Next-to-leading order npi calculations

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Outline

Introduction

strong coupling $\rightarrow$ can't use standard perturbation theory

- need non-perturbative techniques

different approaches (for example):

• lattice calculations

$\rightarrow$ *continuum and infinite volume limits*

• continuum methods

- Schwinger-Dyson equations
- *n*-particle irreducible (*n*pi) effective theories
- renormalization group (RG)

$\rightarrow$ *truncation*

issues:

- physics (?)
- symmetries (gauge invariance)
- renormalization
- computational advantages

this talk: npi using a renormalization group approach

I will discuss mostly symmetric scalar φ^4 theory

Introduction to npi

2pi for scalar theories:

generating functional with local and bi-local sources

$$Z[J,B] = e^{iW[J,B]} = \int \mathcal{D}\varphi e^{i(S[\varphi]+J_i\varphi_i+\frac{1}{2}\varphi_i B_{ij}\varphi_j)}$$

short-hand notation:

$$\int dx \int dy \; \varphi(x)B(x,y)\varphi(y) \to \varphi_i B_{ij}\varphi_j \to B\varphi^2$$

Legendre transform:

$$\begin{aligned}\Gamma[\phi, G] &= W[J, B] - J_i\phi_i - \frac{1}{2}B_{ij}\phi_i\phi_j \\ &= S_{\rm cl}[\phi] + \frac{i}{2}{\rm Tr}\ln G^{-1} + \frac{i}{2}{\rm Tr}\, G_0^{-1}(G - G_0) + \Gamma_2[\phi, G]\end{aligned}$$

$\Gamma[\phi, G]$ is a functional of the 1- and 2-point functions

ϕ and G determined self-consistently from equations of motion

variational principle (in the absence of sources)

$$\frac{\delta\Gamma}{\delta\phi} = \frac{\delta\Gamma}{\delta G} = 0$$

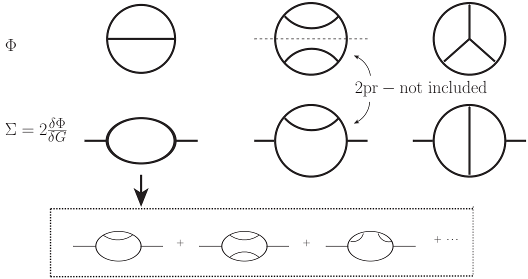
compare to $\Gamma[\phi]$ = 1pi effective action:

- $\Gamma[\phi, G]$ depends on the self consistent propagator

$\rightarrow$ truncated $\Gamma[\phi, G]$ includes an infinite resummation of diagrams

$\rightarrow$ non-perturbative

- $\Gamma[\phi, G]$ is 2pi - no double counting



npi effective action

npi Γ is a functional of n-point functions

3pi $\Gamma[\phi, G, U]$, 4pi $\Gamma[\phi, G, U, V] \cdots$

n-point functions determined self-consistently from the eom's

$\Rightarrow$ hierarchy of coupled equations

- no exact solution method is available
- use approximation techniques: truncate the effective action

Key features:

- non-perturbative
  infinite resummations of selected classes of diagrams
- action based approximation
  $\rightarrow$ symmetries of original theory
- renormalizable ?
  renormalization of 2pi effective action is understood

  - can't apply same method to higher order approximations

  - proposal: renormalization group (RG) method

Why to we need higher order npi?

- hierarchy: if i truncate at L loops

  $\rightarrow$ the $(n \geq L)$ effective actions are the same

  – or the n-loop $n$pi effective action is complete

  J. Berges, Phys. Rev. D70, 105010 (2004).

- n-loop $n$pi effective action preserves gauge invariance to the order of the truncation

  A. Arrizabalaga and J. Smit, Phys.Rev. D66, 065014 (2002),

  MEC, G. Kunstatter and H. Zaraket, Eur. Phys. J. C 42, 253 (2005).

- example: 3d $SU(N)$ Higgs model 3-loop 2pi

  $\rightarrow$ strong gauge dependence

  G. Moore and M. Abraao-York, JHEP 1410, 105 (2014).

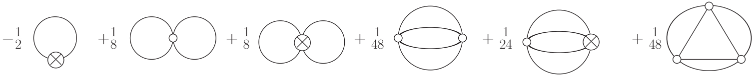
- in gauge theories LO transport coefficients require 3-loop 3pi (collinear singularities / LPM / vertex corrections)
  MEC and E. Kovalchuk, Phys. Rev. D80, 085013 (2009).
- example: 4pi symmetric ϕ^4 theory 3 dimensions - 4-loop
  - 4pi contributions important at large couplings
  MEC, WeiJie Fu, P. Mikula and D. Pickering, PRD89, 025013 (2014).
- example: 2pi symmetric ϕ^4 theory in 4d - 2/3/4 loops
  convergence much better than perturbation theory
  but strong deviations at large coupling
  – indicates a break down of 2pi approximation
  – need for vertex corrections
  MEC, B.A. Meggison and D. Pickering, arXiv:1603.02085.

examples of variational eom's

$$\Phi = \Phi_{\rm no\cdot int} + \Phi_{\rm int} \qquad\qquad (\Phi = i\Gamma)$$

$$\Phi_{\rm no\cdot int} = -\frac{1}{2}\phi G^{-1}_{\rm no\cdot int}\phi - \frac{1}{2}{\rm Tr}\ln G^{-1} - \frac{1}{2}{\rm Tr}\, G^{-1}_{\rm no\cdot int}\, G$$

$\Phi_{\rm int}$ 4-loop 2pi (symmetric)

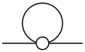
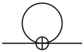
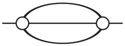


npi 2-point functions

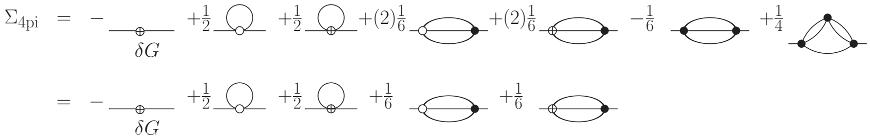
$$\left.\frac{\delta\Phi}{\delta G}\right|_{\substack{\phi=0\\ G=\tilde{G}}} = 0 \quad \rightarrow \quad \tilde{G}^{-1} = G_0^{-1} + \Sigma$$

$$\Sigma = 2\left.\frac{\delta\Phi_{\rm int}}{\delta G}\right|_{\substack{\phi=0\\ G=\tilde{G}}}$$

3 loop 2pi

$\Sigma_{2\rm pi} \;=\; -$  δG $+\frac{1}{2}$  $+\frac{1}{2}$  $+\frac{1}{6}$ 

4-loop 4pi

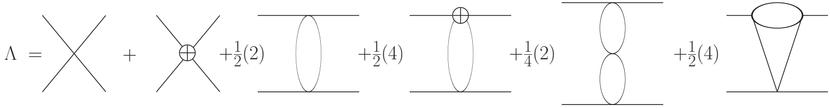


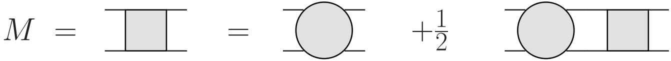
MEC and Yun Guo, PRD 83, 016006 (2010); PRD 85, 076008 (2012).

npi 4-vertices

Bethe-Salpeter vertex (4-loop 2pi)

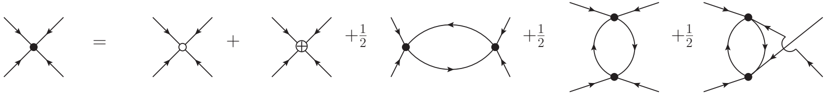
$$\Lambda = 4\frac{\delta^2\Phi_{\rm int}}{\delta G^2}\bigg|_{\substack{\phi=0\\ G=\tilde G}} \qquad M = \Lambda + \frac{1}{2}\,\Lambda\tilde G^2 M$$





Variational 4-vertex (4-loop 4pi)

$$\frac{\delta\Phi_{\rm int}}{\delta V} = 0$$



2pi or 4pi:
coupled self-consistent eom's for the 2- and 4-point functions

2pi and 4pi in 3-dimensions

*** no vertex counter-terms*

method:

- rotate to Euclidean space
- N^d symmetric lattice: lattice spacing $a = 2\pi/(Nm)$
  each momentum component is discretized:

$$Q_i = \frac{2\pi}{aN} n_i = m\, n_i\,, \quad n_i = -\frac{N}{2} + 1, ..., \frac{N}{2}$$

  periodic boundary conditions
- numerical iterative method $\rightarrow$ search for fixed points

J. Berges, Sz. Borsányi, U. Reinosa, and J. Serreau, PRD **71***, 105004 (2005)* 99

Memory constraints:

\# points in phase space of a vertex is $N^{l \times d}$

l is the number of independent momenta and d is the dimension

for V: $l = 3$, $d = 3$, $N_{\max} = 12 \quad \Rightarrow \quad 5.16 \times 10^9$ points

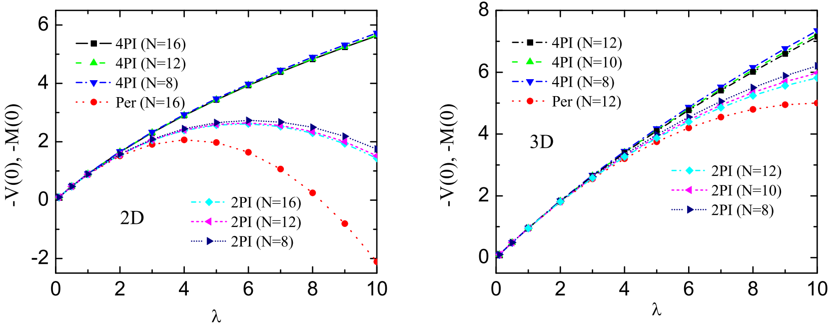
trick: reduce the phase space using the symmetries of the vertex

- V is symmetric under interchange of legs and dirns in $\vec{p}$ space
- don't need to calculate all points

table: size of phase space and # of needed representative points

N	$N^{3 \cdot (d=3)}$	# of reprs
6	10,077,696	11,424
8	134,217,728	129,502
10	1,000,000,000	913,661
12	5,159,780,352	4,608,136

results



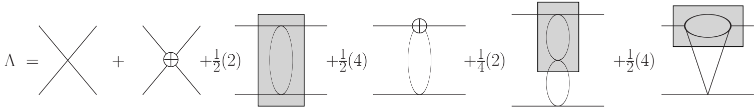
for certain momentum configurations M and V are close together

- this happens when s-channel contributions are big

MEC, Wei-Jie Fu, P. Mikula and D. Pickering, Phys. Rev. D **89**, *025013 (2014)*

Npi renormalization – 4 dimensions

4-loop 2pi 4-kernel Λ



the 2-loop diagrams contain nested 1-loop subdivergences
→ two 1-loop counter-terms must cancel two different 1-loop boxes

⇒ can see there is no one $\delta\lambda_1$ that works

How to see the problem:

variational principle [2pi effective action] $\rightarrow$ set of eom's

expand these eom's $\rightarrow$ infinite set of diagrams

compare with n-pt fcns from the 1pi effective action

$\rightarrow$ some diagrams are missing - some have different $\mathcal{S}$ factors

different combinatorics $\Rightarrow$ renormalization not the same

Remember: the goal of npi is not to include everything but to include what is hopefully the physically important contros

Resolution for 2pi: 2 counter-terms

$\delta\lambda_{\rm bb}$ absorbs divergences in Λ except for global divergence

different $\delta\lambda_{\rm et}$ absorbes global divergences in Λ

chain the Λ's in the s-channel to make M

$\rightarrow$ new divergences in M absorbed by adjusting $\delta\lambda_{\rm et}$

KEY: no new divergences from boxes drawn thru Λ (it's 2pi)

conclusion: need 2 ct's . . . sounds bad . . . BUT

1) they both come from the action

2) at $L \rightarrow \infty$ loops they are equal

H. van Hees, J. Knoll, Phys. Rev. **D65**, *025010 (2002);*
J-P Blaizot, E. Iancu, U. Reinosa, Nucl. Phys. **A736**, *149 (2004);*
J. Berges, Sz. Borsányi, U. Reinosa, J. Serreau, Annals Phys. **320**, *344 (2005).*

comment:
3-loop level $\rightarrow$ only need one ct (Λ has only global divergence)

generic procedure:

(1) determine $\delta\lambda_{\rm et}$ to make Λ finite

(2) adjust $\delta\lambda_{\rm et}$ to make M finite

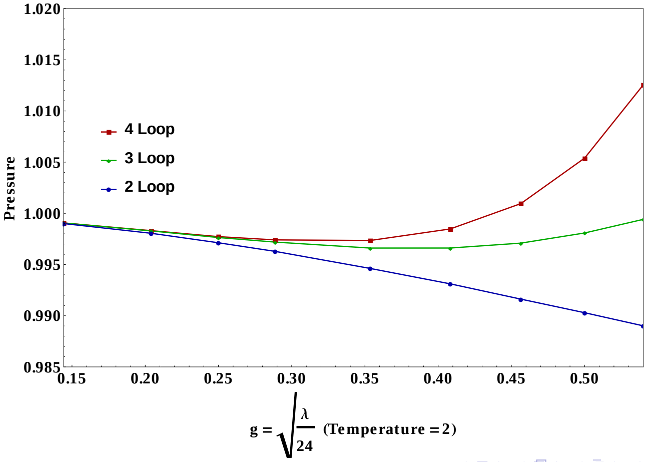
$\rightarrow$ combine in one step

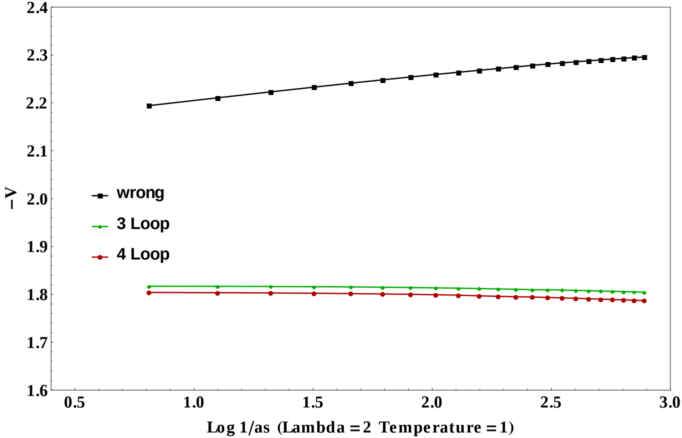
$\Rightarrow$ the structure with 2 cts can only be verified at the 4-loop level

4-loop 2pi - Numerical Method

- Euclidean space
- discretized Cartesian co-ordinates
- momentum phase space is managable
  $\rightarrow$ fast Fourier transforms for speed

Numerical Results - arXiv:1603.02085





The symmetric vertex as a function of momentum cutoff

Conclusions so far

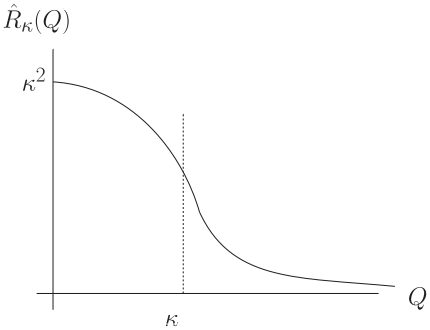
- 4pi versus 2pi in 3d:
  BS vertex doesn't capture all physics in 4pi variational vertex
- 2pi in 4d:
  4-loop contro impt at large $\lambda \to$ breakdown of 2pi expansion

$\Rightarrow$ need for higher order approximations $(n > 2)$pi

Must develop another method to renormalize

Renormalization group method

add to the action a non-local regulator term $\Delta S_\kappa[\varphi] = -\frac{1}{2}R_\kappa\varphi^2$



$$R_\kappa = \frac{Q^2}{e^{Q^2/\kappa^2} - 1}$$

$R_\kappa(Q) \sim \kappa^2$ for $Q \ll \kappa$

fluctuations $Q \ll \kappa$ suppressed

$R_\kappa(Q) \to 0$ for $Q \geq \kappa$

fluctuations $Q \gg \kappa$ unaffected

family of theories indexed by the continuous parameter κ

fluctuations are smoothly taken into account as κ is lowered to zero

$\kappa \to \infty$ regulated action $\to$ classical action

$\kappa \to 0$ regulated action $\to$ full quantum action

J.-P. Blaizot, A. Ipp, N. Wschebor, Nucl. Phys. A **849**, *165 (2011)*
J.-P. Blaizot, J.M. Pawlowski and U. Reinosa, Phys. Lett. B **696**, *523 (2011)*

generating functionals defined in the usual way

$$Z_\kappa[J,B] = \int [d\varphi] \exp\left\{ i\Big(S[\varphi] - \frac{1}{2}\hat{R}_\kappa\varphi^2 + J\varphi + \frac{1}{2}B\varphi^2 + \cdots \Big)\right\}$$

calculate 1pi, 2pi, $\cdots$ effective action

action depends on κ: Φ_κ

$$\text{action flow eqn:} \qquad \partial_\kappa \Phi_\kappa = \frac{1}{2}\partial_\kappa R_\kappa\, G$$

C. Wetterich, Phys. Lett., B 301, 90 (1993).

Method

n-point functions depend on κ

(1) choose an uv scale $\kappa = \mu$ (defn of bare parameters)

theory is classical at this scale (all fluctuations suppressed)

$\rightarrow$ *n*-point functions are known functions of the bare parameters

(2) derive a hierarchy of differential 'flow' equations

$\rightarrow$ relate κ dependent *n*-point functions and their derivatives wrt κ

(3) solve flow equations starting from bc's at $\kappa = \mu$

$\rightarrow$ obtain the *n*-point fcns at $\kappa = 0$ (the quantum solutions)

Tuning

Conventional calculation

definitions of the physical parameters $\leftrightarrow$ RConds
$\rightarrow$ extract bare parameters (ct's) directly from RConds

RG calculation

definition of physical parameters ($\kappa = 0$)
$\rightsquigarrow$ constrains initial conditions on the flow equations ($\kappa = \mu$)

tuning procedure:
– choose the physical parameters
– make a guess for bare parameters from which to start the flow
– solve flow equations - find produced values of physical parameters
– adjust the bare parameters and repeat

Hierarchy of flow equations

definitions of kernels: $\Phi^{(n,m)}_{\mathrm{int}\cdot\kappa} = 2^m \frac{\delta^n}{\delta\phi^n}\frac{\delta^m}{\delta G^m}\Phi_{\mathrm{int}}\Big|_{\substack{G=G_\kappa\\ \phi=0}}$

rename important kernels:

$$\Phi^{(0,1)}_{\mathrm{int}\cdot\kappa} = \Sigma\,, \quad \Phi^{(0,2)}_{\mathrm{int}\cdot\kappa} = \Lambda\,, \quad \Phi^{(0,3)}_{\mathrm{int}\cdot\kappa} = \Upsilon$$

derivatives of action flow eqn:

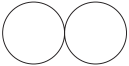
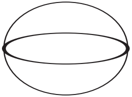
$$\partial_\kappa \Phi^{(n,m)}_{\mathrm{int}\cdot\kappa}\Big|_{\substack{G=G_\kappa\\ \phi=o}} = \frac{1}{2}\int dQ\, \partial_\kappa\,(R_\kappa + \Sigma^{01}_\kappa)\, G^2_\kappa(Q)\, \Phi^{(n,m+1)}_{\mathrm{int}\cdot\kappa}(Q,\quad)\Big|_{\substack{G=G_\kappa\\ \phi=o}}$$

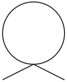
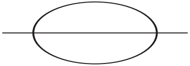
$\Rightarrow$ infinite hierarchy of coupled flow eqns for the n-point kernels

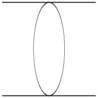
important features of these flow equations

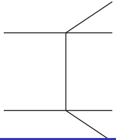
- truncate when action is truncated
- preserve symmetries of the action

3-loop 2pi - symmetric

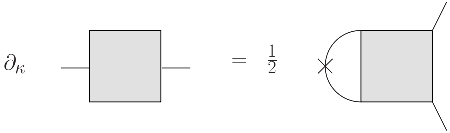
$\Phi_{\text{int}} = \frac{1}{8}$  $+ \frac{1}{48}$ 

$\Sigma = 2\frac{\delta\Phi}{\delta G} = \frac{1}{2}$  $+ \frac{1}{6}$ 

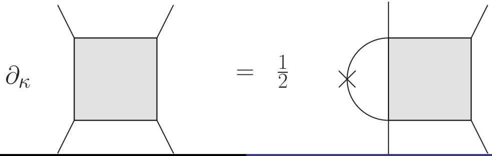
$\Lambda = 2\frac{\delta^2\Phi}{\delta^2 G} =$  $+(2)\frac{1}{2}$ 

$\Upsilon = 2\frac{\delta^3\Phi}{\delta^3 G} =$ (4) 

$$\Sigma \text{ flow equation: } \partial_\kappa \Sigma(P) = \frac{1}{2} \int dQ \, \partial_\kappa (R_\kappa + \Sigma) \, G_\kappa^2 \, \Lambda(Q, P)$$



$$\Lambda \text{ flow equation: } \partial_\kappa \Lambda(P, K) = \frac{1}{2} \int dQ \, \partial_\kappa (R_\kappa + \Sigma) \, G_\kappa^2 \, \Upsilon(Q, P, K)$$



TRUNCATION

8 leg kernel = const

$\partial_\kappa \Upsilon$ is an exact differential $\rightarrow$ don't need the Υ flow equation

integration constant $\Upsilon_{\kappa=\mu} = 0$ - no 6-vertex in Lagrangian

CONSISTENCY

must show bc's $(\kappa = \mu) \quad \Leftrightarrow \quad$ RC's $(\kappa = 0)$

Λ flow equation $\rightarrow \Lambda_\kappa(P,Q) + C$

C is a κ independent integration constant

apply RC $\quad \Rightarrow \quad$ choose $C = -\lambda - \Lambda_0(0,0)$

$$\longrightarrow \quad -\lambda + \Lambda_\kappa(P,Q) - \Lambda_0(0,0)$$

rewrite:
$$\underbrace{-\lambda + \big[\Lambda_\kappa(0,0) - \Lambda_0(0,0)\big]}_{-\lambda_\kappa \;\; \leftarrow \;\; \text{running coupling}} + \big[\Lambda_\kappa(P,Q) - \Lambda_\kappa(0,0)\big]$$

rearrange:
$$-\lambda_\mu + \underbrace{\big[\Lambda_\kappa(P,Q) - \Lambda_\mu(0,0)\big]}_{\rightarrow\, 0 \text{ as } \kappa \rightarrow \mu \gg \{P,Q\}}$$

Σ flow equation $\rightarrow G_\kappa^{-1} = P^2 + m^2 + \Sigma_\kappa(P) + C$

$$\text{RC's on 2-pt fcn} \quad G_0^{-1}(0) = m^2\,, \quad \frac{d}{dP^2} G_0^{-1}\bigg|_{P=0} = 1$$

$\rightarrow$ choose $C = -(\Sigma_0(0) + P^2\Sigma_0'(0))$

show that with this choice of C the limit $\kappa \rightarrow \mu \gg P$ gives

$$G_\mu^{-1} = Z_\mu(P^2 + m_\mu^2)$$

with Z_μ and m_μ momentum independent

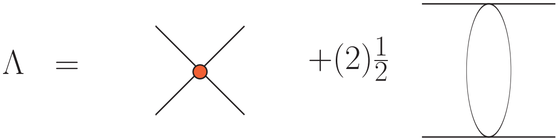
$\rightarrow$ all divergent contributions can be absorbed into defns of m_μ, Z_μ

comment: the flow equation $\partial_\kappa \Lambda$ doesn't have to be solved

(1) Λ has only a global (1-loop) divergence

(2) Integration constant (λ_μ) comes from bare lagrangian

$$\Lambda_\kappa(P,Q) = -\lambda_\mu + \frac{\lambda^2}{2}\int dQ\, G_\kappa(Q)\big[G_\kappa(Q+P-K) + G_\kappa(Q+P+K)\big]$$



satisfies

$$\partial_\kappa \Lambda(P,K) = \frac{1}{2}\int dQ \partial_\kappa \left(R_\kappa + \Sigma\right) G_\kappa^2 \, \Upsilon(Q,P,K)$$

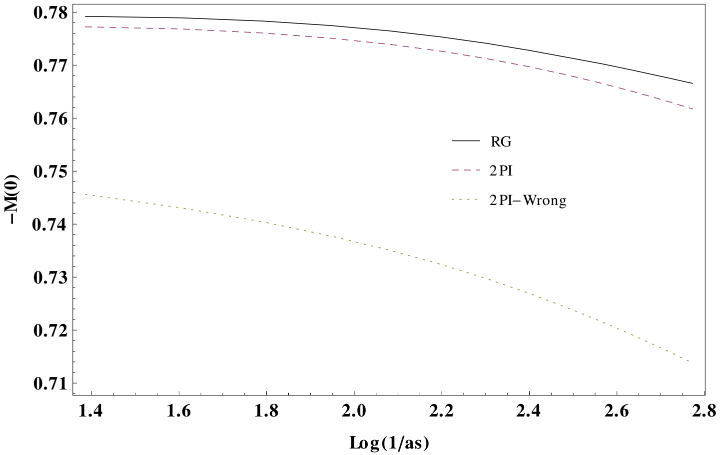
with

$$\begin{aligned}\Upsilon(Q,P,K) \;=\; & -\lambda^2 \big(G_\kappa(Q+P+K) + G_\kappa(Q+P-K) \\ & + G_\kappa(Q-P+K) + G_\kappa(Q-P-K)\big) \,.\end{aligned}$$

and

$$\lim_{\kappa\to\mu} \Lambda(P,K) = -\lambda_\mu$$

vertex as a function of the momentum cutoff (T=2 and λ=1)



4-loop 4pi (symmetric)

action $\Phi[G, V]$

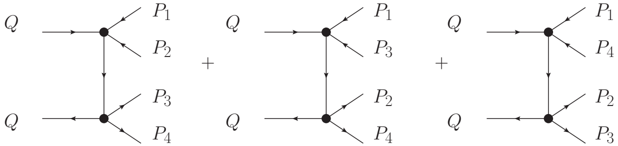
$\rightarrow$ functional of self-consistently determined 2- and 4-pt functions

$\Rightarrow$ two interdependent hierarchys of flow equations

V flow equation

$$\partial_\kappa V(P_1, P_2, P_3) = \frac{1}{2}\int dQ\,\partial_\kappa\left[R_\kappa(Q) + \Sigma(Q)\right] G_\kappa^2(Q)\Phi^{011}(Q; P_1, P_2, P_3)$$

$$\Phi^{011} = 4!\,2\,G^{-4}\frac{\delta}{\delta V}\frac{\delta}{\delta G}\Phi_{\text{int}} =$$



V flow eqn has same structure as the 3-loop 2pi Λ flow eqn

choose integration constant C so:

(1) RC $V_{\kappa=0}(0) = -\lambda$ satisfied at $\kappa = 0$

(2) overall divergence $\rightarrow$ $\vec{p}$ independent bare coupling at $\kappa = \mu$

Σ flow equations

$$\partial_\kappa \Sigma(P) = \frac{1}{2}\int dQ\, \partial_\kappa\big(\Sigma(Q) + R_\kappa(Q)\big)\, G_\kappa^2(Q)\, \Lambda(P,Q)$$

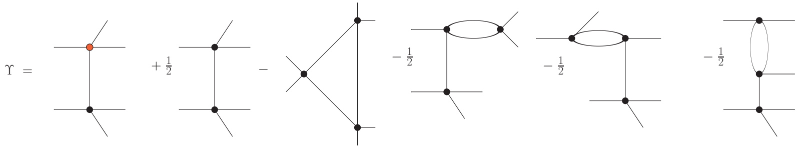
$$\partial_\kappa \Lambda(P,K) = \frac{1}{2}\int dQ\, \partial_\kappa\big(\Sigma(Q) + R_\kappa(Q)\big)\, G_\kappa^2(Q)\, \Upsilon(P,K,Q)$$

Λ as given by the functional derivs contains 2-loop diagrams

- can't substitute it on the rhs of the first equation

- would give $\vec{p}$ dependent sub-divergences $\not\to$ bare parameters

$\Rightarrow$ instead we must solve Λ flow equation



– the kernel Υ has divergent 1-loop contributions

– also has a tree vertex

Q: can i replace it with λ_μ and truncate the hierarchy here ?

combinatorics:
the RC on M 'tunes' to λ_μ which cancels the 1-loop divs in Υ

4pi RG - Numerical Method · · ·

fundamental problem

3d 4pi calculation: "repr" function to reduce the phase space

4d 2pi calculation: used fft's to avoid nested summations

⇒ these two are incompatible

options

- spherical co-ordinates to reduce phase space
- spherical fft
- interpolation from cartesian fft

Conclusions

Higher order npi calculations are needed (in some situations)

- transport coefficients
- thermodynamic quantities at large coupling

RG method is a promising approach: tested on 3-loop 2pi level

4-loop 4pi calculations are in progress