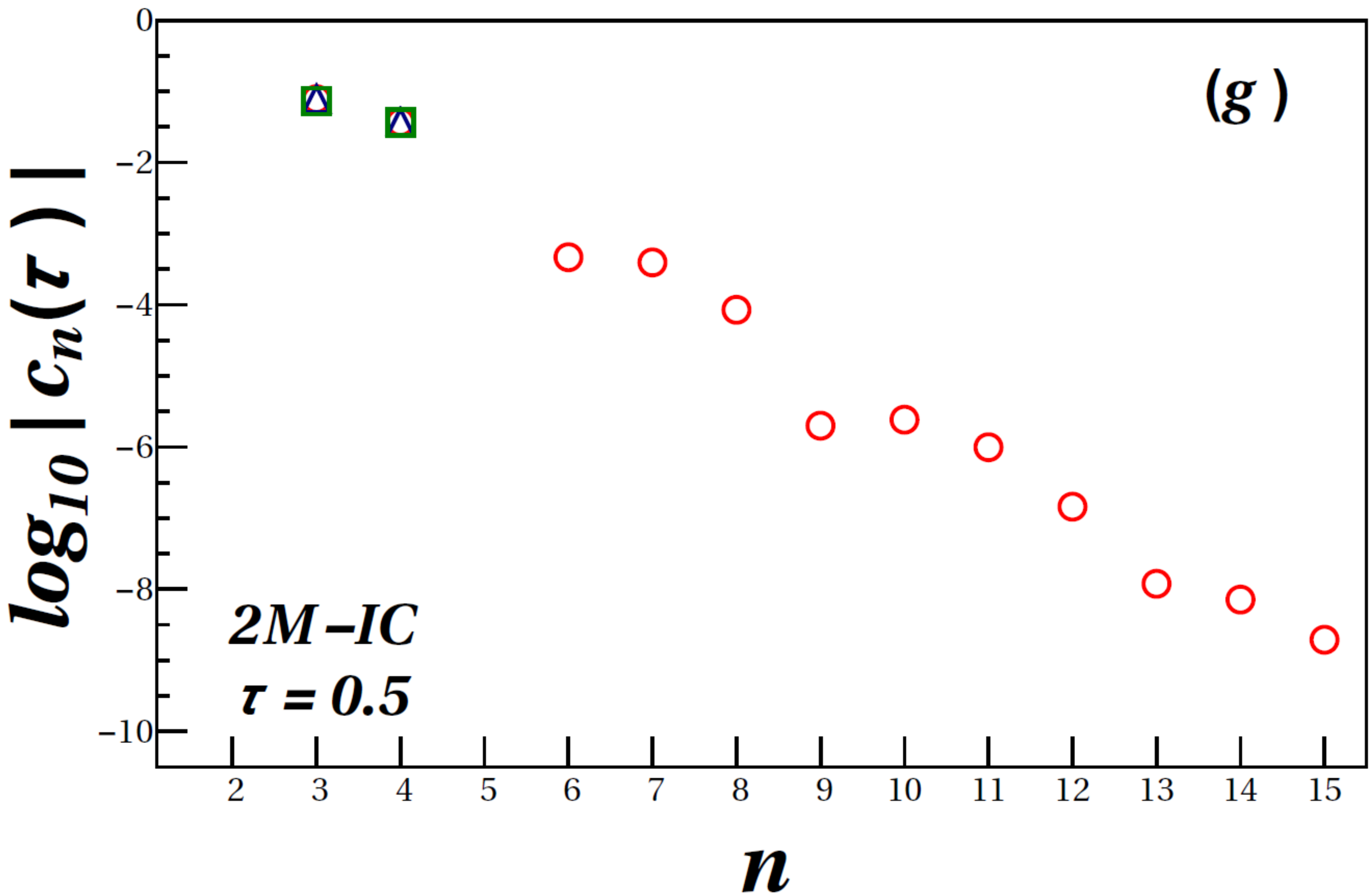


Comparing Laguerre moments



Slowest modes are well described by the linear approximation and fairly well by the RTA.

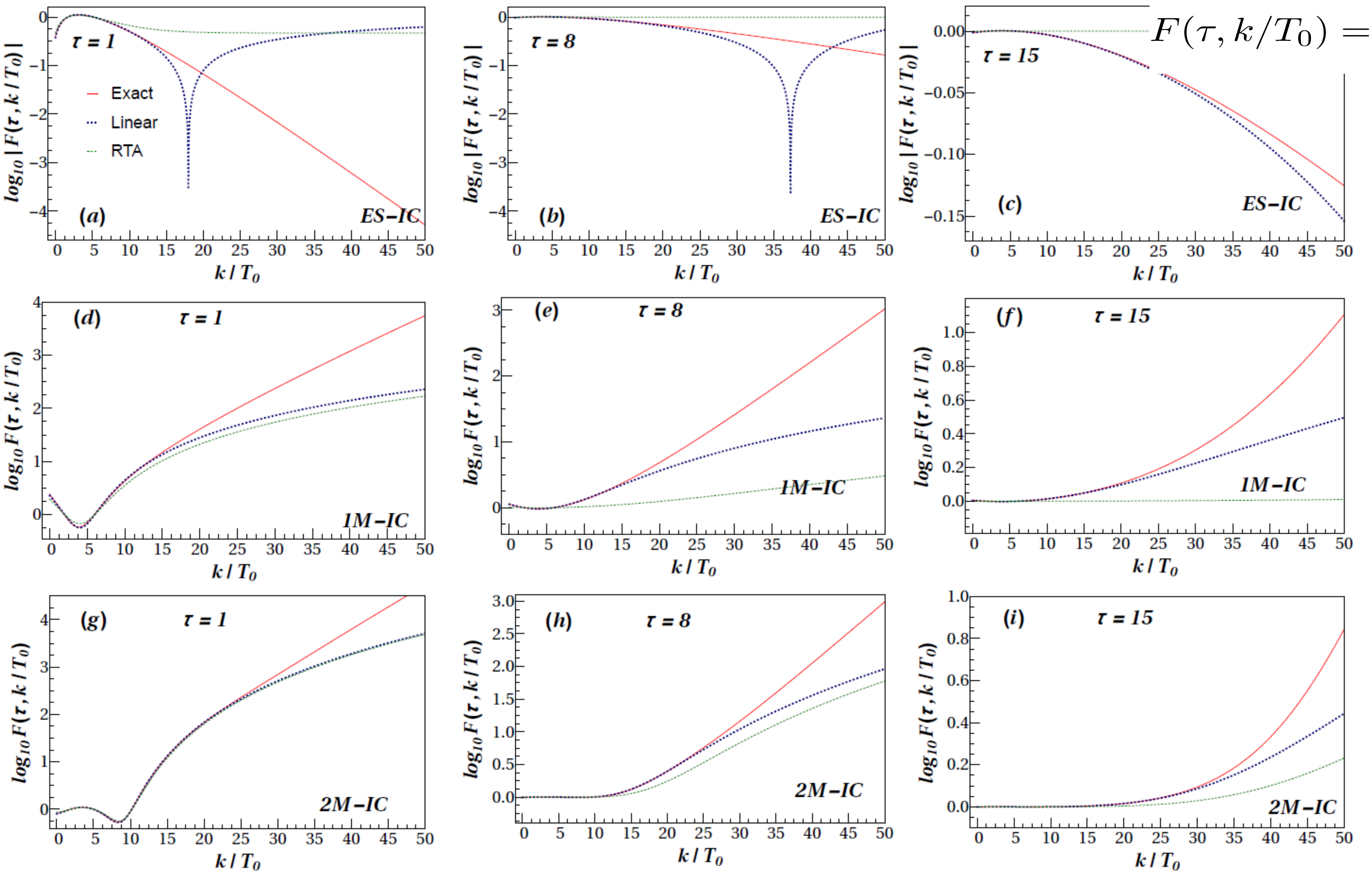
Comparing Laguerre moments



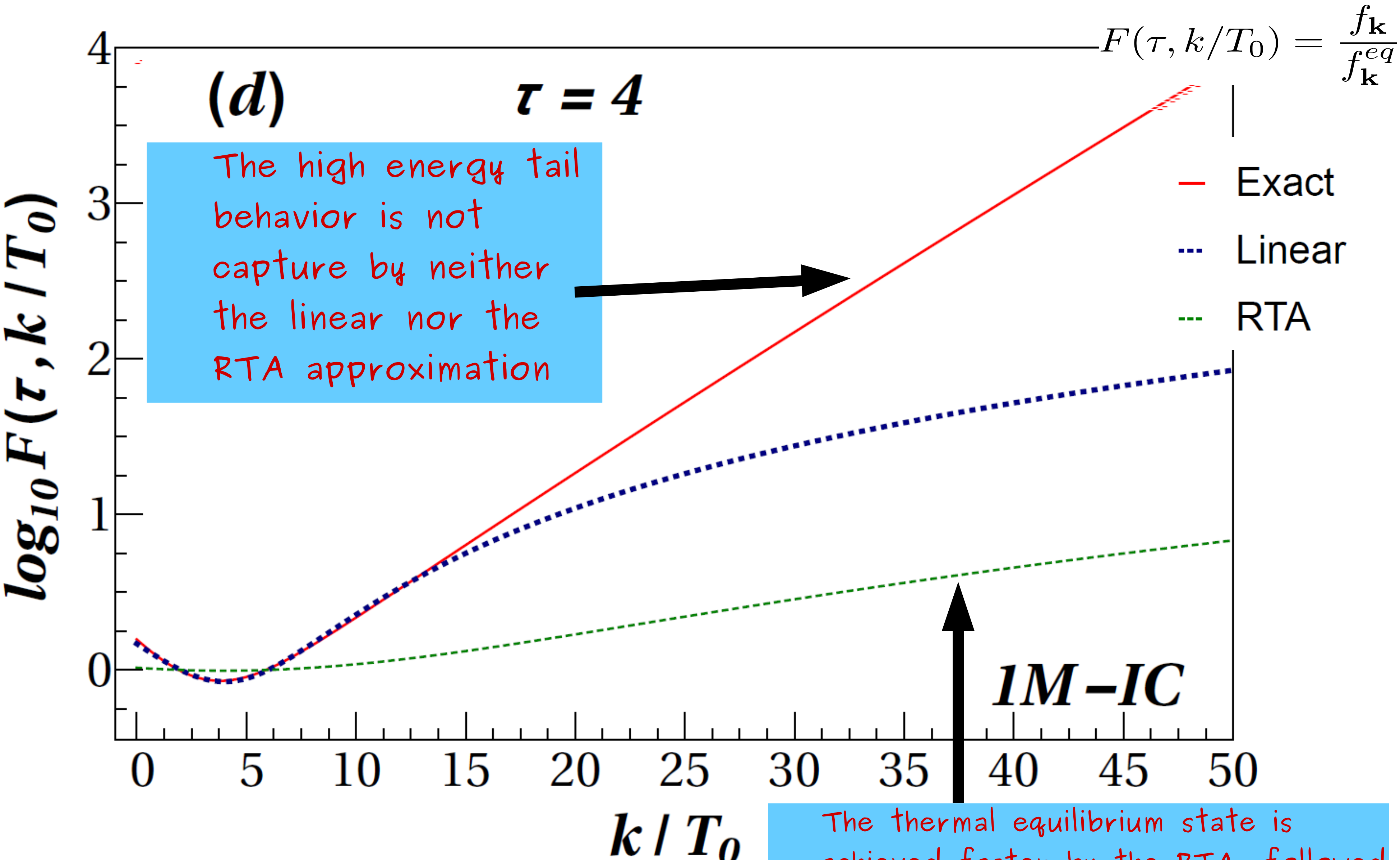
Mode by mode is responsible for exciting higher modes for n>2. This effect is absent in the RTA and linear approximation.

Linear vs. non-linear distribution

$$F(\tau, k/T_0) = \frac{f_{\mathbf{k}}}{f_{\mathbf{k}}^{eq}}$$



Linear vs. non-linear distribution



The high energy tail behavior is not capture by neither the linear nor the RTA approximation

The thermal equilibrium state is achieved faster by the RTA, followed by the linear approx. and then the full nonlinear distribution fucntion

Linear vs. non-linear distribution

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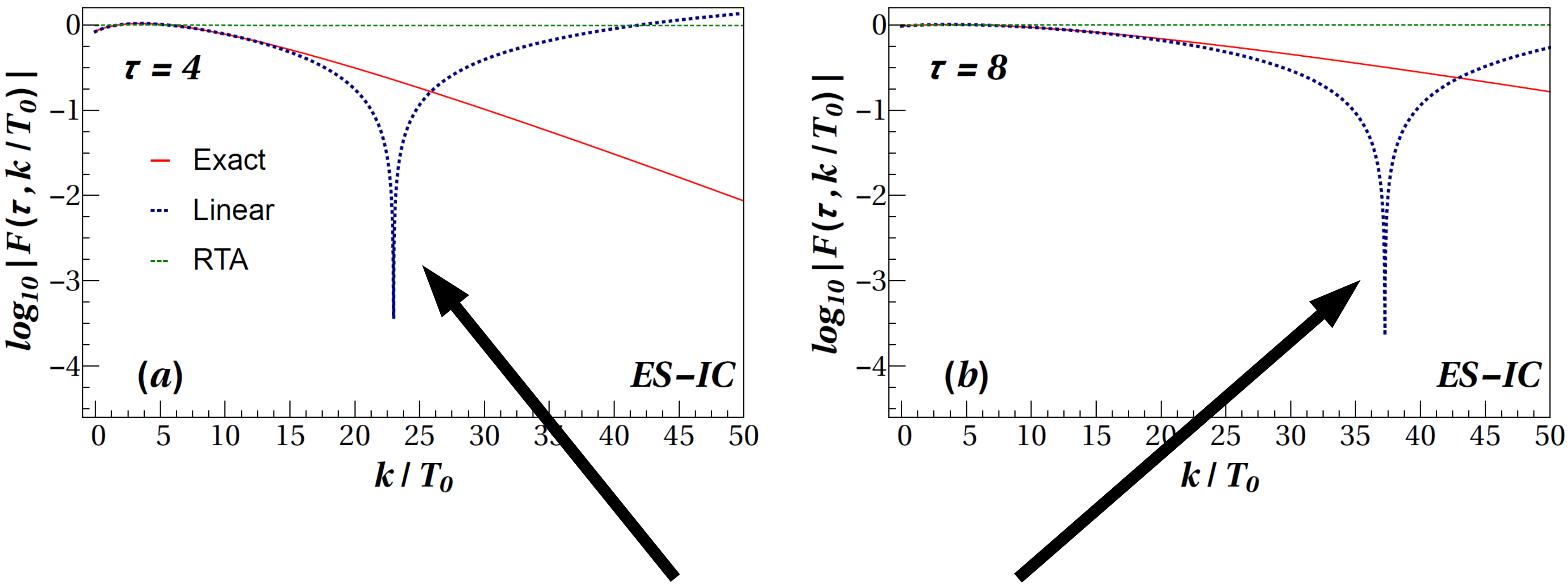
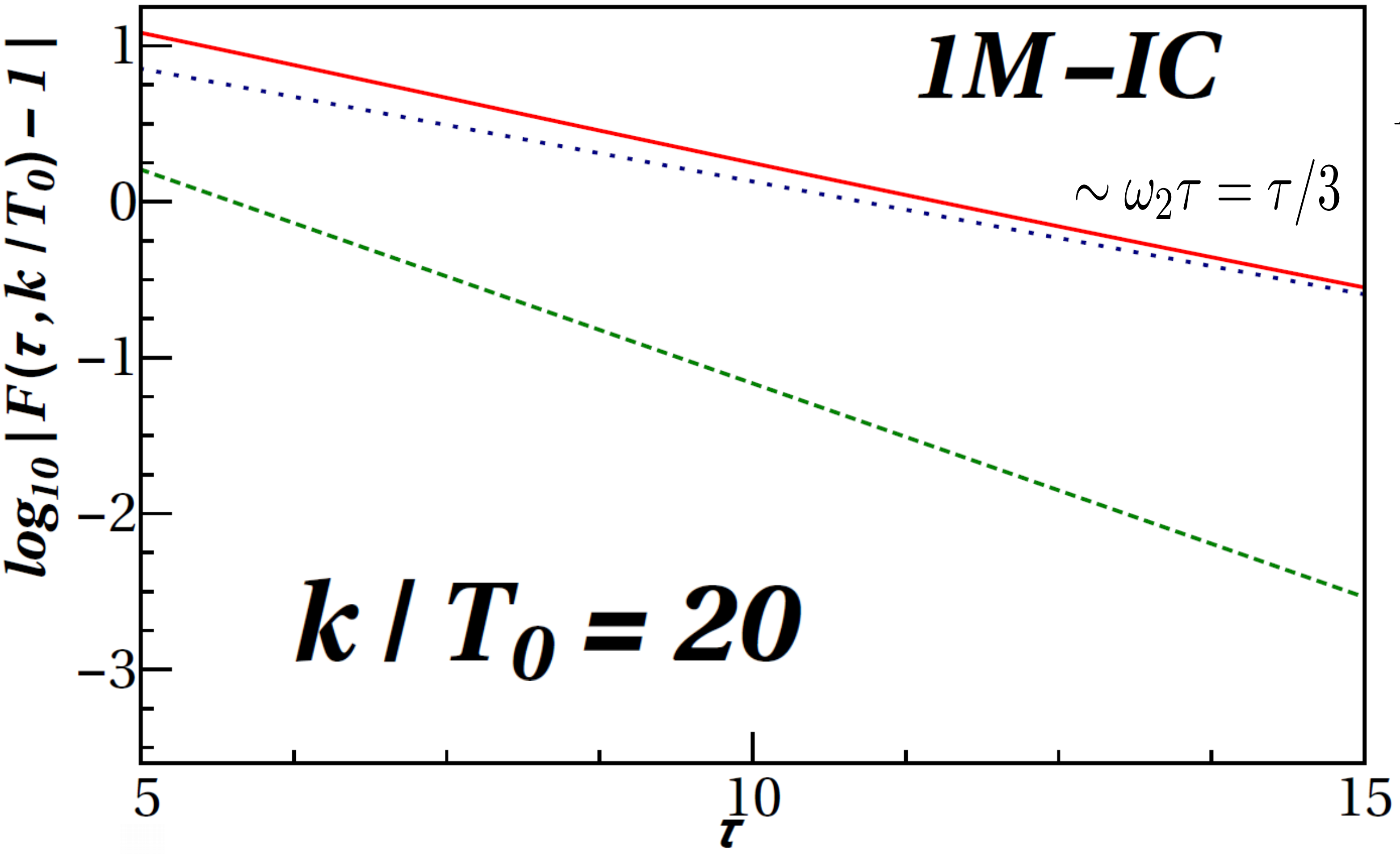


- The linear approximation can lead to unphysical results since f < 0 for far-from equilibrium initial conditions
- Similar findings observed for the exact solution of the RTA Boltzmann equation in a system undergoing Gubser flow

Denicol et. al. PRD90, 125026 (2014), Heinz & Martinez, NPA943, 26 (2015)

Linear vs. non-linear distribution

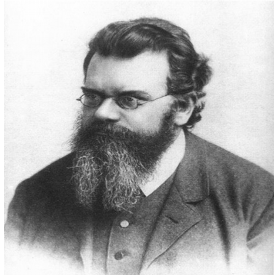


$$F(\tau, k/T_0) = \frac{f_{\mathbf{k}}}{f_{\mathbf{k}}^{eq}}$$

- For the RTA the rate of convergence to equilibrium is $1/\alpha$ for all momentum.
- For the linear and nonlinear Boltzmann equation, the rate of convergence to equilibrium is controlled by the lowest (and slowest) non vanishing non-hydrodynamic moment

Entropy and non hydro modes

Entropy and non hydro modes



- Entropy is a statistical quantity (L. Boltzmann)

$$S^{\mu}(t) \equiv -\left[\int_k k^{\mu}\, f_{\mathbf{k}}\,(\ln f_{\mathbf{k}} - 1)\right]$$

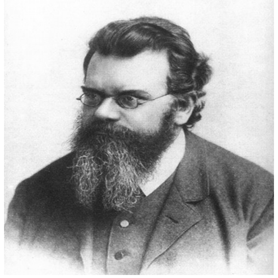
- From the four entropy flow one can study the entropy production rate $D_{\mu}S^{\mu}(t)$

- If we use the generic solution of the Boltzmann equation in the FRW we get

$$D_{\tau}\mathcal{S}(\tau) = -\frac{\rho_0(0)}{2}\sum_{n=2}^{\infty} a_n(\tau) D_{\tau} c_n(\tau)$$

$$a_n(\tau) = \int_0^{\infty} dx\, x^2\, e^{-x}\mathcal{L}_n^{(2)}(x)\,\ln\left[1+\sum_{n=2}^{\infty} c_n(\tau)\mathcal{L}_n^{(2)}(x)\right].$$

Entropy and non hydro modes



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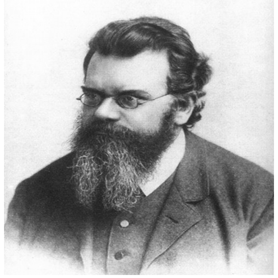
Remember that $$a_n(\tau) = \int_0^\infty dx\, x^2\, e^{-x} \mathcal{L}_n^{(2)}(x) \ln\left[1 + \sum_{n=2}^{\infty} c_n(\tau)\mathcal{L}_n^{(2)}(x)\right].$$

- n=0,1 are determined by conservation laws (hydro modes)
- n $\geqslant$2 correspond to higher moments whose evolution equation is obtained from the Boltzmann equation (non-hydro modes)

Entropy and non hydro modes



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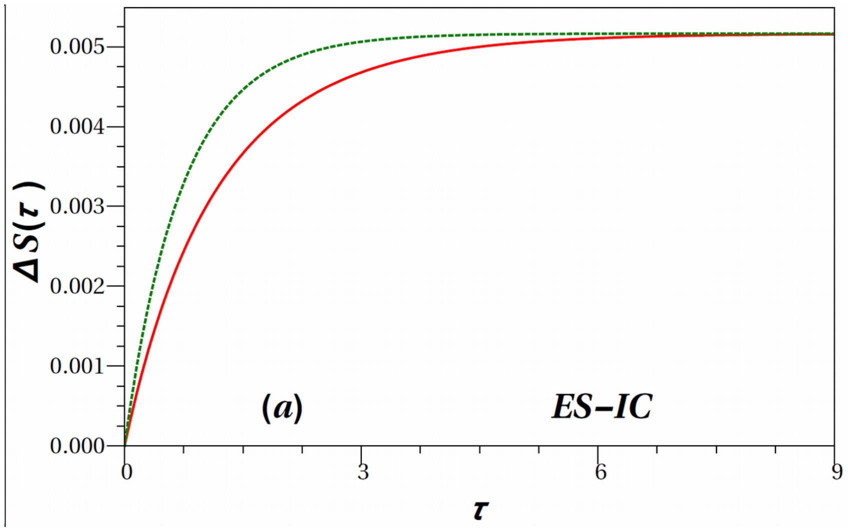
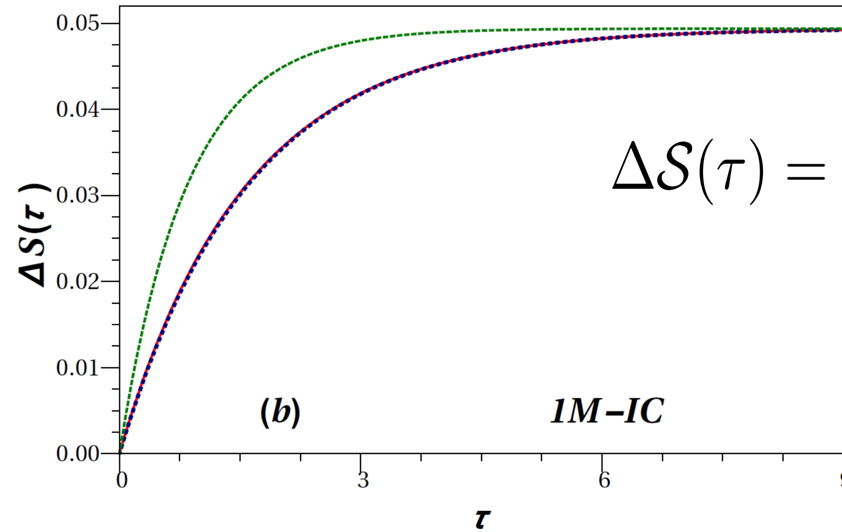
Thus

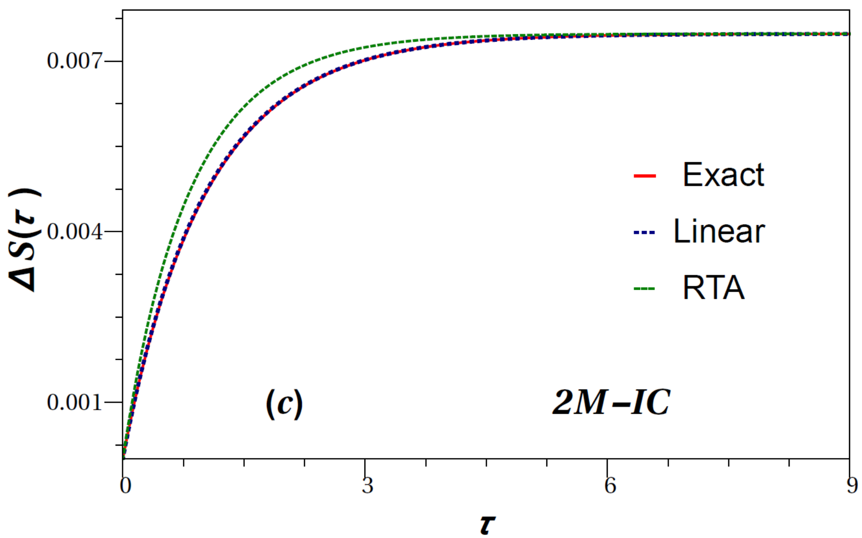
- The energy momentum tensor and particle density evolve according to the ideal hydrodynamics.
- Entropy can be produced due to the presence of non-hydrodynamical modes

Entropy production





$$\Delta\mathcal{S}(\tau) = \frac{\mathcal{S}(\tau) - \mathcal{S}(0)}{\mathcal{S}(0)}$$



- The approach of the total entropy towards its equilibration value is fastest in the RTA.
- The rate of entropy production slows down at the rate of the lowest initially non hydro Laguerre moment for the linear and nonlinear Boltzmann equation
- There are no sizable differences for the total entropy produced between the linear and nonlinear Boltzmann eq.

  $\Rightarrow$ High energy tails do not contribute significantly to the total entropy produced by the system.
- The amount of entropy produced depends on the initial state

Conclusions

- The mathematical problem of solving the nonlinear Boltzmann equation is recast into an infinite set of nonlinear ordinary differential equations for the moments of the distribution function.
- The expansion of the FLRW spacetime is slow enough for the system to move towards (and not away from) local thermal equilibrium, it is not sufficiently slow for the system to actually ever reach complete local equilibrium.
- The evolution of the high energy tails is not captured entirely neither by the linear nor RTA Boltzmann equation. Linear approximation probes additional high momentum details which are not resolved by the RTA approximation.
- Equilibration is achieved faster in the RTA, followed, in turn, by the linear and a fully nonlinear Boltzmann equation.
- Asymptotic behavior of the linear and nonlinear Boltzmann equation is not universal. Thermalization happens at a rate corresponding to the slowest initially nonzero non-hydrodynamical mode.
- Non-hydrodynamical modes decouple completely from the slowest hydrodynamical degrees of freedom. This results in the system flowing as an ideal fluid while at the same time producing entropy.

Outlook

It would be interesting

- Extend these studies for highly anisotropic systems:

  In heavy-ions: Bjorken and Gubser flow.

  In cosmology: Bianchi universes.

- Study of weak turbulence (work in progress): adding source and sink term to the Boltzmann equation. The nonrelativistic case was studied by Nazarenko et. al. (2012)

Furthermore

- The exact analytical solution obtained here can be used as a test of numerical algorithms that solve the Boltzmann equation (e.g. in heavy ion collsions URQMD)