Hydrodynamics at large orders

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based on

1302.0697, 1409.5087, 1503.07514, 1603.05344 + work in progress

Hydrodynamization

M. P. Heller, R. A. Janik and P. Witaszczyk,

Phys. Rev. Lett. 108, 201602 (2012), **1103.3452**

Relativistic hydrodynamics

hydrodynamics is an EFT of the slow evolution of conserved currents in collective media close to equilibrium

DOFs: always local energy density ϵ and local flow velocity u^μ $(u_\nu u^\nu = -1)$

EOMs: conservation eqns $\nabla_\mu \langle T^{\mu\nu}\rangle = 0$ for $\langle T^{\mu\nu}\rangle$ expanded in gradients

$$\langle T^{\mu\nu}\rangle = \epsilon\, u^\mu u^\nu + P(\epsilon)\{\, g^{\mu\nu} + u^\mu u^\nu \,\} - \eta(\epsilon)\,\sigma^{\mu\nu} - \zeta(\epsilon)\{\, g^{\mu\nu} + u^\mu u^\nu \,\}(\nabla\cdot u) + \ldots$$

microscopic input:

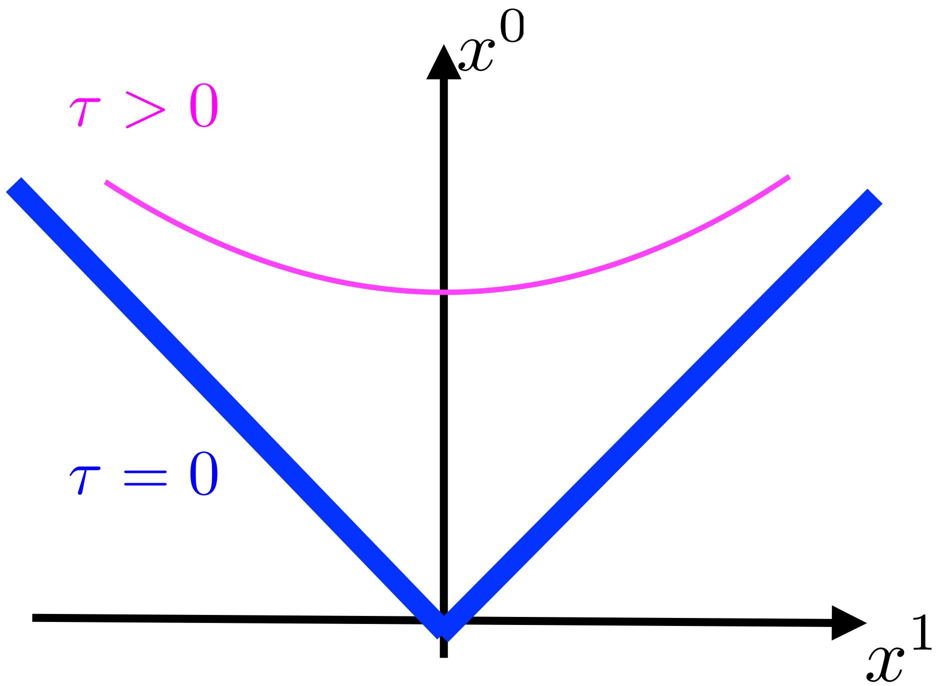
EoS $\left(P(\epsilon) = \frac{1}{3}\epsilon \text{ for CFTs}\right)$

shear viscosity

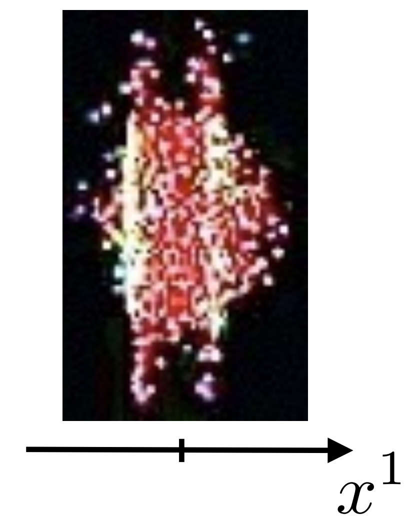
bulk viscosity (vanishes for CFTs)

This talk: behaviour of the gradient expansion at large orders in the number of ∇

Boost-invariant flow [Bjorken 1982]



const x^0 slice:



Boost-invariance: in $\left(\tau \equiv \sqrt{x_0^2 - x_1^2}, \quad y \equiv \operatorname{arctanh}\frac{x_1}{x_0}, \, x_2, \, x_3\right)$ coords no y-dep

In a CFT: $\langle T^\mu_{\ \nu}\rangle = \operatorname{diag}\left\{-\mathcal{E}(\tau), -\mathcal{E} - \tau\,\dot{\mathcal{E}}, \, \mathcal{E} + \frac{1}{2}\tau\,\dot{\mathcal{E}}, \, \mathcal{E} + \frac{1}{2}\tau\,\dot{\mathcal{E}}\right\}$

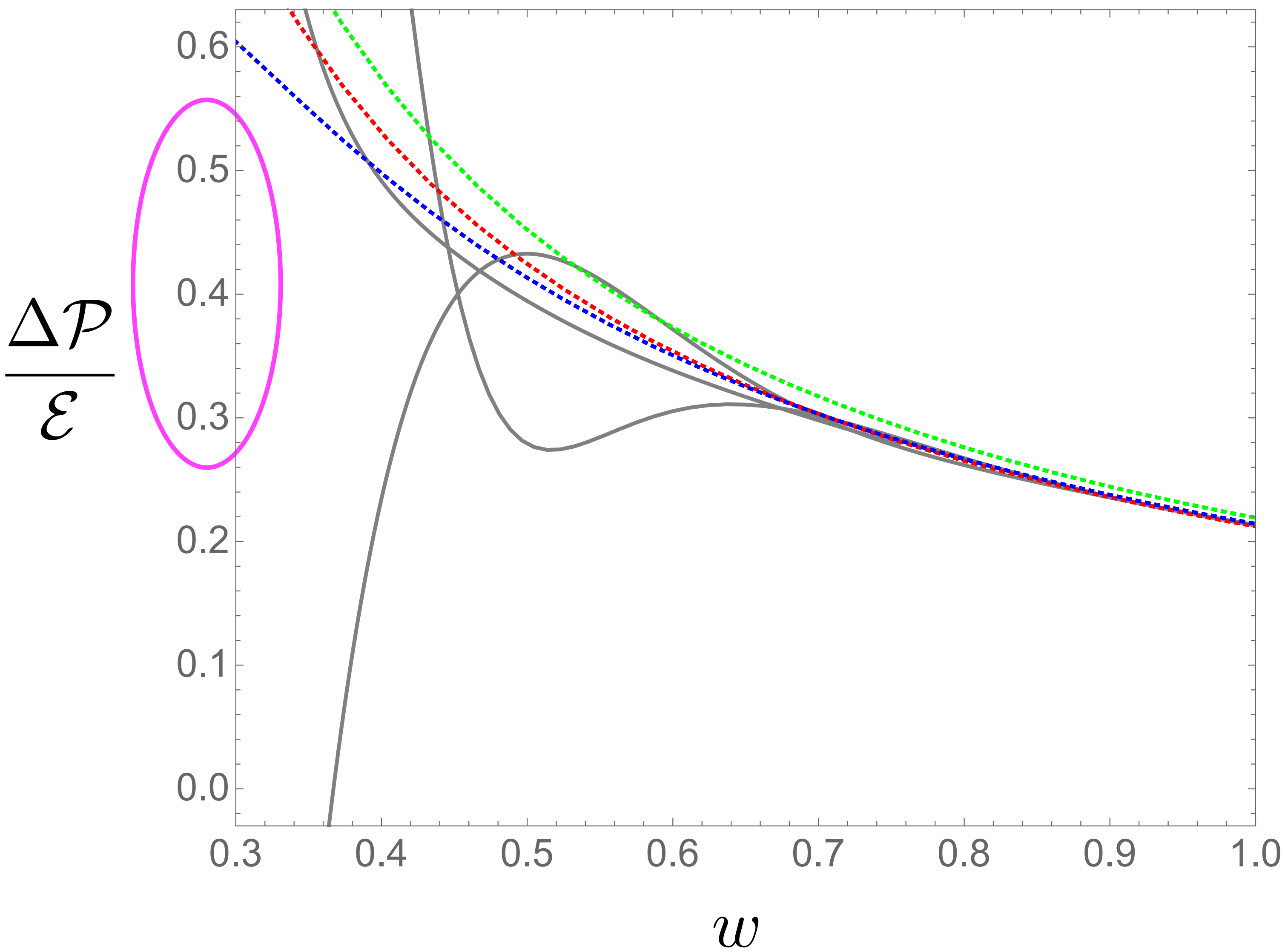
$\Delta\mathcal{P} \equiv \langle T^2_{\ 2}\rangle - \langle T^y_{\ y}\rangle$ $\qquad$ $T \equiv \left(\frac{\mathcal{E}(\tau)}{\frac{3}{8}\pi^2 N_c^2}\right)^{1/4}$

and via scale-invariance $\frac{\Delta\mathcal{P}}{\mathcal{E}}$ is a function of $w \equiv \tau\, T$

Gradient expansion: series in $\frac{1}{w}$.

Hydrodynamization 1103.3452 (see also Chesler & Yaffe 0906.4426, 1011.3562)

Ab initio calculation in N=4 SYM at strong coupling:



Hydrodynamics works despite huge anisotropy captured by $-\eta\,\sigma^{\mu\nu}$

$$\left.\frac{\Delta\mathcal{P}}{\mathcal{E}}\right|_{\text{hydro}} = \frac{0.21}{w} + \frac{0.0069}{w^2} - \frac{0.0049}{w^3} + \ldots$$

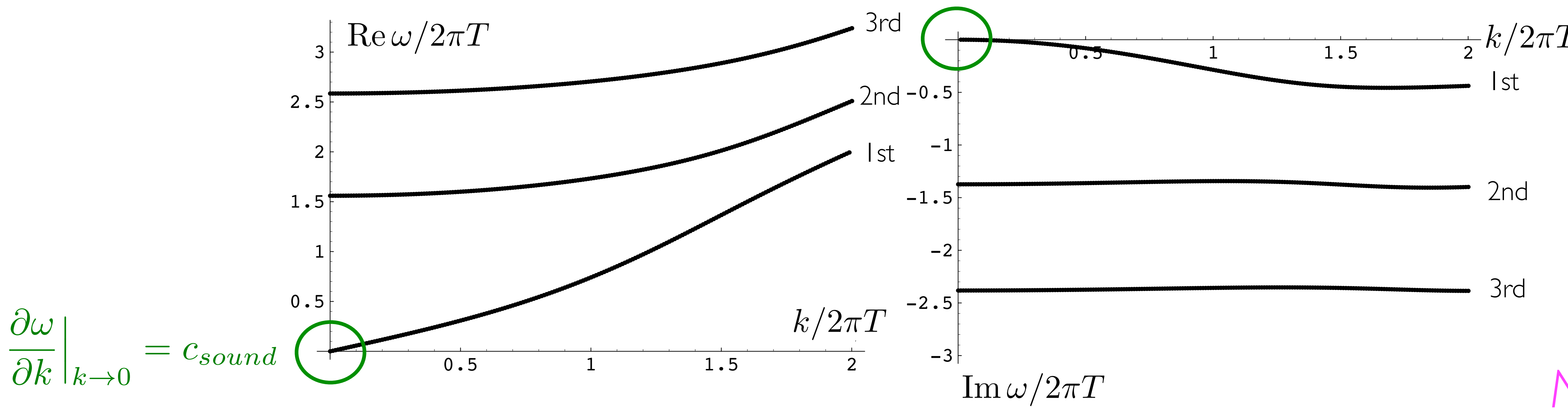
Why hydrodynamization can occur?

M. P. Heller, R. A. Janik and P. Witaszczyk,

Phys. Rev. Lett. 110, 211602 (2013), **1302.0697**

Excitations in strongly-coupled plasmas

see, e.g. Kovtun & Starinets [hep-th/0506184]



N=4 SYM

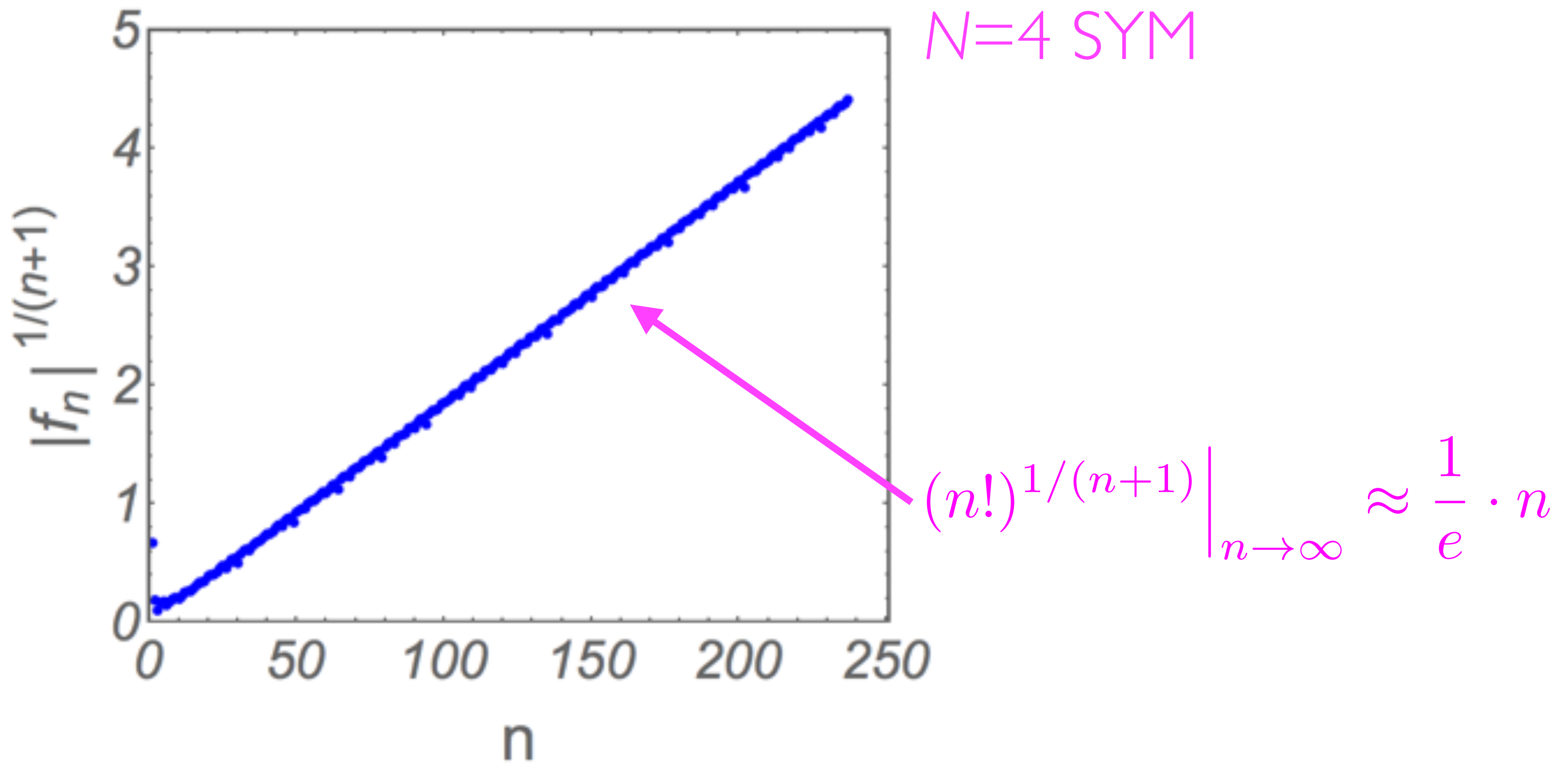
$\omega(k) \to 0$ as $k \to 0$: slowly dissipating modes (hydrodynamic sound waves)

all the rest: far from equilibrium (QNM) modes damped over $t_{therm} = \mathcal{O}(1)/T$

Hydrodynamic gradient expansion is divergent

In 1302.0697 we computed $f(w) \equiv \frac{2}{3} + \frac{1}{6}\frac{\Delta\mathcal{P}}{\mathcal{E}}$ up to $O(w^{-240})$:

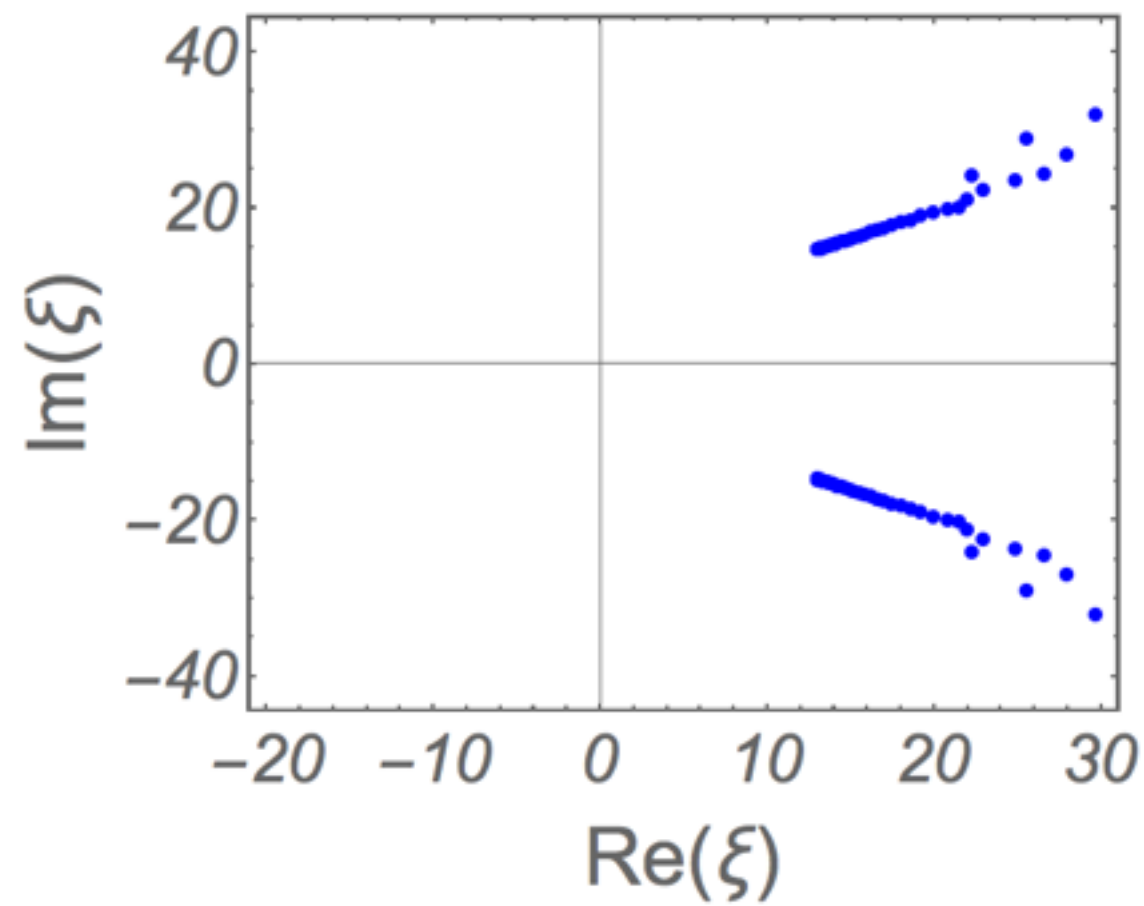
$$f(w) = \sum_{n=0}^{\infty} f_n w^{-n} = $$
$$= \frac{2}{3} + \frac{1}{9\pi} w^{-1} + \ldots$$



Hydrodynamics and QNMs

1302.0697

Analytic continuation of $f_B(\xi) \approx \sum_{n=0}^{240} \frac{1}{n!} f_n \, \xi^n$ revealed the following singularities:



N=4 SYM

Branch cut singularities start at $\frac{3}{2} \, i \, \omega_{QNM_1}$!

Resumming gradient expansion in MIS theory

M. P. Heller, M. Spaliński,

Phys. Rev. Lett. 115, 072501 (2015), **1503.07514**

The boost-invariant MIS theory

1503.07514

$$(\tau_\Pi \mathcal{D} + 1)\, \Pi_{\mu\nu} = -\eta \sigma_{\mu\nu} + \dots$$

$C_\eta = \frac{\eta}{s}$ and $C_{\tau_\Pi} = \tau_\Pi T$

$$f' = -\frac{1}{C_{\tau_\Pi}} + \frac{2}{3\, C_{\tau_\Pi}\, f} + \frac{16}{3\, w} - \frac{16}{9\, w\, f} + \frac{4 C_\eta}{9\, C_{\tau_\Pi}\, w\, f} - \frac{4\, f}{w} + \dots$$

gradient expansion and a QNM

$$f = \sum_{n=0}^{\infty} \frac{f_n}{w^n} + \delta f + \mathcal{O}(\delta f^2)$$

$$\delta f \sim \exp\left(-\frac{3}{2 C_{\tau_\Pi}} w\right) \times \dots$$

general solution has a trans-series form

$$f = \sum_{m=0}^{\infty} (c_{amb} + r)^m \left\{ w^{\frac{C_\eta}{C_{\tau_\Pi}}} \exp\left(-\frac{3}{2 C_{\tau_\Pi}} w \right) \right\}^m \sum_{n=0}^{\infty} a_{m,n} w^{-n}$$

Divergent gradient expansion at weak coupling

M. P. Heller, A. Kurkela & M. Spaliński,

work-in-progress

RTA kinetic theory

Natural language to talk about weakly coupled media is the Boltzmann equation:

$$p^\mu \partial_\mu g(x,p) = C[g(x,p)] \;\; \text{with} \;\; \langle T^{\mu\nu} \rangle (x) = \int_{\text{momenta}} g(x,p)\, p^\mu p^\nu$$

LO $C[g(x,p)]$ for gauge theories is complicated. We will use instead

$$C[g(x,p)] = -\frac{p^\mu u_\mu}{\hat{\mathcal{T}}} \Big\{ g(x,p) - g_0(x,p) \Big\} \;\; \text{with} \;\; g_0(x,p) = e^{\frac{u_\mu p^\mu}{T}}$$

This equation is, typically, highly nonlinear due to $\langle T^{\mu\nu} \rangle \, u_\nu = -\mathcal{E}(T)\, u^\mu$

CFTs: $p^\mu p_\mu = 0$ and $\hat{\mathcal{T}} = \dfrac{\gamma}{T}$. Note that γ can be scaled-away (we set it to 1).

Hydrodynamics in kinetic theory

We search for solution to

$$\partial_\tau g(\tau, p^y, p^\perp) = T(\tau)\left\{g(\tau, p^y, p^\perp) - \exp\left(-\frac{1}{T(\tau)}\sqrt{\tau^2\,(p^y)^2 + (p^\perp)^2}\right)\right\}$$

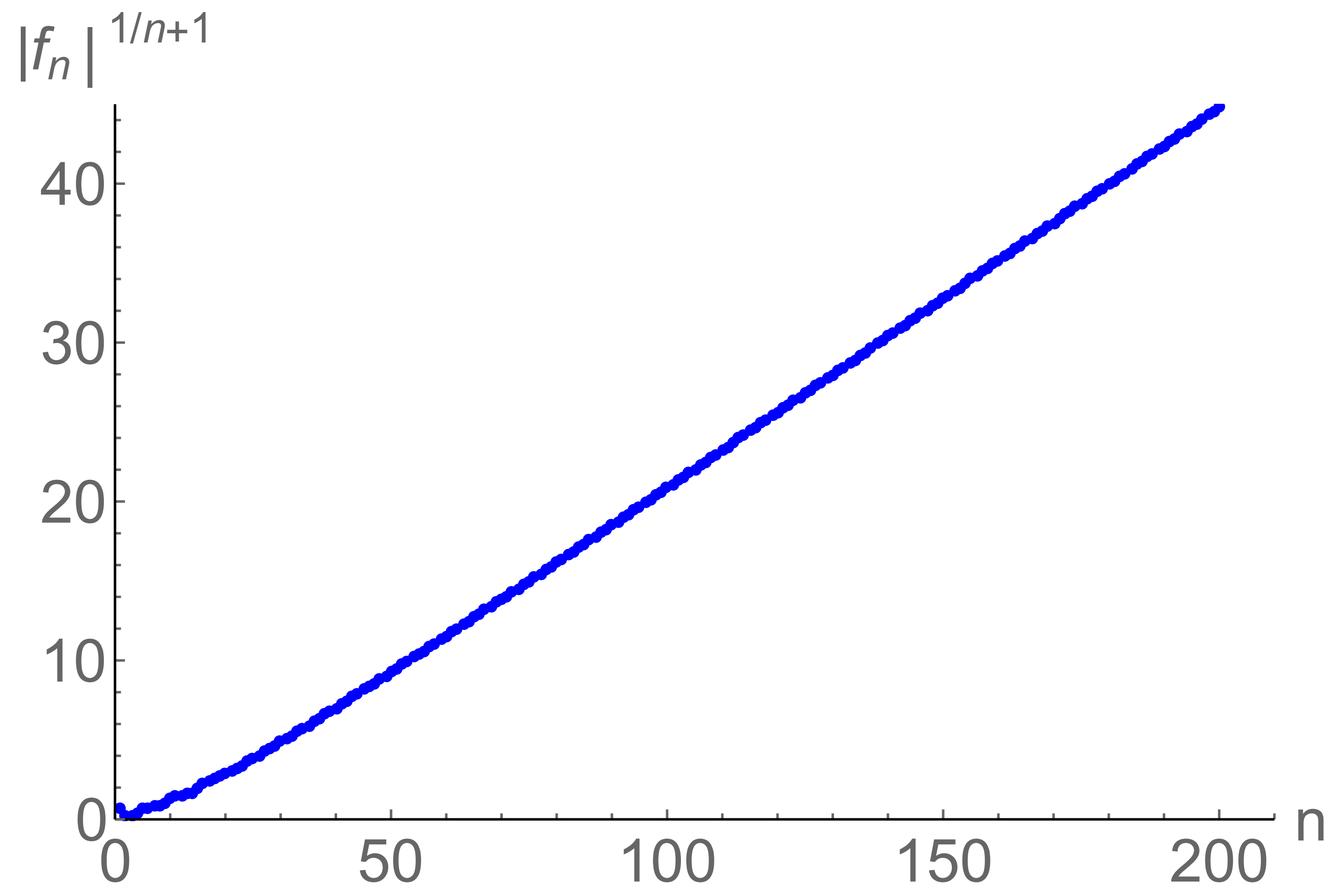
of the form $g(\tau, p^y, p^\perp) = \exp\left(-\frac{1}{T}\sqrt{\tau^2\,(p^y)^2 + (p^\perp)^2}\right)\left\{1 + \sum_{n=1}^{\infty} \boxed{\frac{1}{(\tau T)^n}} g_n\left(\frac{\tau\, p^\eta}{T(\tau)}, \frac{p^\perp}{T(\tau)}\right)\right\}$

$$= \frac{1}{w^n}$$

Landau matching allows us to obtain

$$f(w) = \frac{2}{3} + \frac{1}{6}\frac{\Delta\mathcal{P}}{\mathcal{E}} = \frac{2}{3} + \boxed{\frac{4}{45}}\frac{1}{w} + \frac{16}{945}\frac{1}{w^2} + \ldots$$

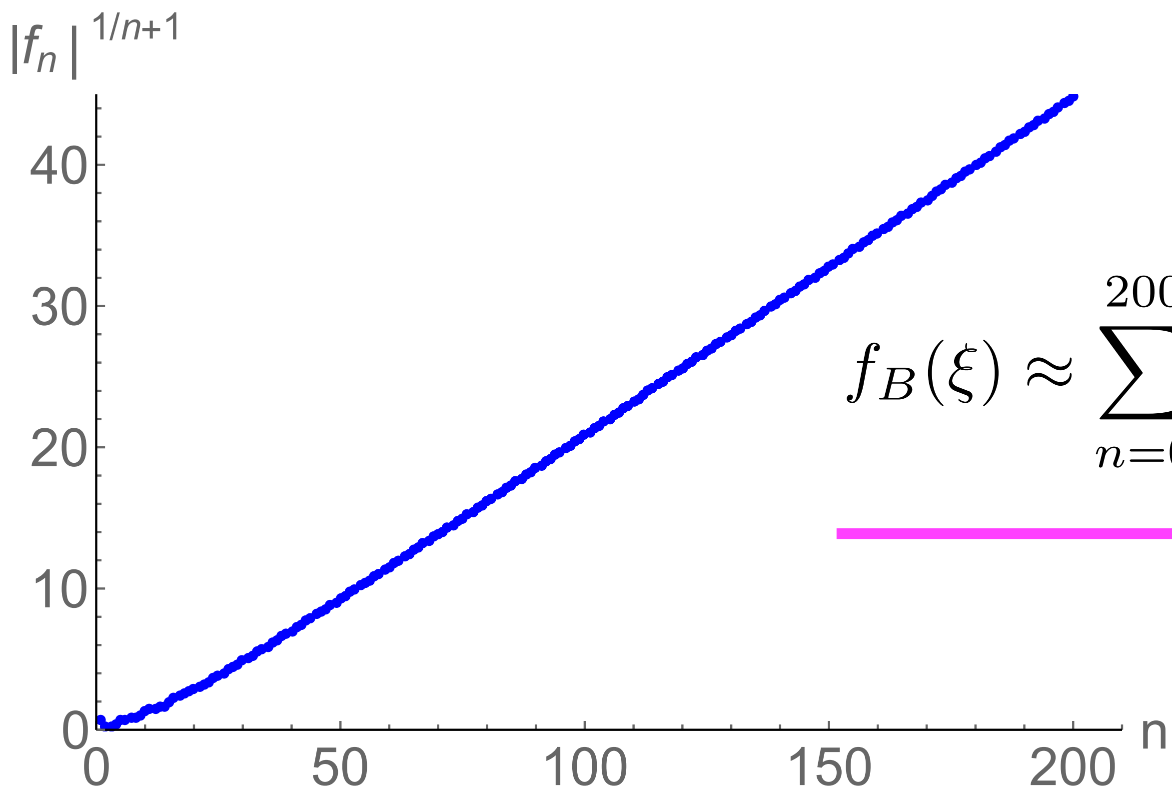
$$\frac{\eta}{s} = \frac{1}{5}\gamma$$



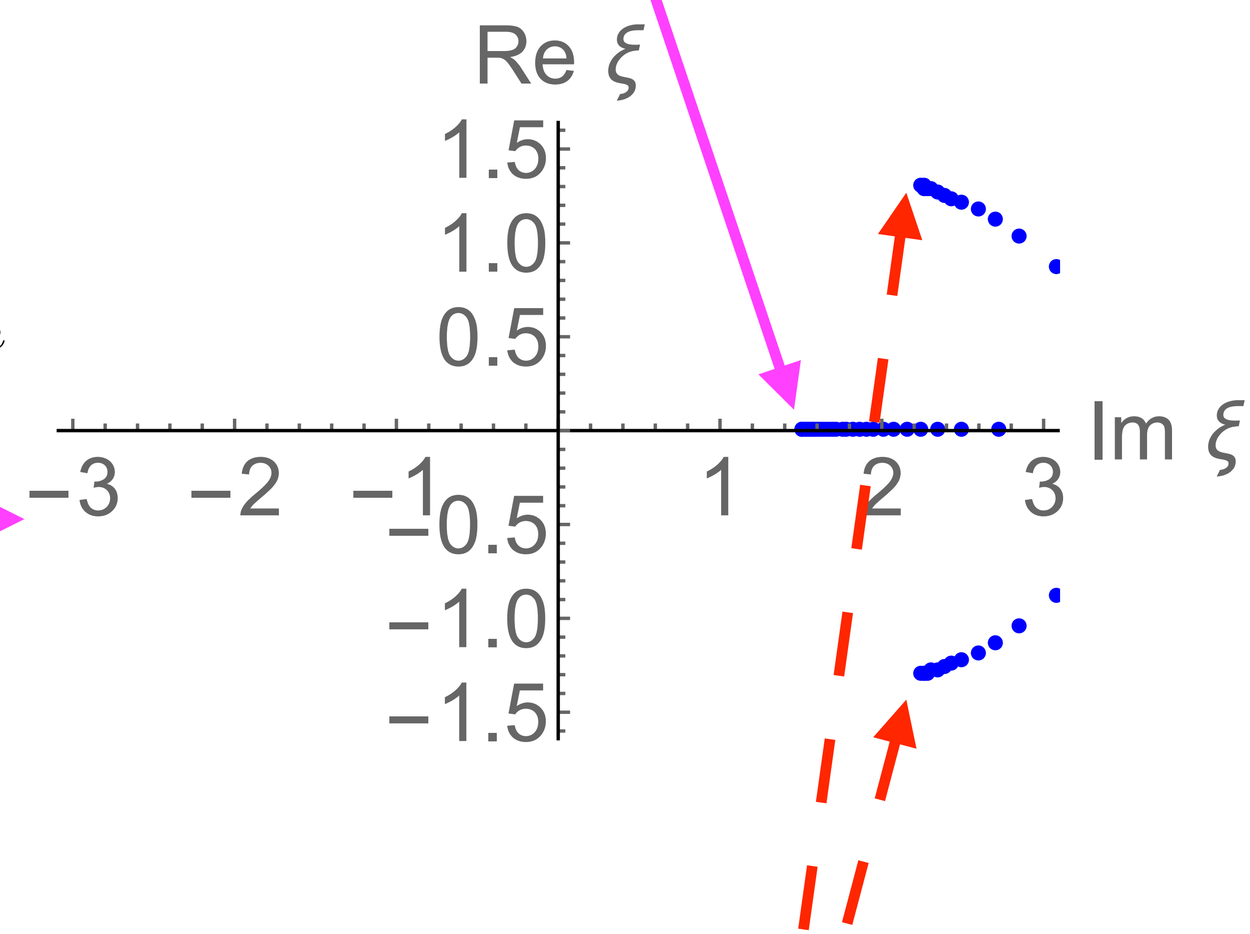
QNM in kinetic theory

$$\xi_{sing} = \frac{3}{2\gamma} \rightarrow \delta f \sim \exp\left(-\frac{3}{2\gamma} w\right) \times \ldots$$

$\surd$ G_R



$$f_B(\xi) \approx \sum_{n=0}^{200} \frac{1}{n!} f_n \, \xi^n$$



$$\delta f \sim \exp\left(-\frac{2.25}{\gamma} \pm \frac{1.3}{\gamma}\, i\right)$$

???

Gradient expansion in FRLW cosmology

A. Buchel, M. P. Heller & J. Noronha

1603.05344

Holographic dual to FRLW cosmology

Another symmetric flow: comoving plasma in a FRLW universe: $ds^2 = -dt^2 + a(t)^2 d\vec{x}^2$

For a CFT, equilibrium state in $\mathbf{R}^{1,3}$ $\xrightarrow{\text{conformal trafo}}$ equilibrium comoving plasma in FRLW:

$$\epsilon = \frac{3}{8}\pi^2 N_c^2 T^4 + \frac{3N_c^2\,\dot{a}^4}{32\pi^2 a^4} \quad \text{and} \quad P = \frac{1}{3}\epsilon - \frac{N_c^2 \dot{a}^2 \ddot{a}}{8\pi^2 a^3} \quad \text{with} \quad T = \frac{\mu}{4\pi a}$$

This is an all-order hydro answer in a CFT. Dual: slicing of AdS-Schwarzschild:

see also Siopsis et al. 0809.3505

$$ds_5^2 = 2dt\,(dr - A dt) + \Sigma^2\, d\boldsymbol{x}^2 \quad \text{with} \quad A = \frac{r^2}{8}\left(1 - \frac{\mu^4}{r^4 a^4}\right) - \frac{\dot{a}}{a} r \quad \& \quad \Sigma = \frac{ar}{2}$$

No entropy production: $r_{hor} = \frac{\mu}{a}$ & $A_{hor} = \frac{\mu^3}{8}$. To introduce it → ~~hydro~~ (see Rozali et al. 1505.03901) / → ~~conformal~~

Hydrodynamic entropy production in FRLW

To ~~conformal~~ we consider: $S_{EH} + \# \int d^5x \sqrt{-g} \left\{ (\partial\phi)^2 - \frac{\Delta(\Delta-4)}{L^2}\phi^2 + \ldots \right\}$ with

$\phi = r^{4-\Delta} J + \ldots$ and we work to LO in $\delta = \frac{J}{T^{4-\Delta}}$

For $\mathcal{N} = 2^*$: $\Delta = 2$ and $J \sim m_b^2$ or $\Delta = 3$ and $J \sim m_f$

We get $\frac{dA_{hor}}{dt} \sim \left(\partial_r \Sigma^3\right) \times \left(\partial_t \phi + A\, \partial_r \phi\right)^2 \Big|_{r=r_{hor}} + O(\delta^3)$

In the rest of my talk we focus on $\left(\partial_t \phi + A\, \partial_r \phi\right) \Big|_{r=r_{hor}}$ alone

Key idea behind 1603.05344 : solve EOM$_\phi$ (to express) in derivatives of $a(t)$

Numerical holography at the precision frontier

We find $\phi \;= \delta \quad a^{4-\Delta}(4-\Delta)\sum_{n=0}^{\infty}\frac{\mathcal{T}_{n,\Delta}[a]}{\mu^n}\,F_{\Delta,n}\left(\frac{\mu}{ar}\right)$

$$\mathcal{T}_{\Delta,n} = a\,\dot{\mathcal{T}}_{\Delta,n-1} + (4-\Delta)\,\dot{a}\,\mathcal{T}_{\Delta,n-1}$$

for $a = e^{H\,t}$:

$$\mathcal{T}_{\Delta,n}[a] = \Gamma(n+4-\Delta)a^n H^n$$

$$\overbrace{0 = F''_{\Delta,n} + \frac{(z^4+3)F'_{\Delta,n}}{z(z^4-1)} + \frac{(\Delta-4)\Delta F_{\Delta,n}}{z^2(z^4-1)}}^{M_{ij}F^j_{\Delta,n}}$$

$$\underbrace{+\frac{8}{z^4-1}\left(F'_{\Delta,n-1} - \frac{3}{2z}F_{\Delta,n-1}\right)}_{-B_i}$$

divergent gradient expansion provided (✓)
$F_{\Delta,n}\left(\frac{\mu}{ar}\right)$ does not decay too fast with n

we solve $M_{ij}F^j_{\Delta,n} = B_i$ + 2 bdry cond
where ij are indices of pseudospectral representation of $z = \dfrac{\mu}{a\,r}$ coordinate

$n_{max} = 300 \longrightarrow$ 150 points & 1000 digits

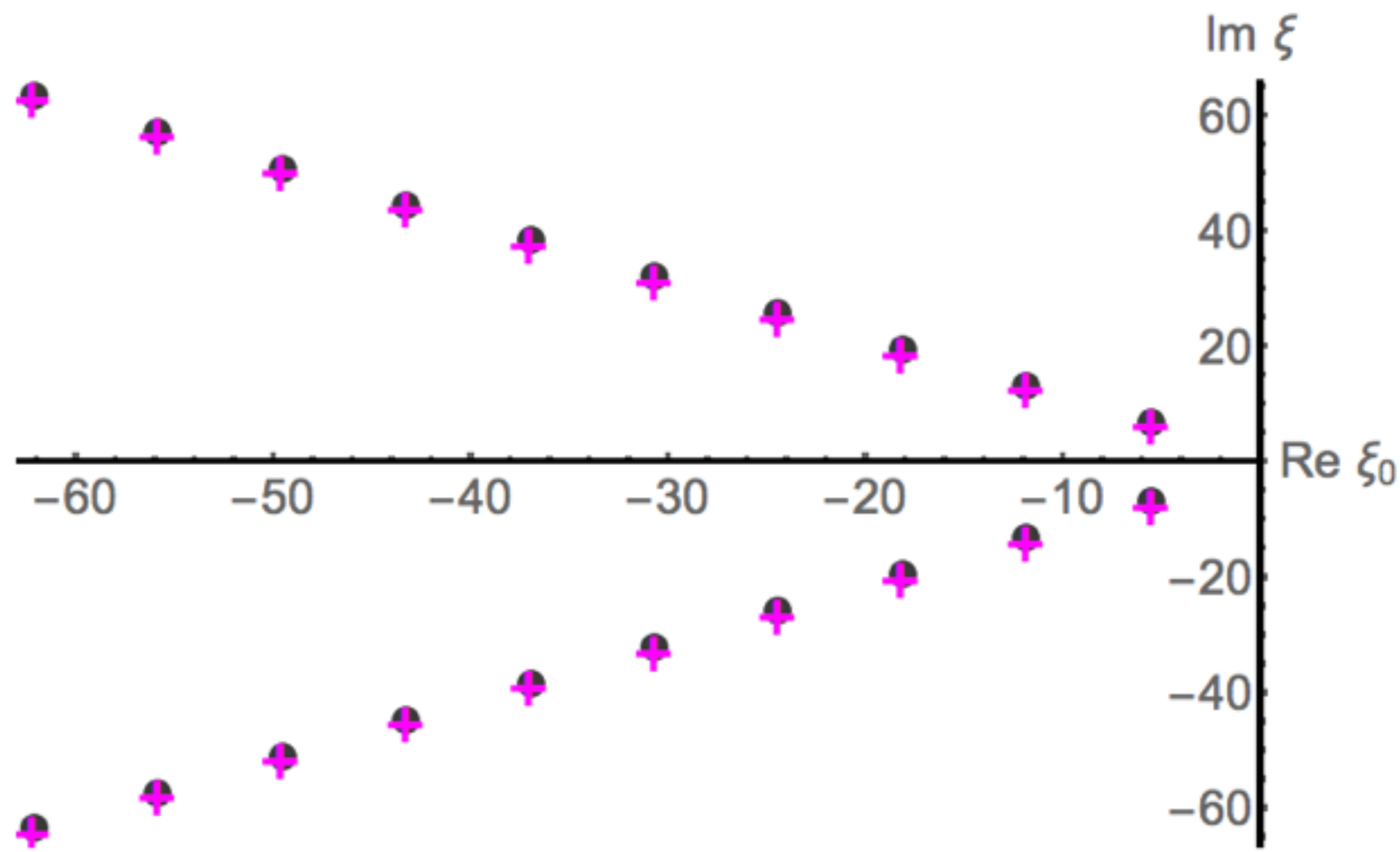
Key result (for $\Delta = 3$)

$$\frac{dA_{hor}}{dt} \sim (\partial_r \Sigma^3) \times (\partial_t \phi + A\,\partial_r \phi)^2 \Big|_{r=r_{hor}} + O(\delta^3) \sim \left(\sum_{n=0}^{\infty} c_n \xi^n\right)^2 + O(\delta^3) \text{ with } \xi = \frac{H}{T}$$

$$\sum_{n=0}^{300} \frac{c_n}{n!} \xi^n \approx \frac{\sum_{m=0}^{150} d_m\, \xi^m}{\sum_{l=0}^{150} e_l\, \xi^l}$$

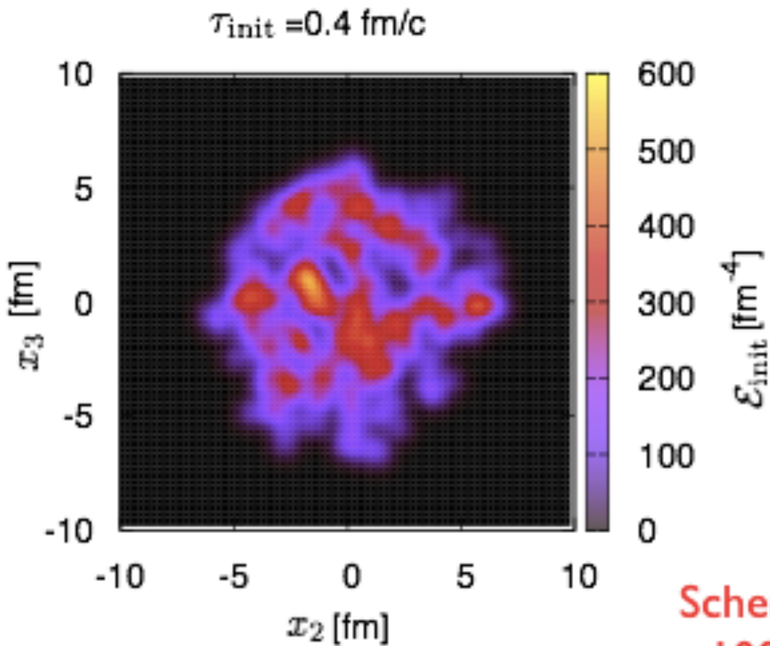
● singularities of Borel trafo

\+ 10 lowest QNM ω's



Executive summary

pheno:
hydrodynamization
(cold atoms???)



Schenke et al.
1009.3244

hydrodynamic gradient expansion diverges → precision calculations in NumHol

towards genericity:
2 flows
∞ + 1 QFTs + MIS + RTA

new connections:
resurgent series (also in QM & QFT)

Support

Evolution equations for relativistic viscous fluids

$\nabla_\mu\{\epsilon\, u^\mu u^\nu + P(\epsilon)\{\, g^{\mu\nu} + u^\mu u^\nu\,\} - \eta(\epsilon)\,\sigma^{\mu\nu}\} = 0$ is acausal.

Remedy: make $\Pi^{\mu\nu} = \langle T^{\mu\nu}\rangle - (\epsilon\, u^\mu u^\nu + P(\epsilon)\{g^{\mu\nu} + u^\mu u^\nu\})$ a new DOF, e.g.

$$(\tau_\Pi \mathcal{D} + 1)\,\Pi^{\mu\nu} = -\eta\sigma^{\mu\nu}$$

Small perturbations obey Maxwell-Cattaneo equation

$$\partial_t^2 \delta u_z - \frac{\eta}{s}\frac{1}{\tau_\Pi T}\partial_x^2 \delta u_z + \frac{1}{\tau_\Pi}\partial_t \delta u_z = 0$$

Take it seriously:

$$\omega = -i\frac{\eta}{sT}k^2 + \ldots$$

hydrodynamics

and

$$\omega = -i\frac{1}{\tau_\Pi} + i\frac{\eta}{sT}k^2 + \ldots$$

purely imaginary "quasinormal mode"

Generalization that adds $\mathrm{Re}(\omega_{QNM})$: $\left((\frac{1}{T}\mathcal{D})^2 + 2\omega_I\frac{1}{T}\mathcal{D} + |\omega|^2\right)\Pi^{\mu\nu} = -\,\eta|\omega|^2\sigma^{\mu\nu}$

1409.5087

extra