# New formulations of relativistic hydrodynamics

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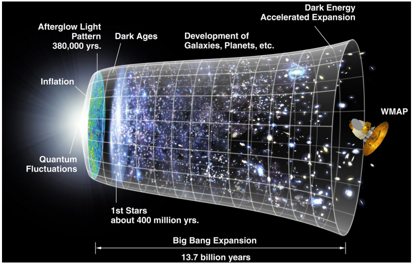
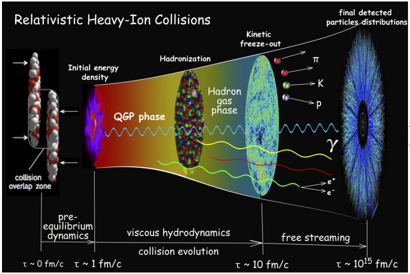
The Big Bang and the Little Bangs
- Non-equilibrium phenomena in cosmology and heavy-ion physics
August 18, 2016

# BIG BANG VS. LITTLE BANGS

# Big Bang vs. Little Bang

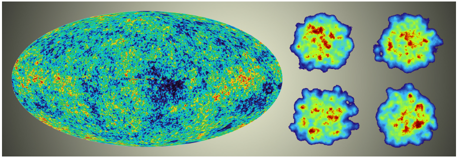



NASA




P. Sorensen, C. Shen

similarities:
- explosive character
- Hubble expansion
- chemical freeze-out (nucleosynthesis (180 s) - hadronization ($30 \times 10^{-24}$ s))
- kinetic freeze-out (charge recombination - microwave radiation, 380000 years - freeze-out of hadron momenta, $45 \times 10^{-24}$ s)
- fluctuations in the initial state



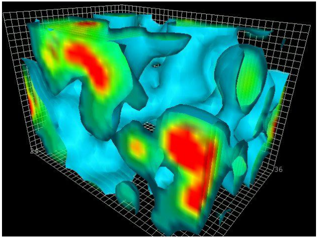
R. Tribble, A. Burrows et al., Implementing the 2007 Long Range Plan

differences:
- slower expansion rate by 18 orders of magnitude
- expansion controlled by gravity vs. expansion controlled by pressure gradients
- larger timescale ($10^9$ years - 10 ys)
- larger distances (ly - 10 fm)

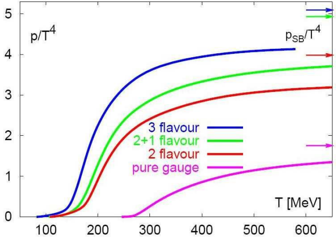
# ROLE OF THE QCD EQUATION OF STATE

crossover vs. first order phase transition

- description of non-perturbative phenomena in QCD is very complicated
$\Rightarrow$ numerical simulations are performed on discretised space-time lattices: lattice QCD (lQCD)
- precise results for vanishing baryon chemical potential
- problems to include the finite baryon chemical potential
- QGP properties:
- "cross-over" phase transition at $T = T_C \approx 155\,\mathrm{MeV}$
- hypothesis of critical point for finite baryon chemical potential
- in the energy range available at RHIC and LHC departures from the Stefan-Boltzmann law can be seen (non-negligible interactions in the plasma)



D. Leinweber (www.physics.adelaide.edu.au)




F. Karsch, E. Laermann, A. Peikert, Phys. Lett. B478, 447 (2000)

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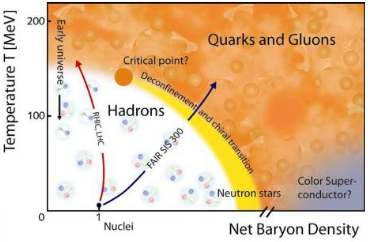


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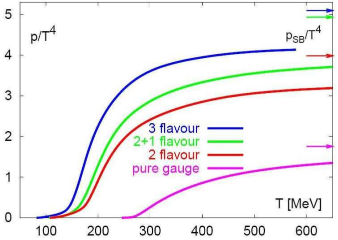
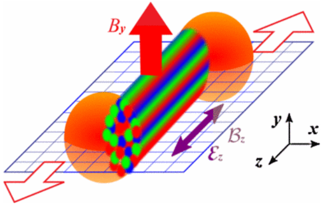
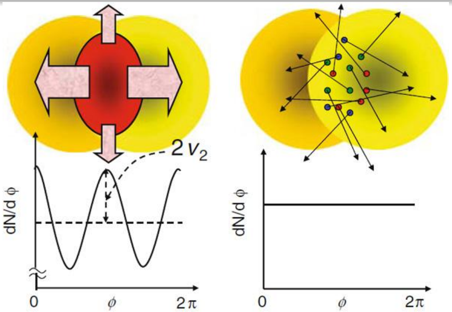



F. Karsch, E. Laermann, A. Peikert, Phys. Lett. B478, 447 (2000)

# "RHIC serves the perfect fluid" (U.Heinz)




K. Fukushima, D. E. Kharzeev, H. J. Warringa, Phys. Rev. Lett. 104, 212001




T. Hirano, N. van der Kolk, A.Bilandzic, Lect.Notes Phys. 785 (2010) 139-178

only indirect studies of QGP are possible
⇒ measurements of the momenta and energies of emitted particles

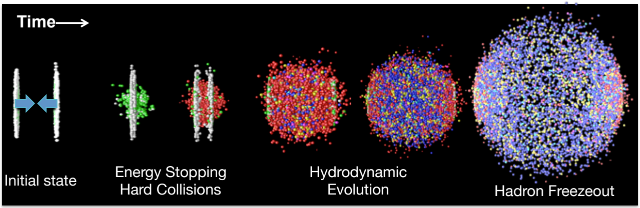
- strong elliptic flow ($v_2$) at RHIC and the LHC
  ⇒ collective behavior
  ⇒ strongly-interacting system
- good description in terms of perfect fluid hydrodynamics

$$\frac{dN}{d\phi} = \frac{N}{2\pi}\left[1 + 2v_1\cos(\phi) + 2v_2\cos(2\phi) + ...\right]$$

$$v_n(p_T, y) = \frac{\int d\phi \cos(n\phi)\frac{dN}{d\phi}}{\int d\phi \frac{dN}{d\phi}} \equiv \langle\cos(n\phi)\rangle$$

**early perfect-fluid calculations included the first-order phase transition**
**→ elliptic flow is insensitive to EOS**

# "Standard model" of evolution of matter produced in heavy-ion collisions




T. K. Nayak, Lepton-Photon 2011 Conference

- initial conditions including fluctuations ($0 < \tau_0 \lesssim 1\,\mathrm{fm}$)
- non-equilibrium phase and thermalization (→hydrodynamization) of matter ($\tau_0 < \tau \lesssim 1\,\mathrm{fm}$)

  ⇒ emission of hard probes: heavy quarks, photons, jets
- hydrodynamic expansion (expansion and cooling) ($1\,\mathrm{fm} \lesssim \tau \lesssim 10\,\mathrm{fm}$)

> microscopic description of such a many body system is very complicated
> ⇓
> effective description in terms of fluid mechanics

- phase transition from QGP to hadron gas (encoded in the equation of state)
- freeze-out and free streaming of hadrons ($10\,\mathrm{fm} \lesssim \tau$)

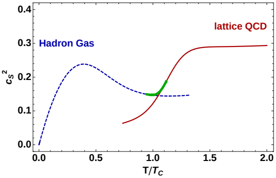
**main assumption: system is in local thermal equilibrium**

- conservation of the baryon number (and other conserved charges), energy and momentum

$$\partial_\mu N^\mu = 0 \qquad N^\mu \equiv n u^\mu$$
$$\partial_\mu T^{\mu\nu}_{id} = 0 \qquad T^{\mu\nu}_{id} \equiv \mathcal{E} u^\mu u^\nu - \Delta^{\mu\nu}\mathcal{P}$$

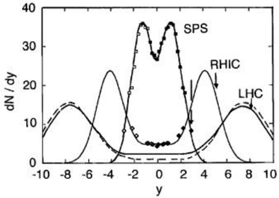
$$\partial_\mu \equiv \frac{\partial}{\partial x^\mu} \qquad \Delta^{\mu\nu} \equiv g^{\mu\nu} - u^\mu u^\nu$$

- 6 independent variables: $(\mathcal{E}, \mathcal{P}, n, u^\mu(3))$
- 6 equations (equation of state $\mathcal{E}(n, \mathcal{P})$)




M. Chojnacki, W. Florkowski, Acta Phys.Polon. B38 (2007) 3249-3262

- in ultra relativistic collisions we may neglect the baryon number conservation




R. Kuiper and G. Wolschin, Annalen Phys. 16, 67 (2007)

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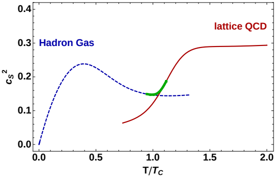


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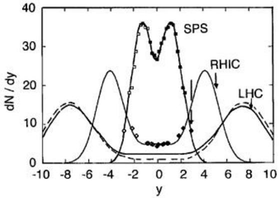
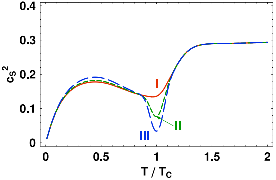
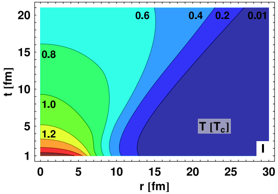


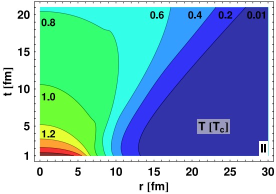

R. Kuiper and G. Wolschin, Annalen Phys. 16, 67 (2007)

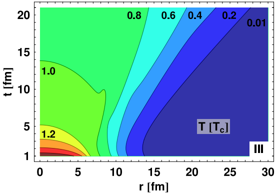
M. Chojnacki, W. Florkowski, Acta Phys.Polon. B38 (2007) 3249, isotherms depending on the form of EOS

EOS can be checked experimentally by looking at the HBT correlations that give information about the space-time extensions of the system




further evidence from complete hydro simulations:
W. Broniowski, M. Chojnacki, WF, A. Kisiel, PRL 101 (2008) 022301
P. Bozek, I. Wyskiel, PRC 79 (2009) 044916; P. Bozek, PRC 83 (2011) 044910

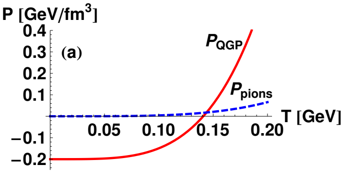
massless gluons and quarks (Stefan-Boltzmann limit)

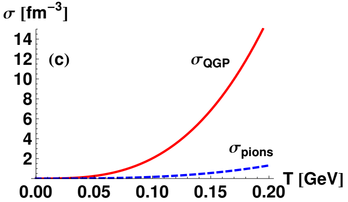
$$\varepsilon_g = \frac{16\pi^2}{30} T^4, \quad P_g = \frac{1}{3}\varepsilon_g, \quad \varepsilon_q + \varepsilon_{\bar q} = 6N_f \left( \frac{7\pi^2}{120} T^4 + \frac{1}{4}\mu^2 T^2 + \frac{1}{8\pi^2}\mu^4 \right), \quad P_q + P_{\bar q} = \frac{1}{3}\left(\varepsilon_q + \varepsilon_{\bar q}\right)$$

$\mu$ is one third of the baryon chemical potential $\mu_B$, $\mu = \frac{1}{3}\mu_B$

gas of quarks and gluons, $B$ – bag constant

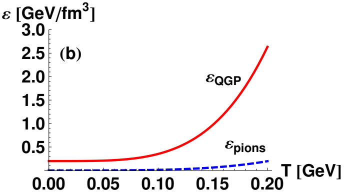
$$\varepsilon_{\rm qgp} = \varepsilon_g(T) + \varepsilon_q(T,\mu) + \varepsilon_{\bar q}(T,\mu) + B, \quad P_{\rm qgp} = P_g(T) + P_q(T,\mu) + P_{\bar q}(T,\mu) - B$$




first order phase transition, $c_s^2 \to 0$ at the critical temperature defined by the condition $P_{\rm qgp} = P_{\rm pions}$

# Early Universe evolution




WF, NPA 853 (2011) 173
isentropic expansion of the Universe
Friedmann equation

$$\frac{d\varepsilon_R}{dt} = -3\sqrt{\frac{8\pi G\varepsilon_R}{3}}(\varepsilon_R + P_R)$$

$\varepsilon_R$ i $P_R$ – total energy density and pressure, combined strong and electroweak sectors
$\varepsilon_R = \varepsilon + \varepsilon_{\rm ew}$, $P_R = P + P_{\rm ew}$

$$\varepsilon_{\rm ew} = g_{\rm ew}\frac{\pi^2}{30}T^4, \quad P_{\rm ew} = g_{\rm ew}\frac{\pi^2}{90}T^4, \quad \sigma_{\rm ew} = \frac{4\varepsilon_{\rm ew}}{3T}$$

temperature can be taken as as independent variable

$$\left[c_s^{-2}\sigma + 3\sigma_{\rm ew}\right]\frac{dT_R}{dt} = -3\sqrt{\frac{8\pi G(\varepsilon + \varepsilon_{\rm ew})}{3}}(\varepsilon + \varepsilon_{\rm ew} + P + P_{\rm ew})$$

# Early Universe evolution

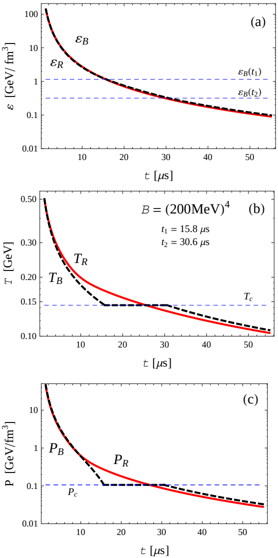
initial condition is set for the temperature $T_R(t_0) = 500\,\mathrm{MeV}$ corresponding energy density

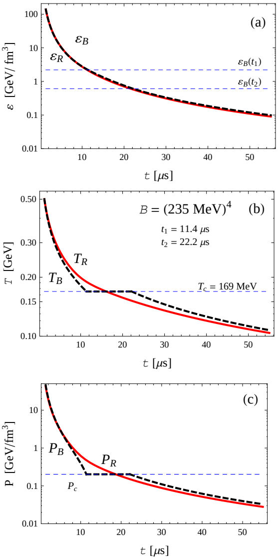
$$\varepsilon_{R,0} = \varepsilon(T_R(t_0)) + \varepsilon_{\rm ew}(T_R(t_0)) = 138\,\mathrm{GeV/fm^3}$$

assuming that for $t < t_0$ the evolution is dominated by radiation one gets

$$t_0 = \sqrt{\frac{3}{32\pi G \varepsilon_{R,0}}} = 1.35\,\mu s$$

# Early Universe evolution

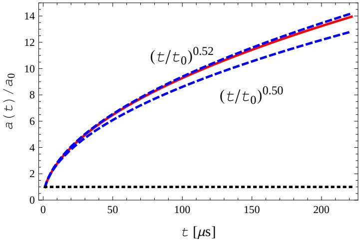


# Early Universe evolution

time evolution of the scale factor $a(t)$

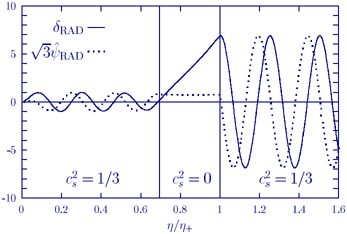
$$a(t) = a(t_0)\exp\left[\int_{t_0}^{t}\sqrt{\frac{8\pi G\varepsilon_R(t')}{3}}\,dt'\right]$$




$$\frac{a(t)}{a(t_0)} = (t/t_0)^p, \quad p = 0.517$$

$H$ – Hubble constant, $H = \frac{da}{a\,dt} = \frac{p}{t}$

Ch. Schmid, D. Schwarz, P. Widerin, PRD59 (1999) 043517




The time evolution of the density contrast and the peculiar velocity of the radiation during the QCD transition in the bag model

not confirmed for realistic QCD EOS

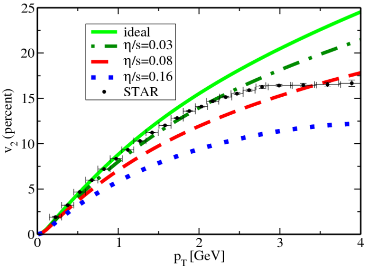
VISCOSITY MATTERS

# QGP as almost perfect fluid

- quantum mechanics (Danielewicz, Gyulassy, 1985) / AdS/CFT (Kovtun, Son, Starinets, 2005) lower bound on viscosity $\eta/\mathcal{S} > 1/4\pi$

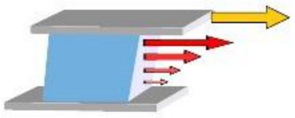
  $\Rightarrow$ one should use relativistic dissipative hydrodynamics

  $\Rightarrow$ better description (assuming small $\eta/\mathcal{S} = \text{const}$)



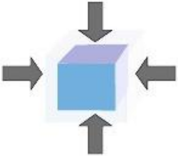

P. Romatschke and U. Romatschke, Phys. Rev. Lett. 99, 172301 (2007)

shear viscosity $\eta$
$\Downarrow$
reaction to a change of shape



$$\pi^{\mu\nu}_{Navier-Stokes} \propto \eta \; (\partial^{\langle\mu} u^{\nu\rangle})$$

bulk viscosity $\zeta$
$\Downarrow$
reaction to a change of volume



$$\Pi_{Navier-Stokes} \propto \zeta \; (\partial_\mu u^\mu)$$

bulk viscosity and pressure vanish for conformal fluids

# Pitch drop experiment

John Mainstone (Wikipedia)

**Pitch**

Start 1927

$1^{st}$ drop 1938

$8^{th}$ drop 2000

$\eta \sim 2\,10^8$ Pa s

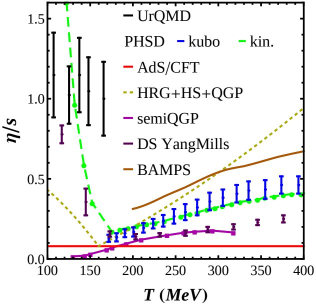
$\sim 10^{11}\eta_{H_2O}$

Wikipedia: The ninth drop touched the eighth drop on 17 April 2014. However, it was still attached to the funnel. On 24 April 2014, Prof. White decided to replace the beaker holding the previous eight drops before the ninth drop fused to them. While the bell jar was being lifted, the wooden base wobbled and the ninth drop snapped away from the funnel.

$$\eta_{\rm qgp} > \eta_{\rm pitch}$$

$\eta_{\rm qgp} \sim 10^{11}$ Pa s, $(\eta/s)_{\rm qgp} < 3/(4\pi)\hbar$ (from experiment)

- $\eta/S$ reaches minimum in the region of the phase transition




S. I. Finazzo et al., arXiv:1412.2968

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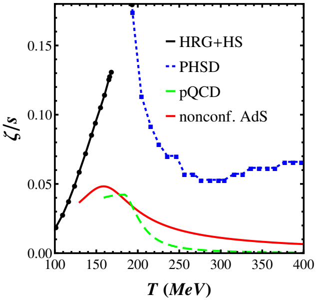


J. Noronha-Hostler

see also: WF, K. Tywoniuk, N. Su, R. Ryblewski, APPB 47 (2016) 1833, arXiv:1509.01242, calculations for the Gribov-Zwanziger plasma

# Dissipative hydrodynamics

**the system is never in local equilibrium (diffusion, friction, heat conduction,...)**

- energy-momentum conservation

$$\partial_\mu T^{\mu\nu}_{vis} = 0 \qquad T^{\mu\nu}_{vis} = \mathcal{E} u^\mu u^\nu - \Delta^{\mu\nu}(\mathcal{P} + \Pi) + \pi^{\mu\nu}$$

- number of equations: 5 + 6 ($\mathcal{E}, \mathcal{P}, u^\mu(3), \Pi, \pi^{\mu\nu}(5)$)
- number of equations: 4 + 1 (equation of state $\mathcal{E}(\mathcal{P})$)
- we need 6 extra equations - different methods possible

$$\dot{\Pi} + \frac{\Pi}{\tau_\Pi} = -\beta_\Pi \theta, \qquad \theta = \partial_\mu u^\mu$$

$$\dot{\pi}^{\langle\mu\nu\rangle} + \frac{\pi^{\mu\nu}}{\tau_\pi} = 2\beta_\pi \sigma^{\mu\nu}, \qquad \sigma^{\mu\nu} \text{- shear tensor constructed from the velocity field } u^\mu$$

**Navier-Stokes equations** — algebraic equations — $T, u^\mu$ are the only hydrodynamic variables

kinetic coefficients: $\tau_\Pi \beta_\Pi = \zeta \rightarrow$ bulk viscosity, $\tau_\pi \beta_\pi = \eta \rightarrow$ shear viscosity

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$$\dot{\Pi} + \frac{\Pi}{\tau_\Pi} = -\beta_\Pi \theta - \delta_{\Pi\Pi} \Pi \theta + \lambda_{\Pi\pi} \pi^{\mu\nu} \sigma_{\mu\nu}$$

$$\dot{\pi}^{\langle\mu\nu\rangle} + \frac{\pi^{\mu\nu}}{\tau_\pi} = 2\beta_\pi \sigma^{\mu\nu} + 2\pi_\gamma^{\langle\mu} \omega^{\nu\rangle\gamma} - \delta_{\pi\pi} \pi^{\mu\nu} \theta - \tau_{\pi\pi} \pi_\gamma^{\langle\mu} \sigma^{\nu\rangle\gamma} + \lambda_{\pi\Pi} \Pi \sigma^{\mu\nu}$$

**Israel-Stewart equations** — $\Pi, \pi^{\mu\nu}$ promoted to dynamic variables — non-hydrodynamic modes are introduced with the appropriate relaxation times $\tau_\Pi, \tau_\pi$

$\tau_\Pi \beta_\Pi = \zeta \rightarrow$ bulk viscosity, $\tau_\pi \beta_\pi = \eta \rightarrow$ shear viscosity

**the system is never in local equilibrium (diffusion, friction, heat conduction,...)**

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$$\dot{\pi}^{\langle\mu\nu\rangle} + \frac{\pi^{\mu\nu}}{\tau_\pi} = 2\beta_\pi \sigma^{\mu\nu} + 2\pi_\gamma^{\langle\mu}\omega^{\nu\rangle\gamma} - \delta_{\pi\pi}\pi^{\mu\nu}\theta \underline{- \tau_{\pi\pi}\pi_\gamma^{\langle\mu}\sigma^{\nu\rangle\gamma} + \lambda_{\pi\Pi}\Pi\sigma^{\mu\nu}}$$

**New approaches** (extra terms allowed by symmetries, shear-bulk coupling $\eta - \zeta$)

$\tau_\Pi \beta_\Pi = \zeta \rightarrow$ bulk viscosity, $\tau_\pi \beta_\pi = \eta \rightarrow$ shear viscosity

**Denicol et al.** — simultaneous expansion in the Knudsen number and inverse Reynolds number

**in the kinetic theory the basic quantity is the one-particle phase-space distribution function**

- perfect-fluid hydrodynamics: local thermal equilibrium

$$f(x,p) = f_{\rm iso}\left(\frac{p^\mu u_\mu}{T(x)}\right) \quad \Rightarrow \quad T^{\mu\nu}_{id} = \int dP\, p^\mu p^\nu f(x,p)$$

- dissipative hydrodynamics: linear deviations from local equilibrium

$$f(x,p) = \underbrace{f_{\rm iso}\left(\frac{p^\mu u_\mu}{T(x)}\right)}_{\rm LO} + \underbrace{\delta f(x,p)}_{\rm NLO} \quad \Rightarrow \quad T^{\mu\nu}_{vis} = T^{\mu\nu}_{id} + \delta T^{\mu\nu}_{id}$$

URHIC $\rightarrow$ EXTREME SPACETIME SCALES
very small systems
very large gradients
very fast expansion
$\Downarrow$
dissipative corrections are substantial

**standard dissipative hydrodynamics assumes that the system is always close to local equilibrium, this is in contrast with microscopic calculations showing that plasma is highly anisotropic in the momentum space (at the early stages of the evolution)**

# NEW DEVELOPMENTS

**Phenomenological formulation**

R. Ryblewski, WF

PRC 83, 034907 (2011), JPG 38 (2011) 015104

1. energy-momentum conservation
   $\partial_\mu T^{\mu\nu} = 0$
2. ansatz for the entropy source, e.g.,
   $\partial(\sigma U^\mu) \propto (\lambda_\perp - \lambda_\parallel)^2/(\lambda_\perp \lambda_\parallel)$

**Kinetic-theory formulation**

M. Martinez, M. Strickland

NPA 848, 183 (2010), NPA 856, 68 (2011)

1. first moment of the Boltzmann equation = energy-momentum conservation
2. zeroth moment of the Boltzmann equation = specific form of the entropy source

**3.** Generalized form of the equation of state based on the Romatschke-Strickland (RS) form

generalization of equilibrium/isotropic distributions, frequently used in the studies of anisotropic quark-gluon plasma (here as a modified Boltzmann distribution in the local rest frame)

$$f_{RS} = \exp\left(-\sqrt{\frac{p_\perp^2}{\lambda_\perp^2} + \frac{p_\parallel^2}{\lambda_\parallel^2}}\right) = \exp\left(-\frac{1}{\lambda_\perp}\sqrt{p_\perp^2 + x\, p_\parallel^2}\right) = \exp\left(-\frac{1}{\Lambda}\sqrt{p_\perp^2 + (1+\xi)\, p_\parallel^2}\right)$$

anisotropy parameter $x = 1 + \xi = \left(\frac{\lambda_\perp}{\lambda_\parallel}\right)^2$ and transverse-momentum scale $\lambda_\perp = \Lambda$

**4.** Energy-momentum tensor (with single anisotropy parameter)

$$T^{\mu\nu} = (\varepsilon + P_\perp)\, U^\mu U^\nu - P_\perp\, g^{\mu\nu} - (P_\perp - P_\parallel) Z^\mu Z^\nu$$

$\varepsilon(\sigma, x)$ — energy density, $P_\perp(\sigma, x)$ — transv. pressure, $P_\parallel(\sigma, x)$ — long. pressure
alternatively one may use: $\varepsilon(\Lambda, \xi)$, $P_\perp(\Lambda, \xi)$, $P_\parallel(\Lambda, \xi)$

$U$ — flow four-vector, $Z$ — beam four-vector, $U^2 = 1$, $Z^2 = -1$, $U \cdot Z = 0$

this form of $T^{\mu\nu}$ follows from the covariant version of RS

$$f_{RS} = \exp\left(-\frac{1}{\Lambda}\sqrt{(p\cdot U)^2 + \xi\,(p\cdot Z)^2}\right), \quad U = (t/\tau, 0, 0, z/\tau), \quad Z = (z/\tau, 0, 0, t/\tau)$$

**5.** Several applications have been made to describe the heavy-ion data within this framework

Kinetic-theory formulation

Perturbative approach

Bazov, Heinz, Strickland
PRC 90, 044908 (2014)

$$f = f_{RS} + \delta f$$

- the leading order is still described by the Romatschke-Strickland form (accounting for the difference between the longitudinal and transverse pressures)
- advanced methods of traditional viscous hydrodynamics are used to restrict the form of the correction $\delta f$ and to derive aHydro equations — non-trivial dynamics included in the transverse plane and, more generally, in (3+1)D case

Molnar, Niemi, Rischke
PRD 93, 114025 (2016), arXiv:1606.09019

Non-perturbative approach

Nopoush, Ryblewski, Strickland,Tinti, WF

$$f = f_{\rm aniso} + ...$$

- all effects due to anisotropy included in the leading order, in the generalised RS form

1. (1+1)D conformal case, two anisotropy parameters
2. (1+1)D non-conformal case, two anisotropy parameters + one bulk parameter
3. full (3+1)D case, five anisotropy parameters + one bulk parameter (shear tensor and bulk pressure)

Boost-invariant and cylindrically symmetric expansion, (1+1)D non-perturbative approach
as much as possible, the momentum anisotropy is included in the leading order

$$f(x,p) = f_{\rm aniso} = f_{\rm iso}\left(\frac{\sqrt{p^\mu \Xi_{\mu\nu} p^\nu}}{\lambda}\right)$$

1. Conformal case, two anisotropy parameters
   L. Tinti, WF, Phys.Rev. C89 (2014) 034907

$$\Xi^{\mu\nu} = U^\mu U^\nu + \xi^{\mu\nu}$$
$$u_\mu \xi^{\mu\nu} = 0 \qquad \xi^\mu_\mu = 0$$
$$\xi^{\mu\nu} = \mathrm{diag}(0, \boldsymbol{\xi}) \qquad \boldsymbol{\xi} \equiv (\xi_x, \xi_y, \xi_z) \quad \text{(in the local rest frame)}$$

2. Non-conformal case, two anisotropy parameters + one bulk parameter
   M. Nopoush, R. Ryblewski, M. Strickland, Phys. Rev. C 90 (2014) 014908

$$\Xi^{\mu\nu} U^\mu U^\nu + \xi^{\mu\nu} - \Delta^{\mu\nu}\Phi$$
$$u_\mu \xi^{\mu\nu} = 0 \qquad u_\mu \Delta^{\mu\nu} = 0 \qquad \xi^\mu_\mu = 0 \qquad \Delta^\mu_\mu = 3$$
$$\xi^{\mu\nu} = \mathrm{diag}(0, \boldsymbol{\xi}) \qquad \boldsymbol{\xi} \equiv (\xi_x, \xi_y, \xi_z) \quad \text{(in the local rest frame)}$$

equations of motion for $\xi_x, \xi_y, \Phi, \lambda, T, u^r$ for (1+1)d case are obtained by taking moments of the Boltzmann equation in the relaxation time approximation

$$p^\mu \partial_\mu f = p^\mu \frac{u_\mu}{\tau_{\rm eq}}(f^{\rm eq} - f) \qquad \rightarrow \qquad \partial_{\mu_1} \int dP\, p^{\mu_1} ... p^{\mu_{n+1}} f = u_{\mu_1} \int dP\, p^{\mu_1} ... p^{\mu_n} \frac{1}{\tau_{\rm eq}}(f^{\rm eq} - f)$$

0th moment (1 eq.)

$$\partial_\mu N^\mu = \frac{u_\mu}{\tau_{\rm eq}}(N^\mu_{\rm eq} - N^\mu)$$

1st moment (2 eq.)

$$\partial_\mu T^{\mu\nu} = \frac{u_\mu}{\tau_{\rm eq}}(T^{\mu\nu}_{\rm eq} - T^{\mu\nu})$$

Landau matching condition for the energy (1 eq.)

$$u_\mu T^{\mu\nu}_{\rm eq} = u_\mu T^{\mu\nu}$$

2nd moment (2 eq.)

$$X^i_\mu X^i_\nu \partial_\lambda \Theta^{\lambda\mu\nu} = X^i_\mu X^i_\nu \frac{u_\lambda}{\tau_{\rm eq}} \left(\Theta^{\lambda\mu\nu}_{\rm eq} - \Theta^{\lambda\mu\nu}\right)$$

two linear combinations of these equations with $i = 0, 1, 2, 3$

$X, Y$ defined in addition to $U$ and $Z$

...

**3A.** (3+1) dimensional framework for leading order anisotropic hydrodynamics
L. Tinti, Phys. Rev. C92 (2015) 014908

Testing different formulations of leading order anisotropic hydrodynamics
L. Tinti, R. Ryblewski, W. Florkowski, M. Strickland, Nucl. Phys. A946 (2016) 29

$$\Xi^{\mu\nu} = u^\mu u^\nu + \xi^{\mu\nu} - \Delta^{\mu\nu}\Phi$$
$$u_\mu \xi^{\mu\nu} = 0 \qquad \xi^\mu_\mu = 0 \quad \text{(5 parameters in } \xi^{\mu\nu}\text{)}$$

**3B.** Anisotropic matching principle for the hydrodynamic expansion, L. Tinti, arXiv:1506.07164

$$T^{\mu\nu} = \int dP\, p^\mu p^\nu f_{\rm aniso}(x,p) = \int dP\, p^\mu p^\nu f_{\rm iso}\left(\frac{\sqrt{p^\mu \Xi_{\mu\nu} p^\nu}}{\lambda}\right)$$

Instead of looking at the moments we can derive first the equations for the pressure corrections, following DNMR (Denicol, Niemi, Molnar, Rischke) strategy used for viscous hydrodynamics

This is the latest development for the leading order, that may be supplemented by NLO terms following the approach by Heinz et al.

# Exact solutions of the Boltzmann equation

new technique: aHydro and viscous hydro predictions are checked against exact solutions of the Boltzmann kinetic equation in the relaxation time approximation, important constraints on the structure of the hydro equations and the form of the kinetic coefficients

G. Baym, Phys. Lett. B138 (1984)

One dimensional expansion

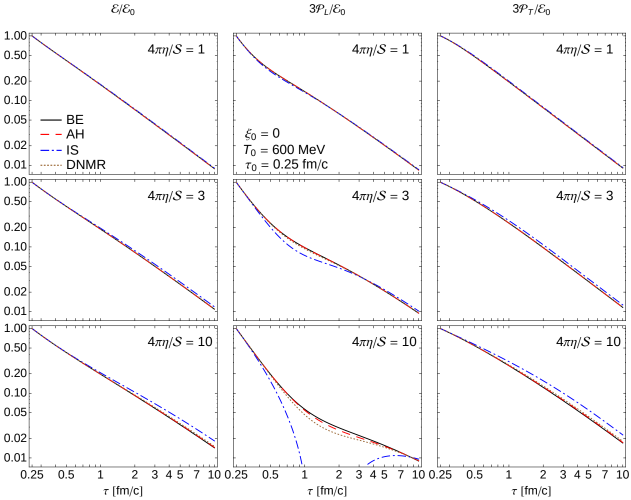
Denicol, Maksymiuk, Ryblewski, Strickland, WF

1. conformal case
2. non-conformal case
3. non-conformal case with quantum statistics
4. mixtures

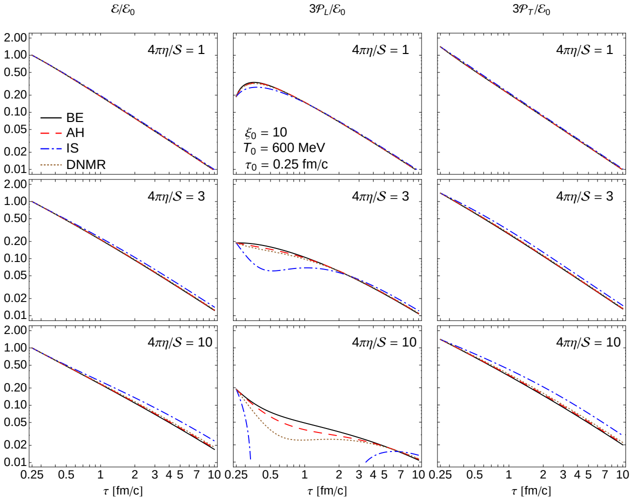
(1+1)D flow with Gubser symmetry

Denicol, Heinz, Martinez, Noronha, Strickland

WF, R. Ryblewski, M. Strickland, Phys.Rev. C88 (2013) 024903
$m = 0$, boost-invariant, transversally homogeneous system, (0+1) case



WF, R. Ryblewski, M. Strickland, Phys.Rev. C88 (2013) 024903
$m = 0$, boost-invariant, transversally homogeneous system, (0+1) case



# BULK PRESSURE EVOLUTION

$$\tau_\Pi \dot{\Pi} + \Pi = -\frac{\zeta}{\tau} - \frac{1}{2}\tau_\Pi \Pi \left[\frac{1}{\tau} - \left(\frac{\dot{\zeta}}{\zeta} + \frac{\dot{T}}{T}\right)\right] \qquad (A)$$

Muronga, Phys. Rev. C69 (2004) 034903
Heinz, Song, Chaudhuri, Phys. Rev. C73 (2006) 034904

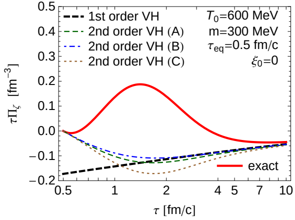
$$\tau_\Pi \dot{\Pi} + \Pi = -\frac{\zeta}{\tau} - \frac{4}{3}\tau_\Pi \Pi \frac{1}{\tau} \qquad (B)$$

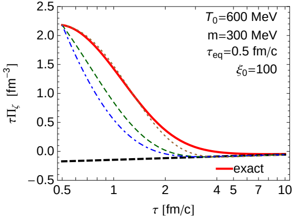
Jaiswal, Bhalerao, Pal, Phys. Rev. C87 (2013) 021901

$$\tau_\Pi \dot{\Pi} + \Pi = -\frac{\zeta}{\tau} \qquad (C)$$

Heinz, Song, Chaudhuri, Phys. Rev. C73 (2006) 034904

WF, E. Maksymiuk, R. Ryblewski, M.Strickland, Phys.Rev. C89 (2014) 054908




exact solution and all 2nd order viscous hydrodynamics variations tend toward the 1st order solution at late times

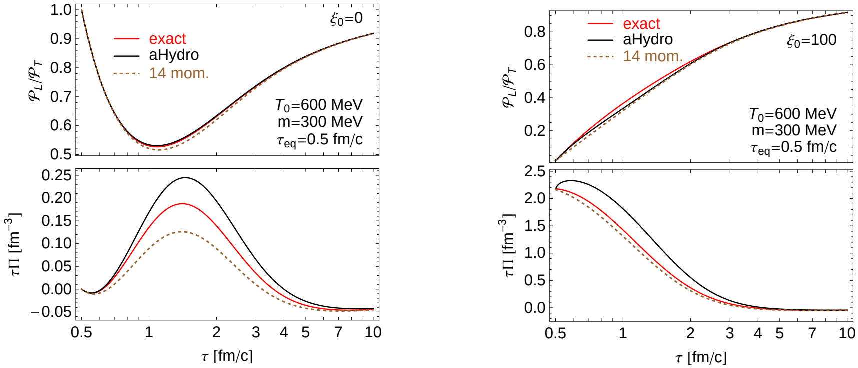
none of the 2nd order viscous hydrodynamics variations seems to qualitatively describe the early-time evolution of the bulk viscous pressure in all cases

there is something incomplete in the manner in which 2nd order viscous hydrodynamics treats the bulk pressure (neglected shear–bulk coupling)

Bulk viscous pressure evolution within full viscous anisotropic hydrodynamics with SHEAR-BULK COUPLING

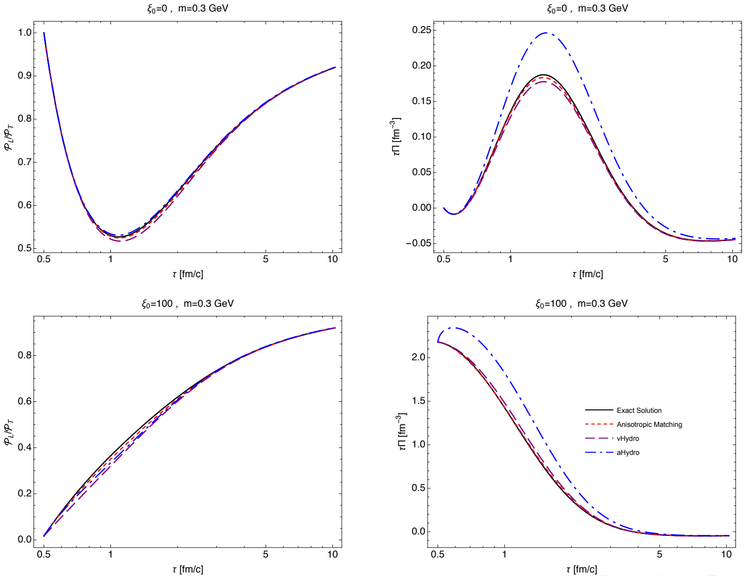
G. Denicol, S. Jeon, C. Gale, Phys.Rev. C90 (2014) 024912

G. Denicol, R. Ryblewski, WF, M.Strickland, Phys.Rev. C90 (2014) 044905




the shear-bulk couplings are extremly important for correct description of the bulk viscous correction

Leonardo Tinti, arXiv:1506.07164, PRC in print



# SUMMARY

- QCD EOS

  known from first principles at $\mu_B = 0$ and restricted by the data analysis of the HBT correlations

  semi hard (large values of $c_s$), faster evolution of the system, shorter time for equilibration of matter produced in heavy-ion collisions

  smooth, precise, background in the Early Universe, does not lead to structures typical for the first order phase transition

- THERMALIZATION (LOCAL EQUILIBRATION) → HYDRODYNAMIZATION

  small shear viscosity to entropy ratio + large gradients (1/t) = large pressure corrections

  AdS/CFT → large pressure corrections

- BULK VISCOSITY

  large at the QCD phase transition, yet to be extracted from heavy-ion collisions, strongly coupled to the shear sector

  to be included (more systematically) in cosmological models or astrophysical calculations?

- RELATIVISTIC VISCOUS HYDRODYNAMICS

  the most popular version of Israel-Stewart has problems (negative pressures and distribution functions), not controlled growth of the shear tensor, especially at the edges of the system, one should use at least a complete 2nd order formulation

  example of anisotropic hydrodynamics shows that one can construct a hydrodynamic theory that is closer to the underlying microscopic theory than the 2nd order formulation