Nonhydro modes in relativistic hydrodynamics

Michał Spaliński
University of Białystok & National Centre for Nuclear Research

Hydrodynamic theories

$$T^{\mu\nu} = \mathcal{E} u^\mu u^\nu + \mathcal{P}(\mathcal{E})(g^{\mu\nu} + u^\mu u^\nu) + \Pi^{\mu\nu}$$
$$\nabla_\alpha T^{\alpha\beta} = 0$$

needed to describe dissipation

- Perfect fluid: $\Pi^{\mu\nu} = 0$
- Navier Stokes: $\Pi^{\mu\nu} = -\eta(\mathcal{E})\sigma^{\mu\nu}$ ← typically acausal
- Mueller; Israel & Stewart; BRS3: $(\tau_\Pi \mathcal{D} + 1)\,\Pi^{\mu\nu} = -\eta\sigma^{\mu\nu} + \ldots$
- H+QNM: $\left((\frac{1}{T}\mathcal{D})^2 + 2\omega_I \frac{1}{T}\mathcal{D} + |\omega|^2 \right) \Pi^{\mu\nu} = -\eta|\omega|^2\sigma^{\mu\nu} - c_\sigma \frac{1}{T}\mathcal{D}\,(\eta\sigma^{\mu\nu}) + \ldots$
- Anisotropic hydrodynamics (same nonhydro sector as MIS)

all coincide at late times, differ at early times

$T \sim \mathcal{E}^{1/4}$

The gradient expansion

This works, because the SST can be expressed as a formal **infinite series**

$$\Pi^{\mu\nu} = -\eta\sigma^{\mu\nu} + \tau_\Pi \mathcal{D}(\eta\sigma^{\mu\nu}) + \ldots$$

The relaxation time appears as a second order transport coefficient

whose form is fixed by symmetries, and it

- defines what we mean by transport coeffficients
- allows comparison between different hydrodynamic theories
- connects the phenomenological and microscopic descriptions

$$\langle T^{\mu\nu}\rangle = \mathcal{E}u^\mu u^\nu + \mathcal{P}(g^{\mu\nu} + u^\mu u^\nu) - \eta\sigma^{\mu\nu} + \tau_\Pi \mathcal{D}(\eta\sigma^{\mu\nu}) + \lambda_1 \left(\sigma^2\right)^{\mu\nu} + \ldots$$

Calculated explicitly in some examples

Definition of u: $\langle T_\mu^\nu\rangle u^\mu = -\mathcal{E}u^\nu$

Linearized perturbations in MIS/BRS3

Consider small deviations from equilibrium

$$\delta\Phi \sim \exp\left(-i(\omega t - \vec{k}\cdot\vec{x})\right)$$

Hydro and nonhydro modes (sound channel)

$$\omega_H^{(\pm)} = \pm\frac{k}{\sqrt{3}} - \frac{2i}{3T}\frac{\eta}{s}k^2 + \ldots \qquad \omega_{NH} = -i\left(\frac{1}{\tau_\Pi} - \frac{4}{3T}\frac{\eta}{s}k^2\right) + \ldots$$

Velocity of propagation

$$v = \frac{1}{\sqrt{3}}\sqrt{1 + 4\frac{\eta/s}{T\tau_\Pi}} \longleftarrow \boxed{v \leq 1 \iff T\tau_\Pi \geq 2\eta/s}$$

The nonhydro sector acts as a **regulator** ensuring causality,

[Baier et al. 0712.2451]

Regulator independence

"Hydro works" $\Longleftrightarrow$ **no sensitivity to the nonhydro sector**

This can fail at early times or in small systems.

E.g. at the linearized level **nonhydro modes dominate** for

$$k > \frac{T}{\sqrt{2(T\tau_{\Pi})(\eta/s)}}$$

Using the causality bound, and and the KSS for eta/s this implies RT<1.

Heuristic arguments for small plasma drops lead to conclusions roughly consistent with [Habich et al. 1512.05354].

So: **what if there is regulator dependence?**

[MS 1607.06381]

Bjorken flow

Energy-momentum tensor:

$$\langle T^{\mu}_{\nu} \rangle = \mathrm{diag}(-\mathcal{E}, \mathcal{P}_L, \mathcal{P}_T, \mathcal{P}_T)$$

$\mathcal{E}(\tau)$

$$\mathcal{P}_L = -\mathcal{E} - \tau\dot{\mathcal{E}} \ , \quad \mathcal{P}_T = \mathcal{E} + \frac{1}{2}\tau\dot{\mathcal{E}}$$

Independent of initial conditions

Large proper-time (gradient) expansion:

$$T(\tau) = \frac{\Lambda}{(\Lambda\tau)^{1/3}}\left(1 + \frac{t_1}{(\Lambda\tau)^{2/3}} + \frac{t_2}{(\Lambda\tau)^{4/3}} + \dots\right)$$

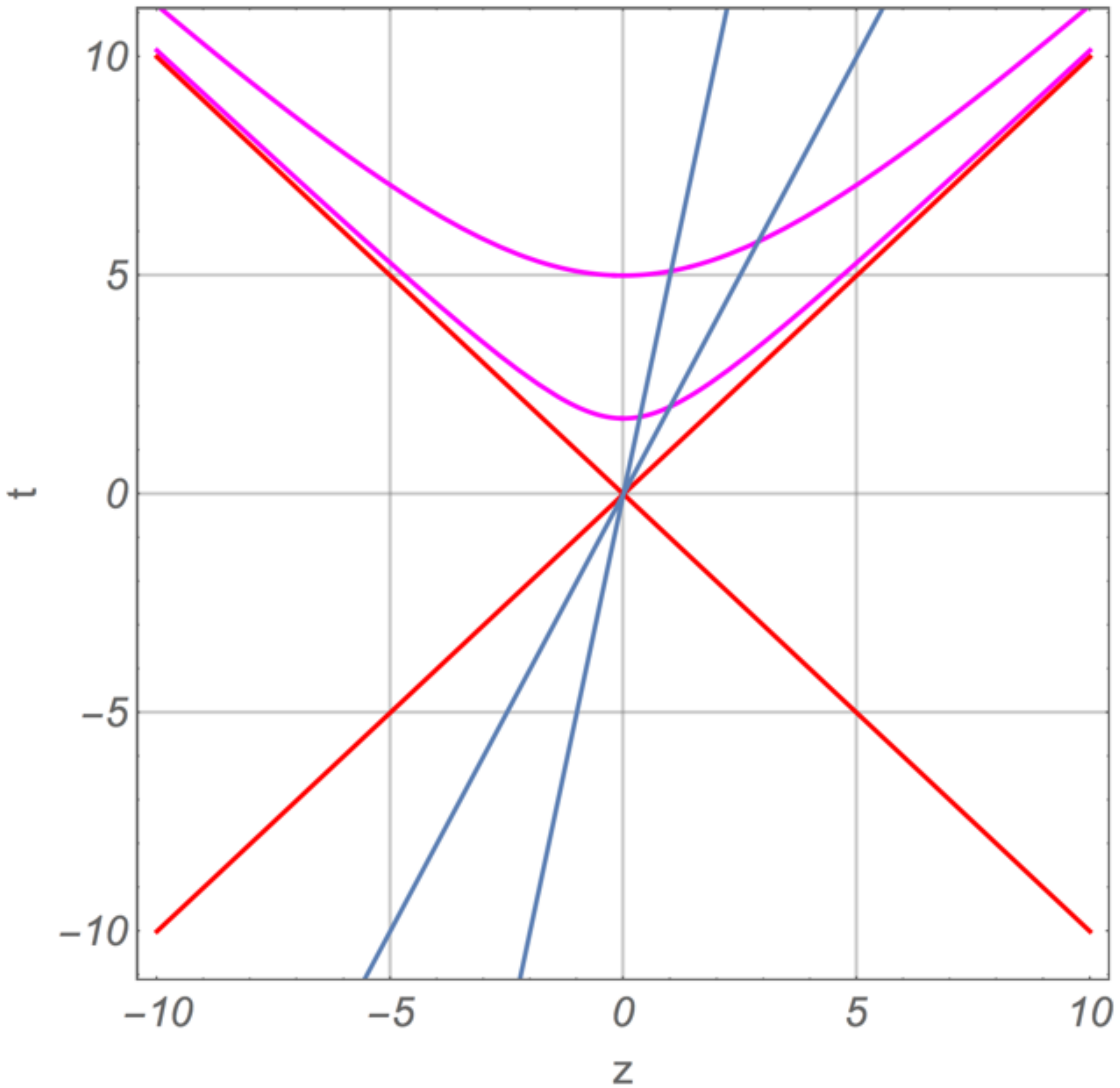
Remnant of initial conditions

Dimensionless variables:

$$w \equiv \tau T, \quad \mathcal{R} \equiv \frac{\mathcal{P}_T - \mathcal{P}_L}{\mathcal{P}}$$

$$f \equiv \frac{2}{3}\left(1 + \frac{\mathcal{R}}{12}\right) = \sum_{n=0}^{\infty} f_n w^{-n}$$

$$t = \tau\cosh\lambda \ , \quad z = \tau\sinh\lambda$$



Large order behaviour

The gradient expansion coefficients have been computed in some microscopic models with the result

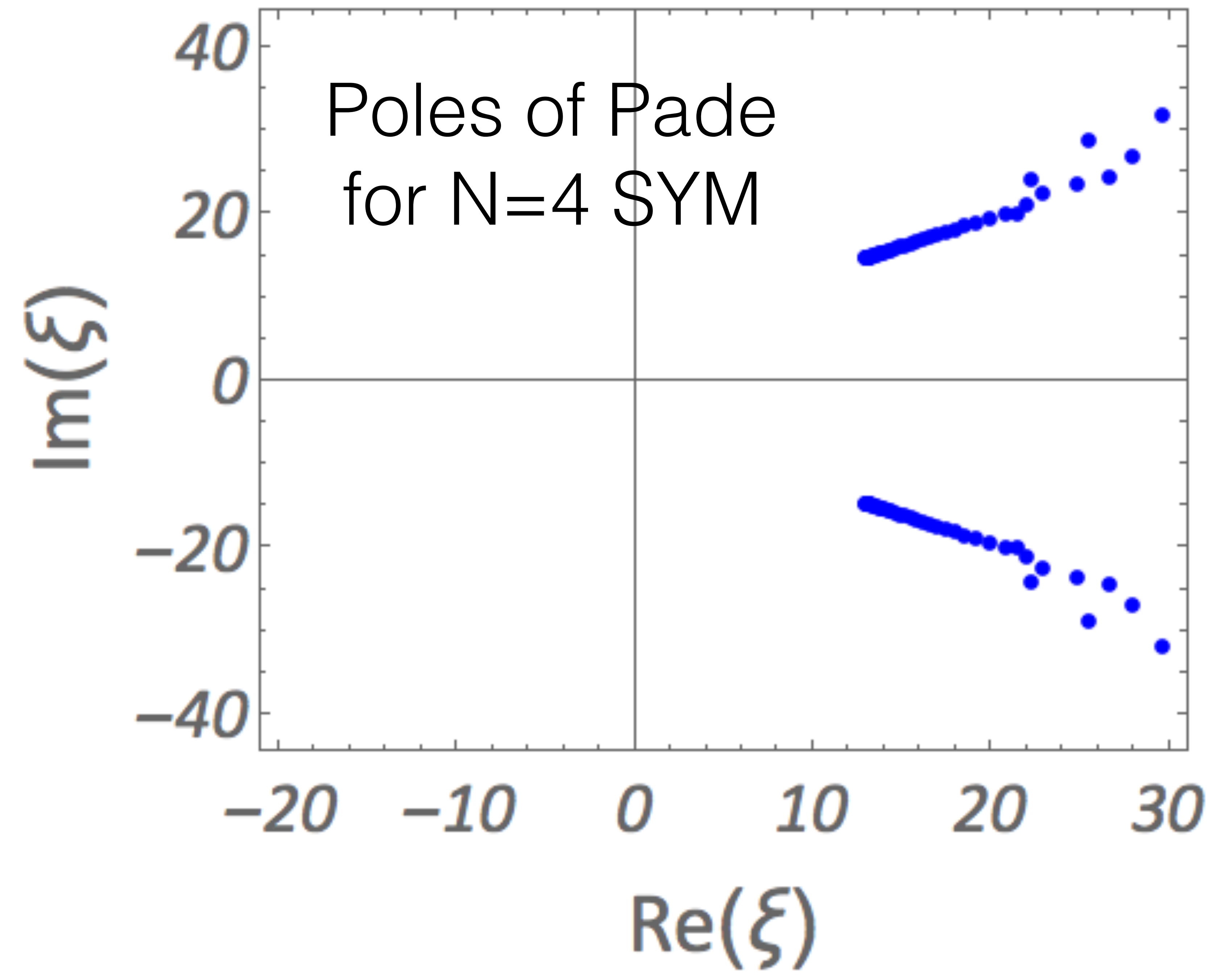
$$f_n \sim n!$$

Similar calculations in hydrodynamics also lead to divergent series

The singularities of the analytic continuation of the Borel transform

$$f_B(\xi) = \sum_{n=0}^{\infty} \frac{f_n}{n!} \xi^n$$

contain **information about nonhydro modes** of the system.



[Heller et al.1302.0697]

Bjorken flow in MIS/BRS3

Evolution equation

$$w f f' + 4 f^2 + \left(\frac{w}{C_{\tau\Pi}} - \frac{16}{3} \right) f - \frac{2w}{3C_{\tau\Pi}} - \frac{4C_\eta}{9C_{\tau\Pi}} + \frac{16}{9} = 0$$

where (not SYM values!)

$$C_{\tau\Pi} = T\tau_\Pi \to 6C_\eta, \quad C_\eta = \eta/s \to 2\frac{1}{4\pi}$$

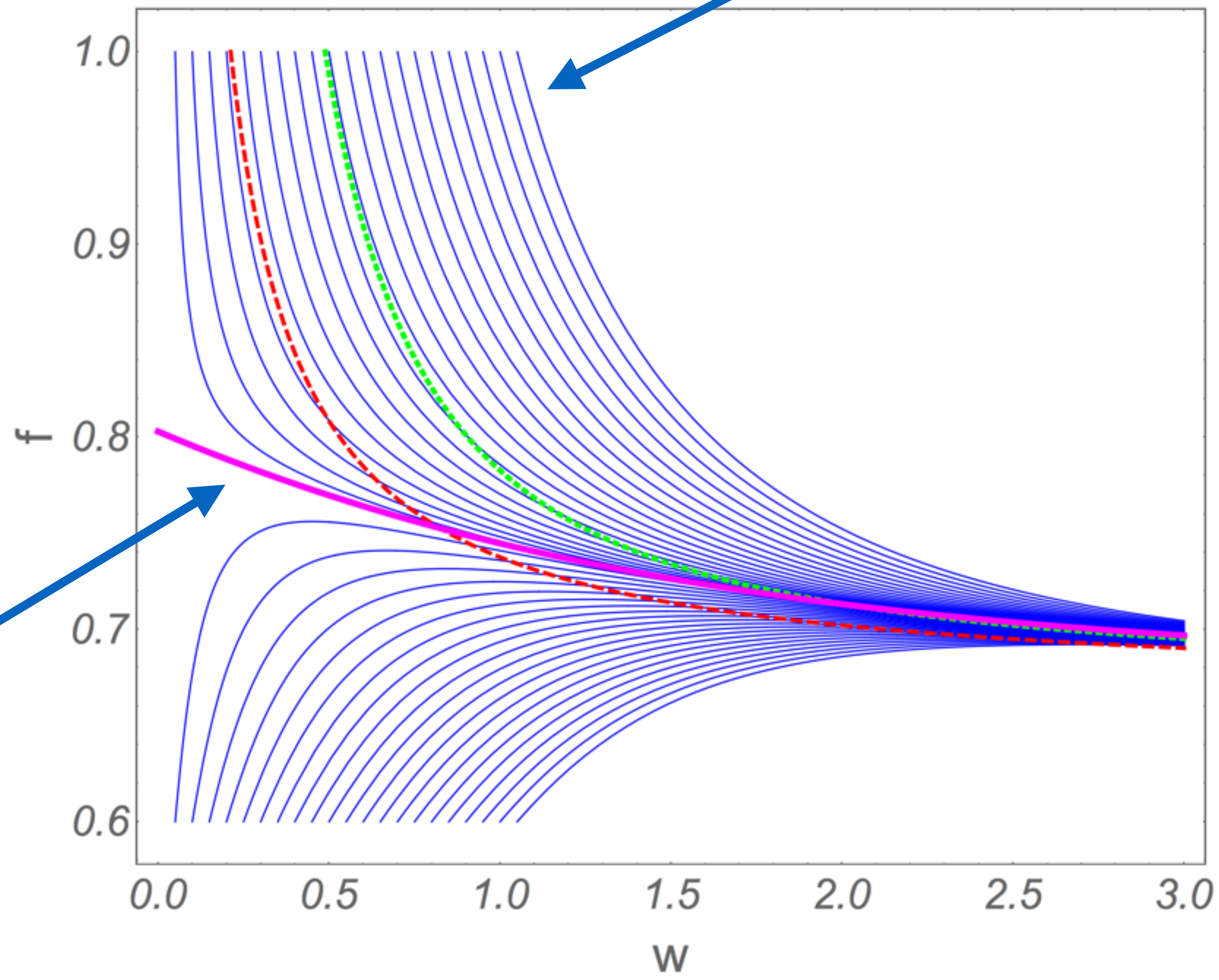
Finite order hydrodynamics:

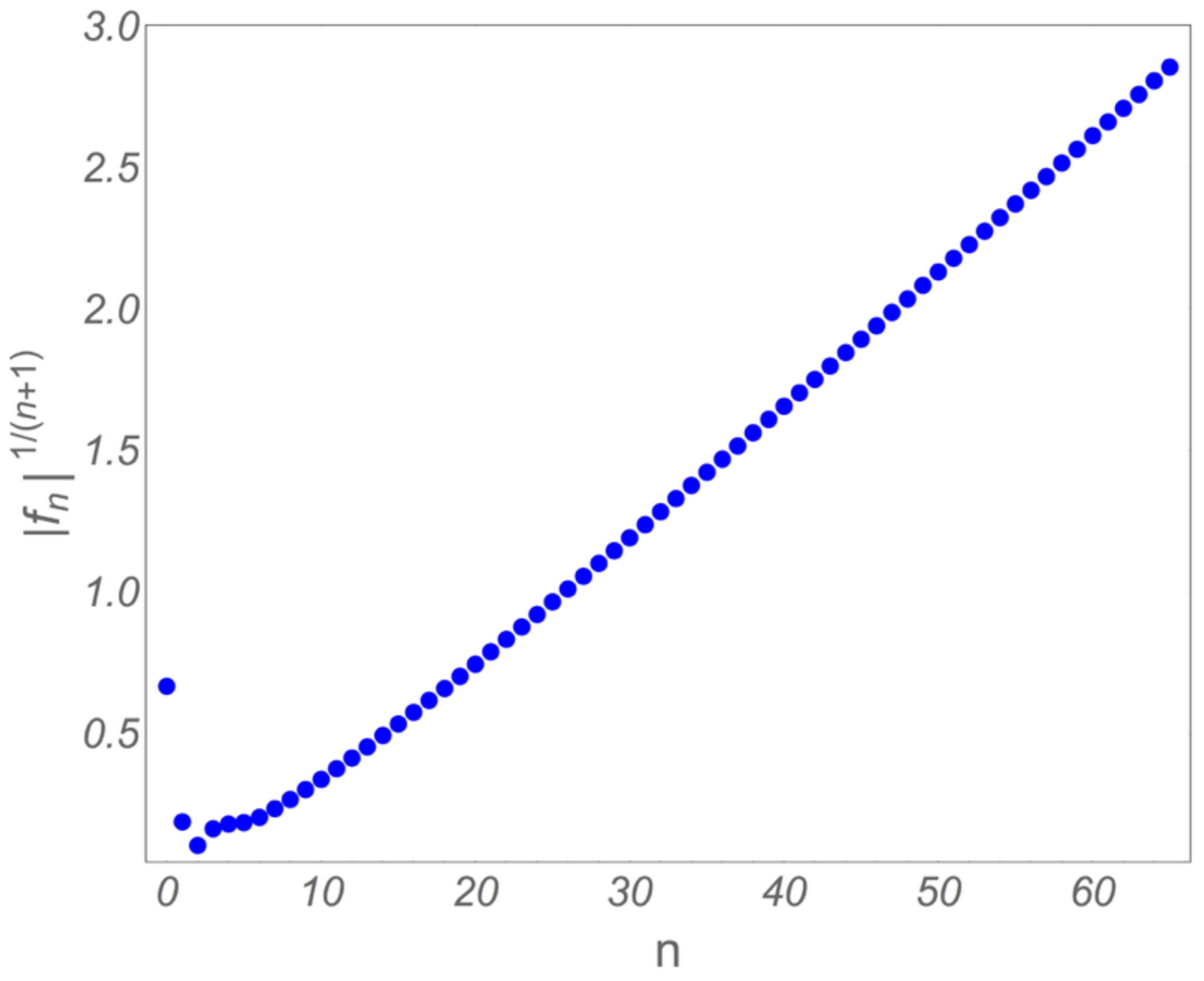
$$f(w) = \boxed{\boxed{\frac{2}{3} + \frac{4C_\eta}{9w}} + \frac{8C_\eta C_{\tau\Pi}}{27w^2}} + \dots$$

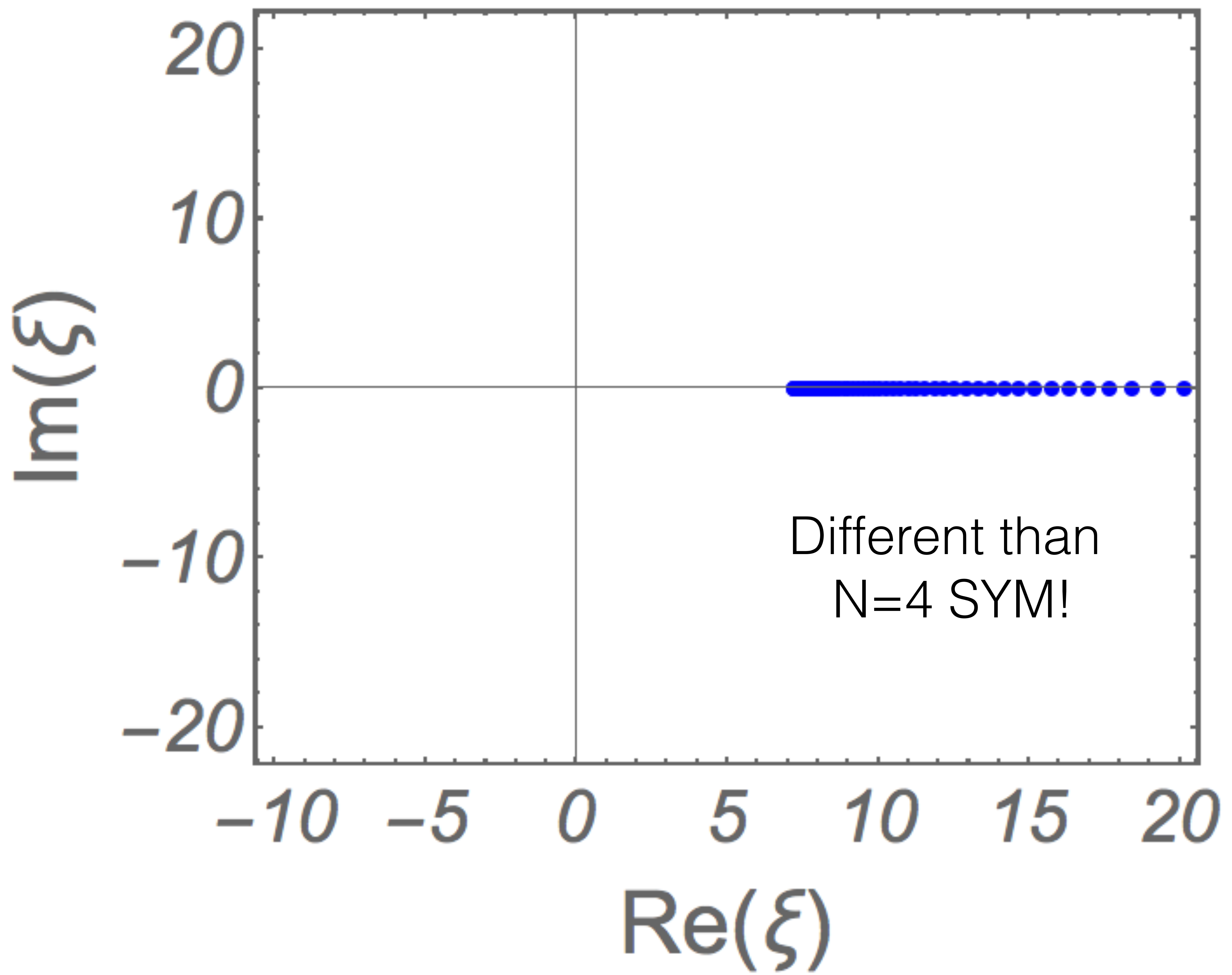
Attractor = "resummed hydro"?

[Heller, MS 1503.07514]

numerical solutions







- The series is asymptotic
- Single purely damped nonhydro mode, decay rate given by cut location
- Resummation ambiguity resolved by resurgence

The result is a transseries

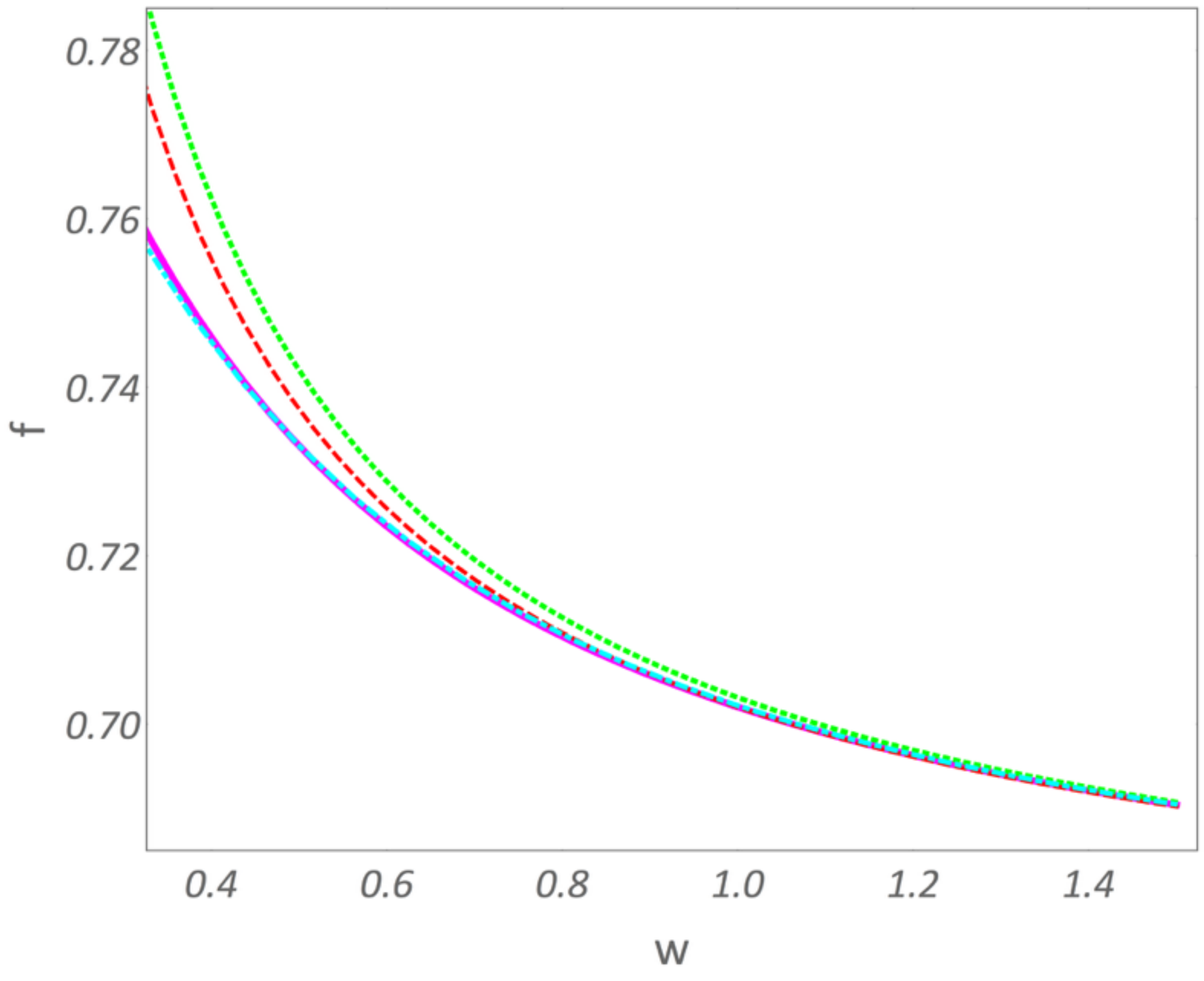
$$f = \sum_{n=0}^{\infty} f_n w^{-n} + c\, e^{-\frac{3}{2C_{\tau\Pi}} w} \left(w^{\frac{C_\eta - 2C_{\lambda_1}}{C_{\tau\Pi}}} \sum_{n=0}^{\infty} f_n^{(1)} w^{-n} \right) + \ldots$$

Different “instanton sectors” are related by resurgence, which fixes Im(c).

This leaves Re(c) as an integration constant.

Matching the attractor requires

$$\mathrm{Re}(c) = 0.049 \neq 0$$



H+QNM

Replace the MIS relaxation equation by

$$\left((\frac{1}{T}\mathcal{D})^2 + 2\Omega_I \frac{1}{T}\mathcal{D} + |\Omega|^2 \right) \Pi^{\mu\nu} = -\eta |\Omega|^2 \sigma^{\mu\nu} - c_\sigma \frac{1}{T}\mathcal{D}\left(\eta\sigma^{\mu\nu}\right) + \ldots$$

$$|\Omega|^2 = \Omega_R^2 + \Omega_I^2$$

"QNM frequency"

Velocity of sound

$$v = \frac{1}{\sqrt{3}}\sqrt{1 + 8c_\sigma \frac{\eta}{s}\Omega_I}$$

so

$$v \leq 1 \iff c_\sigma \leq \frac{1}{4\Omega_I(\eta/s)}$$

Stability (sound channel) requires

$$c_\sigma \geq \frac{|\Omega|^2}{4\Omega_I^2}$$

[Heller et al.1409.5087]

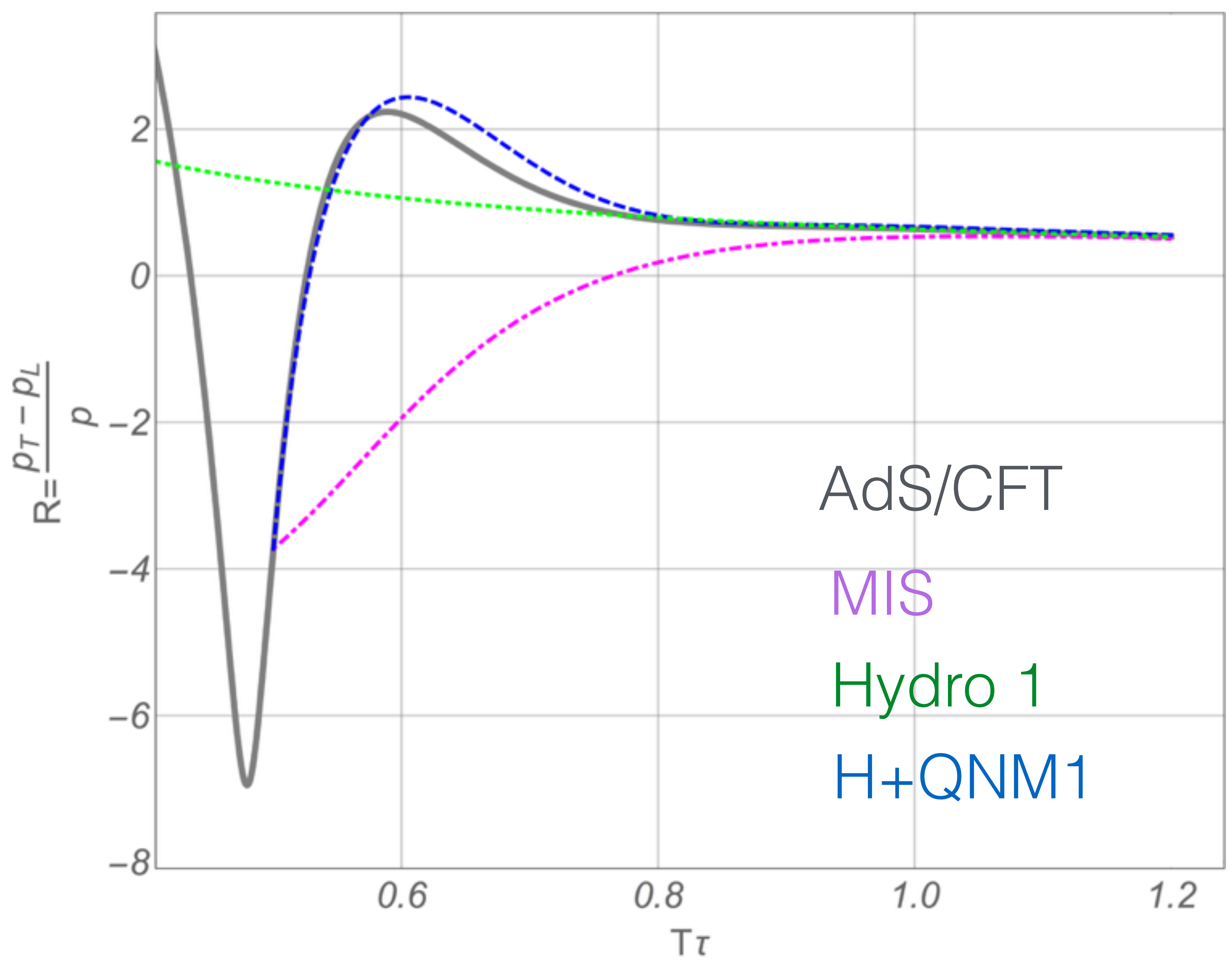
For Bjorken flow

$$w f^2 f'' + \alpha f f' + 12 f^2 f' + w f f'^2 + \frac{\beta + \gamma f + \delta f^2 + 12 f^3}{w} = 0$$

Gradient expansion

$$f = \frac{2}{3} + \frac{4}{9} C_\eta w^{-1} + \frac{16}{27} C_\eta \frac{\omega_I}{|\omega|^2} (c_\sigma - 1) w^{-2} + \dots$$

- Early time behaviour better
- Extra initial conditions are required
- Stringent causality/stability conditions



- Pattern of poles shows a pair of complex conjugate nonhydro mode frequencies - expected.

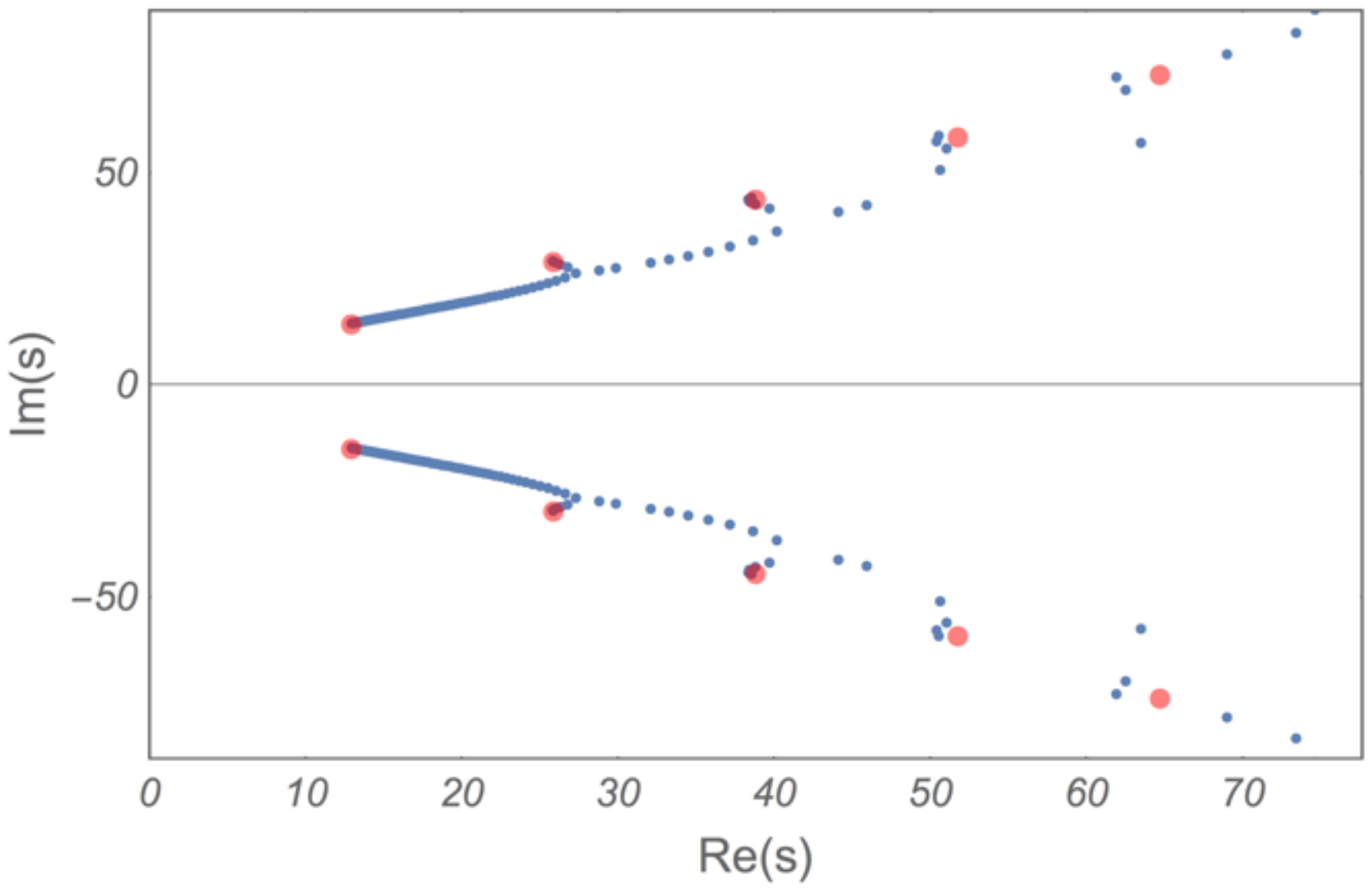
- Two-parameter transseries

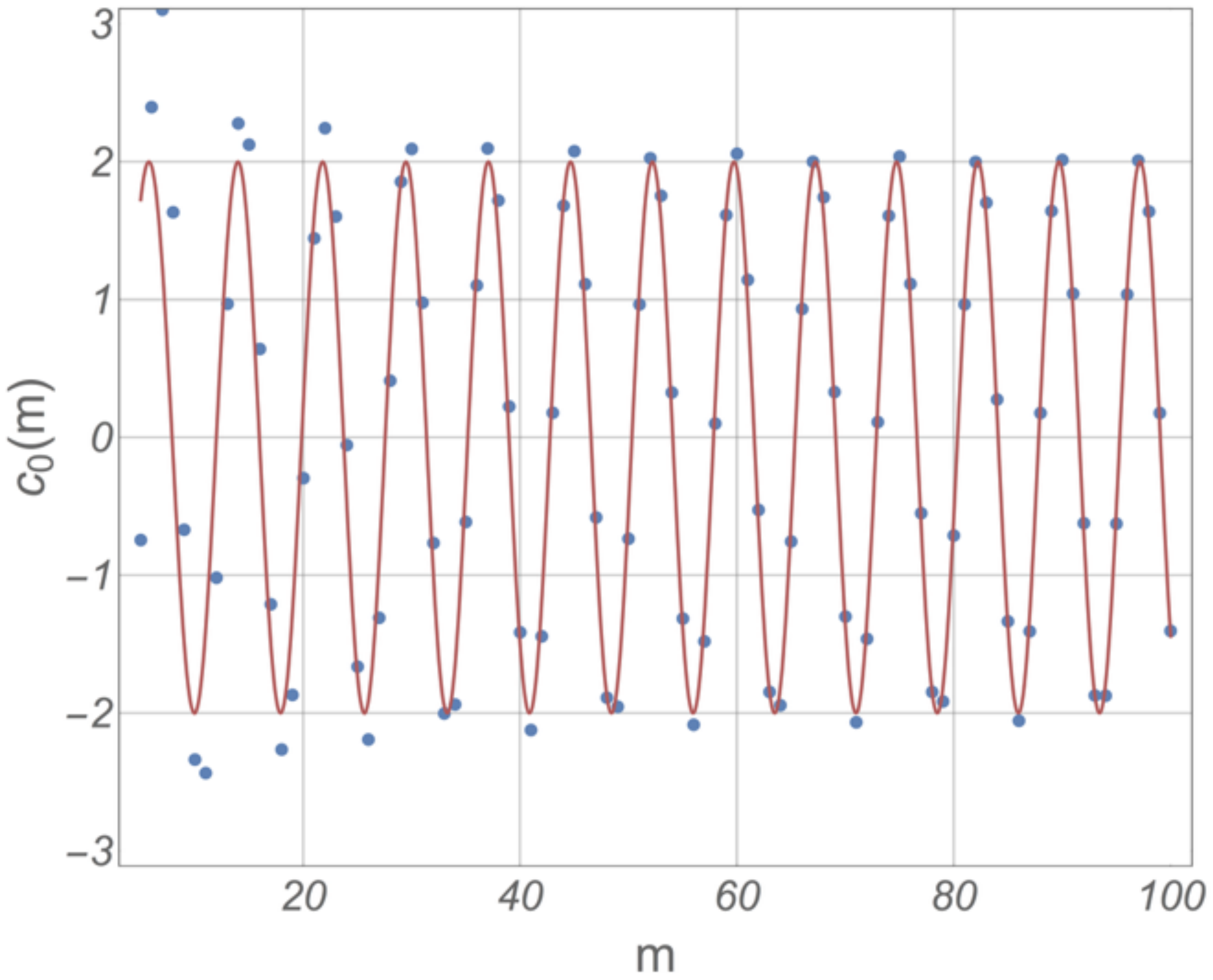
$$f\left(w,\sigma_{\pm}\right)=\sum_{n,m=0}^{+\infty}\sigma_{+}^{n}\sigma_{-}^{m}\mathrm{e}^{-\left(nA_{+}+mA_{-}\right)w}\Phi_{(n|m)}\left(w\right)$$

$$\Phi_{(n|m)}\left(w\right)=w^{\beta_{n,m}}\sum_{k=0}^{+\infty}a_{k}^{(n|m)}w^{-k}$$

- Resurgence relations are satisfied - the hydro series itself contains all the information.

[Aniceto, MS 1511.06358]





Summary

- Relativistic hydrodynamic theories include nonhydrodynamics modes which serve as a **regulator** for causality

- Information about these nonhydrodynamics modes is encoded in the **large order behaviour** of the gradient expansion

- In principle, hydrodynamic theories can be **engineered** to match the nonhydrodynamic sector of a given microscopic theory