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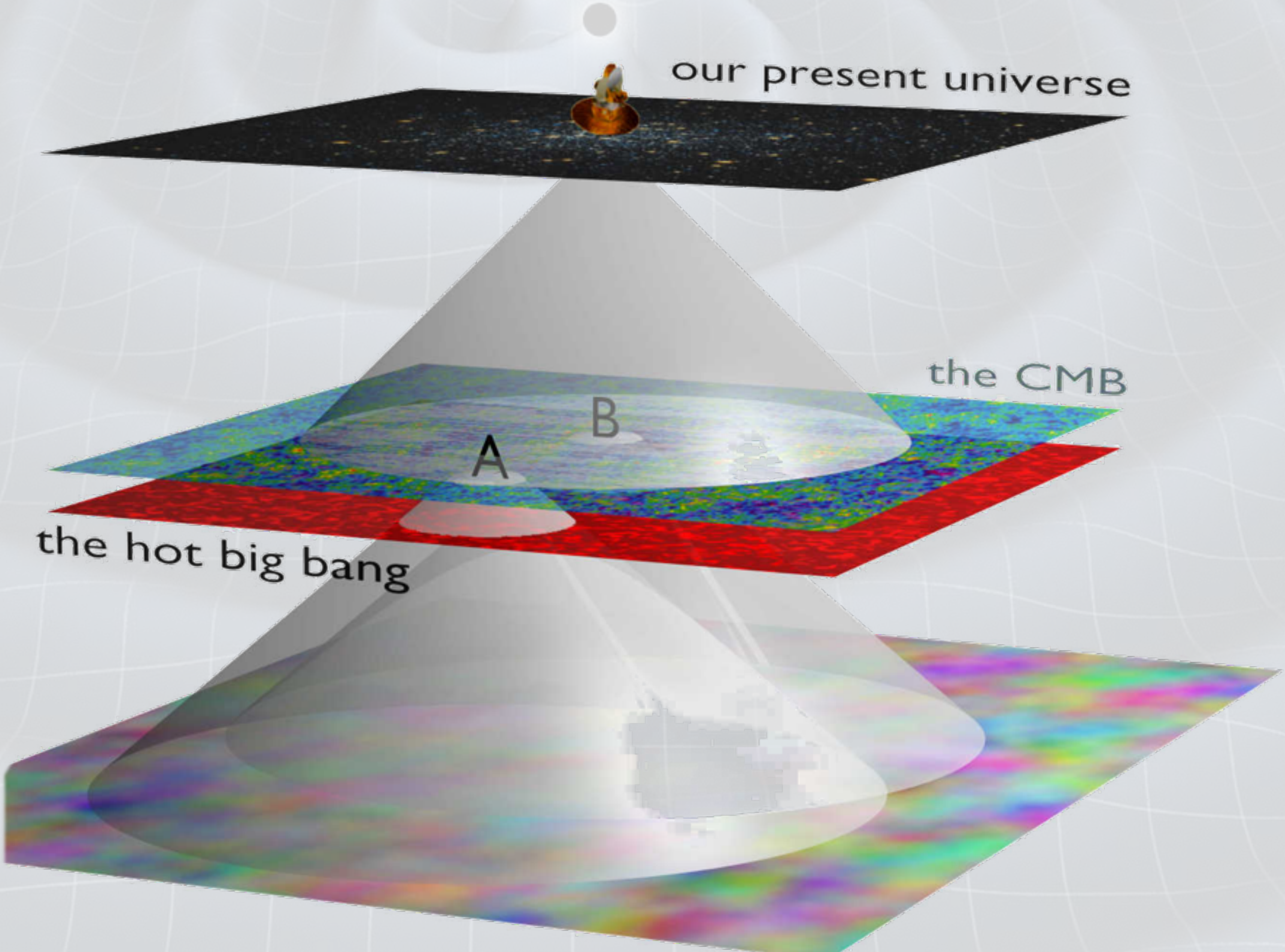
HEATING UP THE BIG BANG

Paul Saffin

Ed Copeland, Richard Easther, Hal Finkel, Zong-Gang Mou, Arttu Rajantie, Paul Tognarelli, Anders Tranberg, Shuang-Yong Zhou

Horizon Problem

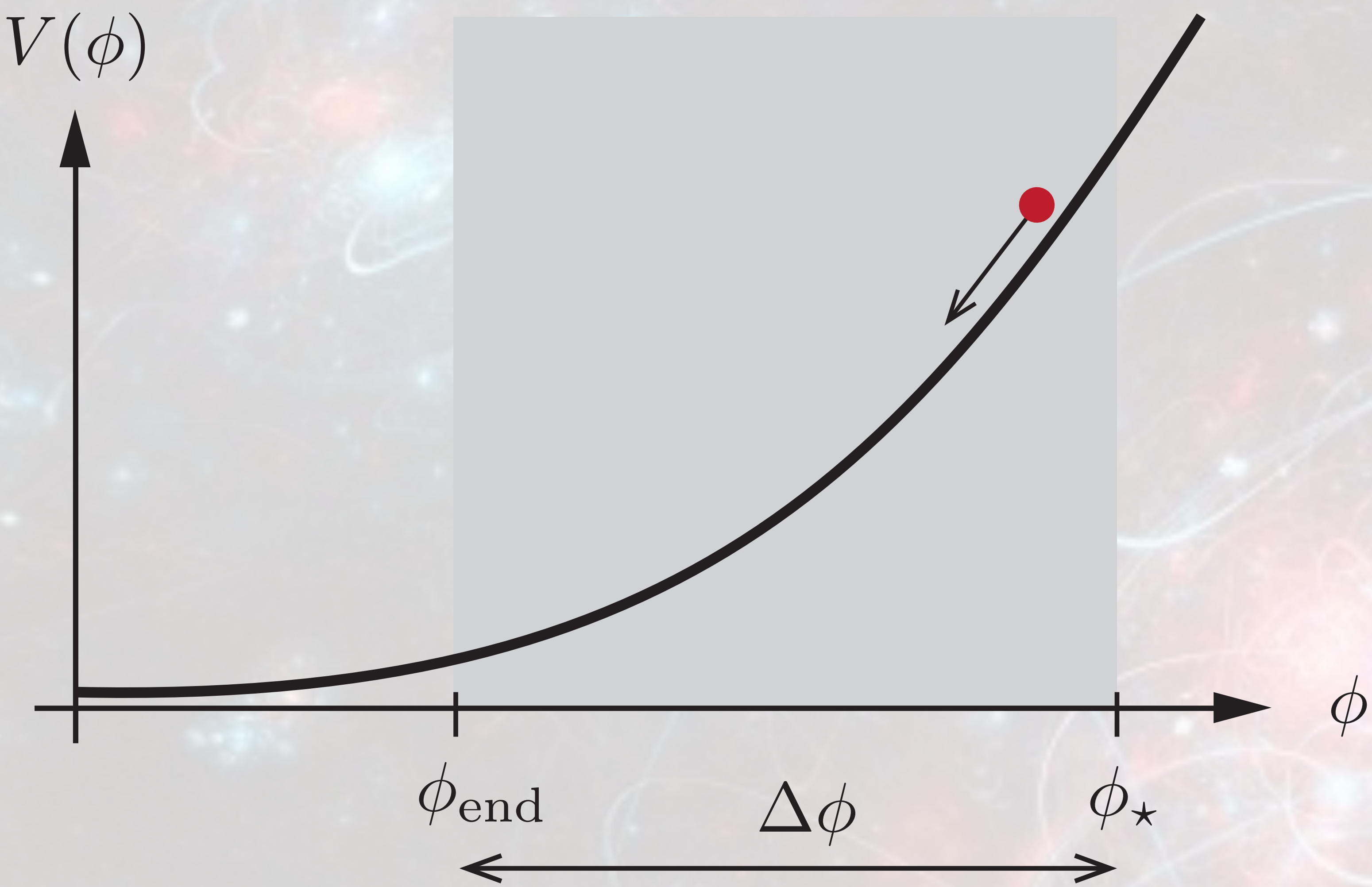


Wang arXiv:1303.1523

Outline

- reheating
- preheating
  - various epochs
  - formation of defects
  - gravitational waves
  - baryogengesis - EWK and otherwise
- conclusions

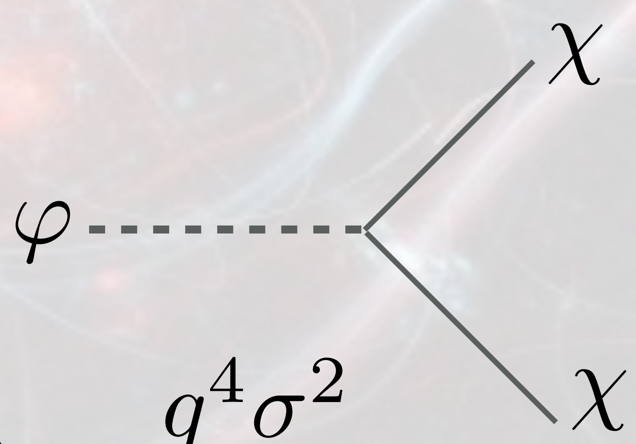
reheating:



$$\epsilon = -\frac{\dot{H}}{H^2} = -\frac{d\, ln\, H}{dN}$$

$$\epsilon \simeq \frac{M_p^2}{2}\left(\frac{V'}{V}\right)^2$$

$$\mathcal{L} = ... - \frac{1}{2}m^2\varphi^2 - 2g^2\sigma\varphi\chi^2$$



$$\Gamma(\varphi \to \chi\chi) = \frac{g^4\sigma^2}{8\pi m}$$

Kofman, Linde, Starobinsky, Albrecht, Steinhardt, Turner, Wilczek

missing physics: Bose enhancement

missing physics: Bose enhancement

$\varphi \to \chi\chi$

$$\langle n_\varphi - 1, n_k + 1, n_{-k} + 1 | a_k^\dagger a_{-k}^\dagger a_\varphi | n_\varphi, n_k, n_{-k} \rangle$$

$$= \sqrt{(n_k + 1)(n_{-k} + 1) n_\varphi}$$

$\chi\chi \to \varphi$

$$\langle n_\varphi + 1, n_k - 1, n_{-k} - 1 | a_k a_{-k} a_\varphi^\dagger | n_\varphi, n_k, n_{-k} \rangle$$

$$= \sqrt{n_k n_{-k} (n_\varphi + 1)}$$

missing physics: Bose enhancement

$\varphi \to \chi\chi$

$$\langle n_\varphi - 1, n_k + 1, n_{-k} + 1 | a_k^\dagger a_{-k}^\dagger a_\varphi | n_\varphi, n_k, n_{-k} \rangle$$

$$= \sqrt{(n_k + 1)(n_{-k} + 1) n_\varphi}$$

$\chi\chi \to \varphi$

$$\langle n_\varphi + 1, n_k - 1, n_{-k} - 1 | a_k a_{-k} a_\varphi^\dagger | n_\varphi, n_k, n_{-k} \rangle$$

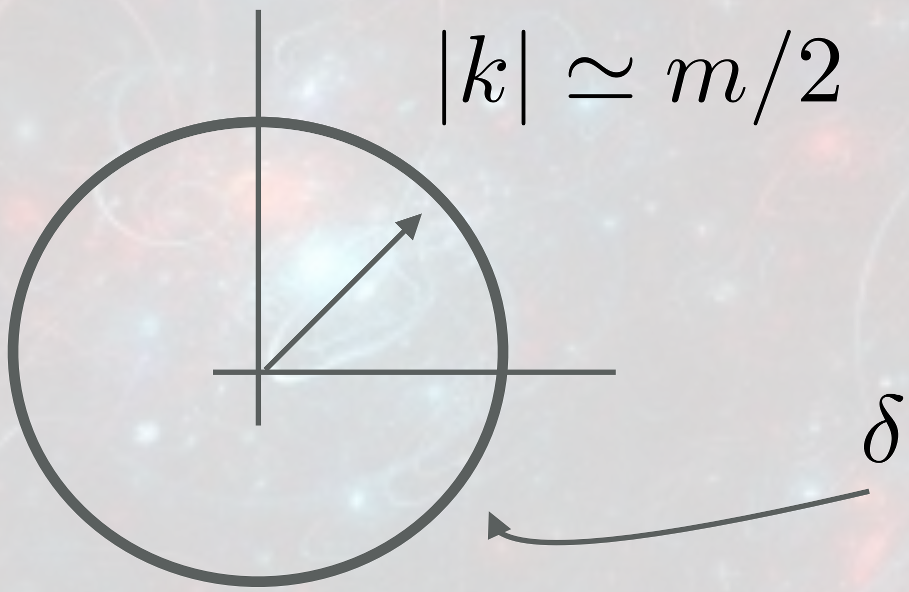
$$= \sqrt{n_k n_{-k} (n_\varphi + 1)}$$

$n_\varphi >> 1$
$n_k = n_{-k}$

$$(\varphi \to \chi\chi) - (\chi\chi \to \varphi) \sim n_\varphi (2n_k + 1)$$

$$\Gamma_{eff} = \Gamma_\chi (2n_k + 1)$$

missing physics: Bose enhancement



$$\delta k \simeq \frac{8g^2\sigma\Phi}{m}$$

$$m^2_{\chi,eff} = m^2_\chi + 4g^2\sigma\Phi$$

$$\frac{1}{a^3}\frac{d}{dt}\left[a^3 n_\chi\right] = \frac{2g^4\sigma^2}{8\pi m}\left[1 + \frac{\pi^2\Phi}{g^2\sigma}\frac{n_\chi}{n_\phi}\right]n_\phi$$

$$n_\chi \sim \exp\left(\frac{2\pi^2 g^2\sigma\Phi}{m^2}N\right)$$

what works against you?

- expansion narrows the resonance band $\delta k \sim \Phi \sim t^{-1}$
- inflaton amplitude decays and narrows resonance
- expansion redshifts modes out of resonance
- rescattering moves modes out of resonance

parametric resonance

$$V(\varphi,\chi) = \frac{1}{2}m^2\varphi^2 + \frac{1}{2}g^2\varphi^2\chi^2$$

$$\varphi(t) = \Phi(t)\sin(mt)$$

$$m^2_{\chi,eff} = g^2\varphi^2$$

$$\ddot{\chi}_k(t) + 3H\dot{\chi}_k(t) + \left(\frac{k^2}{a^2} + g^2\varphi^2\right)\chi_k(t) = 0$$

$$\Phi(t) \sim const \qquad z = mt \qquad A_k = \frac{k^2}{m^2} + 2q \qquad q = \frac{g^2\Phi^2}{4m^2}$$

Kofman, Linde, Starobinsky

parametric resonance

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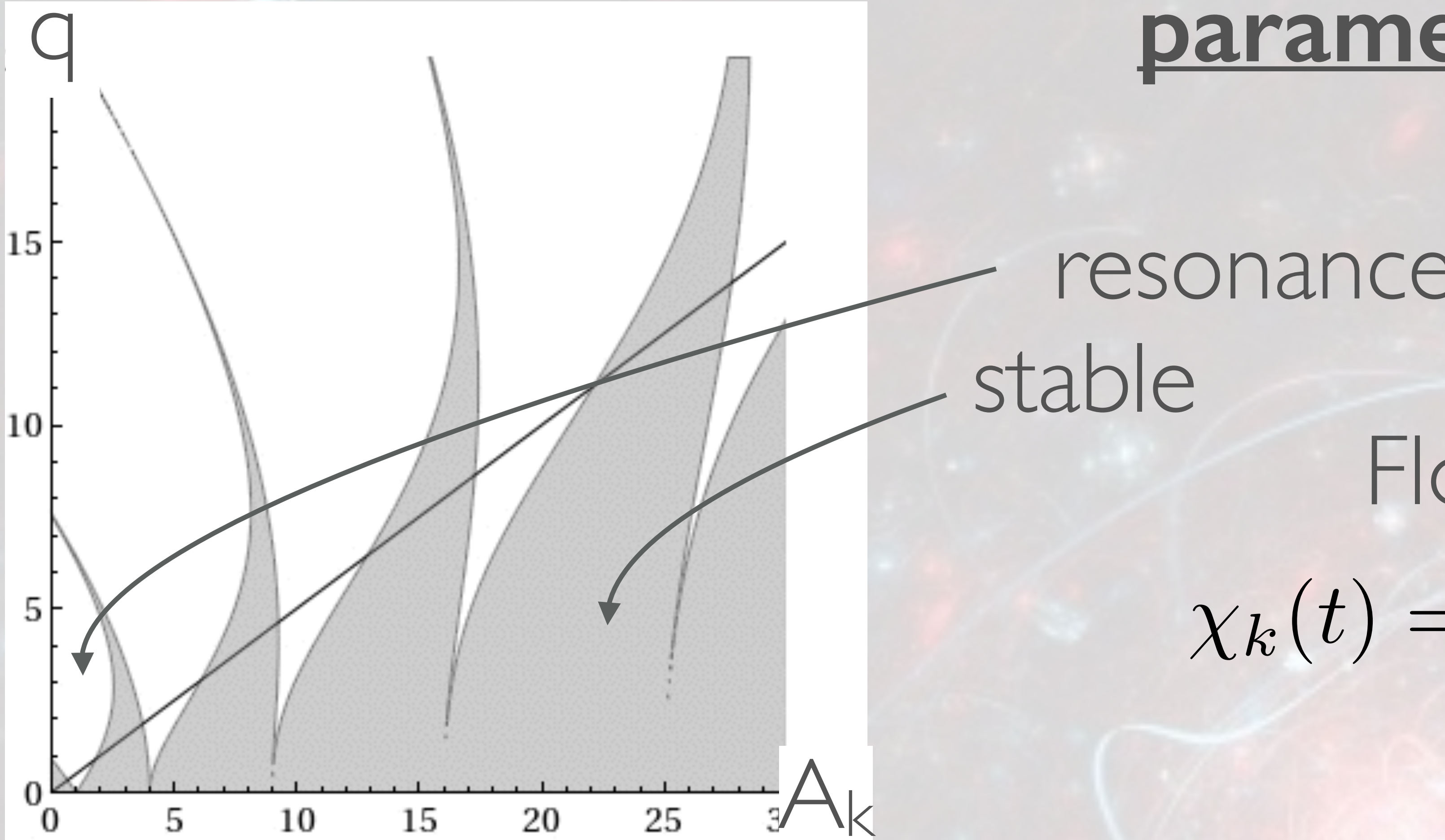
$\Phi(t) \sim const$ $\quad z = mt \quad$ $A_k = \frac{k^2}{m^2} + 2q$ $\quad q = \frac{g^2\Phi^2}{4m^2}$

driven-oscillator Mathieu equation

$$\chi_k'' + \left[A_k - 2q\cos(2z)\right]\chi_k = 0$$

Kofman, Linde, Starobinsky

parametric resonance

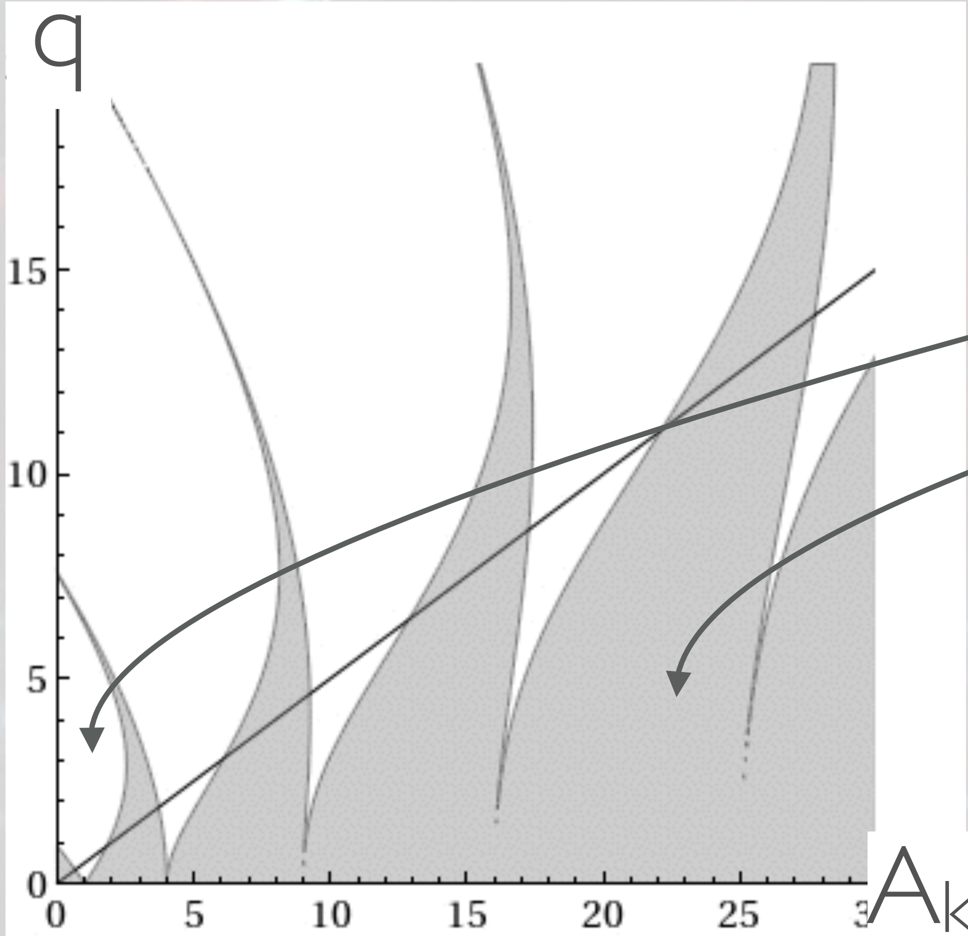


Floquet/Hill/Bloch

$$\chi_k(t) = e^{\mu_k t} P(A_k, q, z)$$

Kofman, Linde, Starobinsky

parametric resonance



resonance

stable

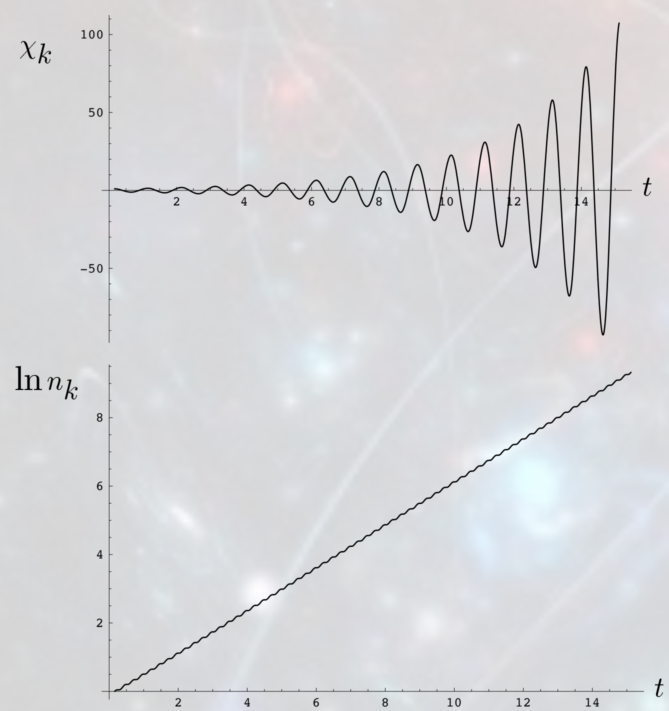
Floquet/Hill/Bloch

$$\chi_k(t) = e^{\mu_k t} P(A_k, q, z)$$

narrow resonance $q << 1$

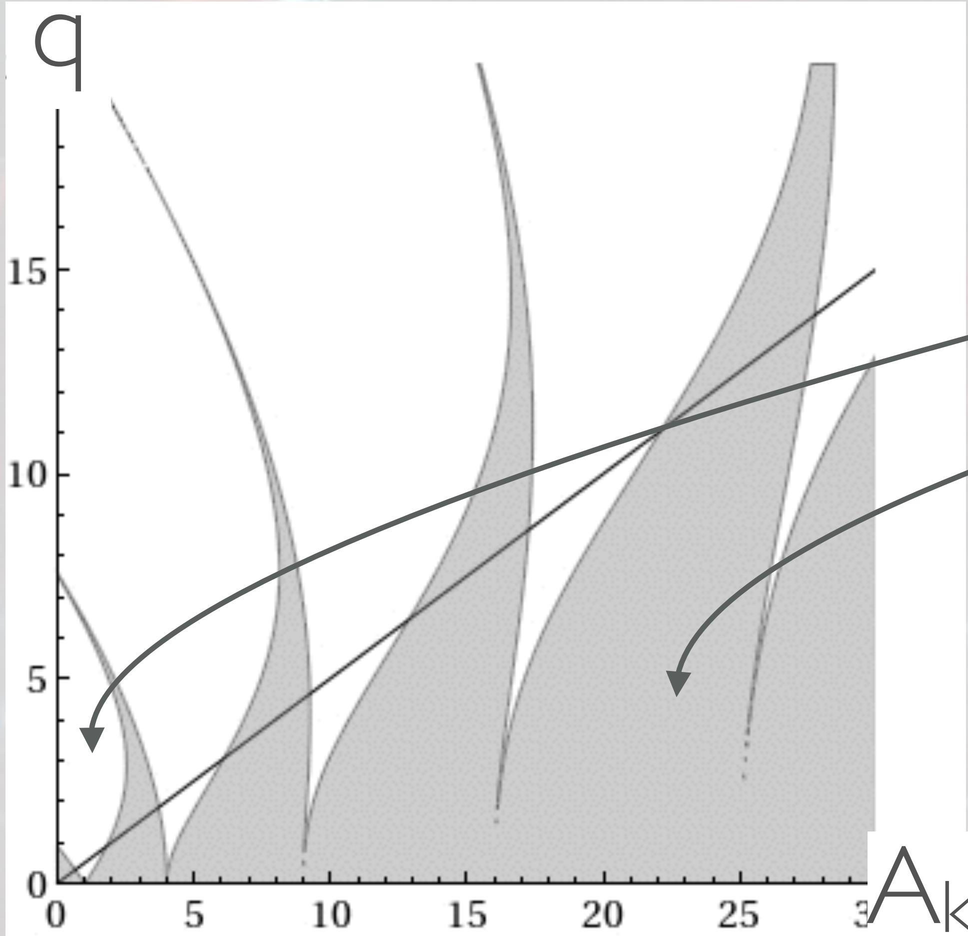
$$A_k^{(n)} \simeq n^2 \qquad \delta A_k^{(n)} \simeq q^n \qquad \Rightarrow \frac{k^2}{m^2} \simeq n^2 - 2q \pm q^n$$

first band:



Kofman, Linde, Starobinsky

parametric resonance



resonance

stable

Floquet/Hill/Bloch

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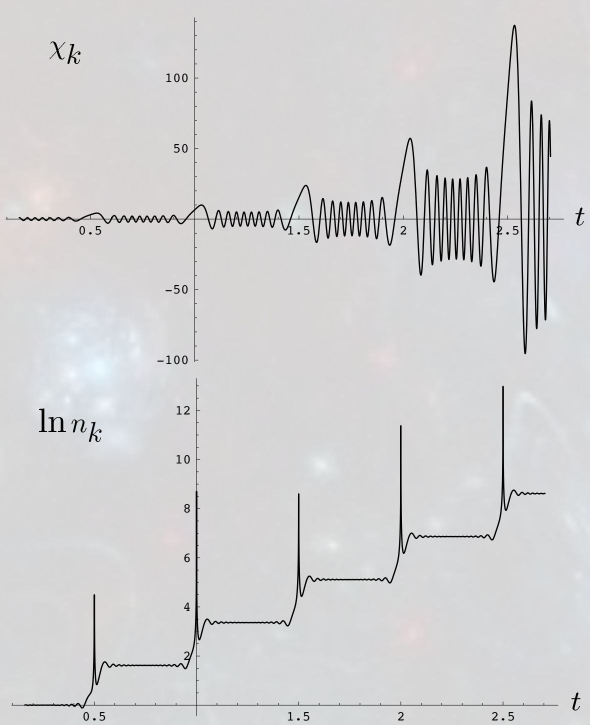
broad resonance $q >> 1$

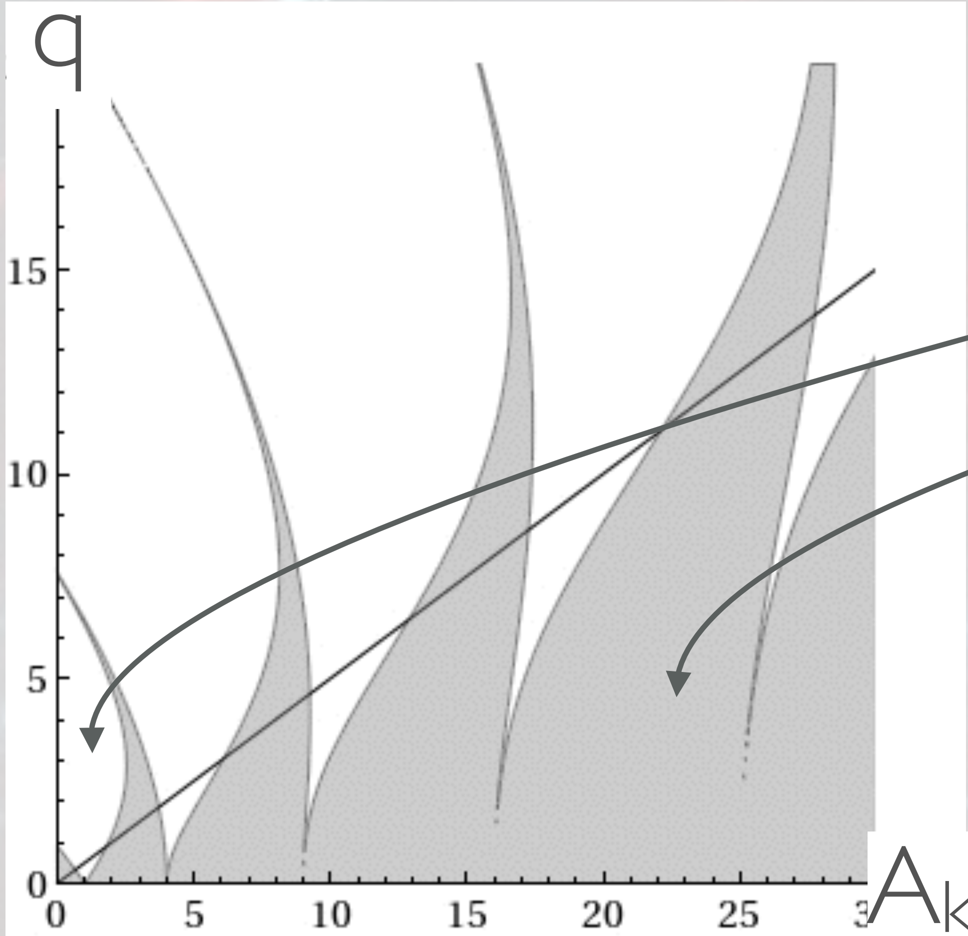


$$m_{\chi,eff} = g\Phi = m\sqrt{q}$$

$$k \lesssim k_\star = \sqrt{gm\Phi}$$

Kofman, Linde, Starobinsky

parametric resonance



resonance

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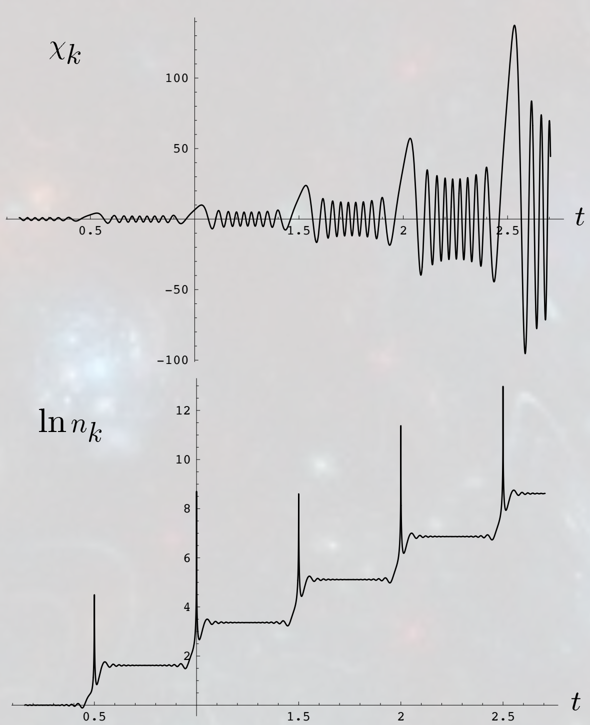

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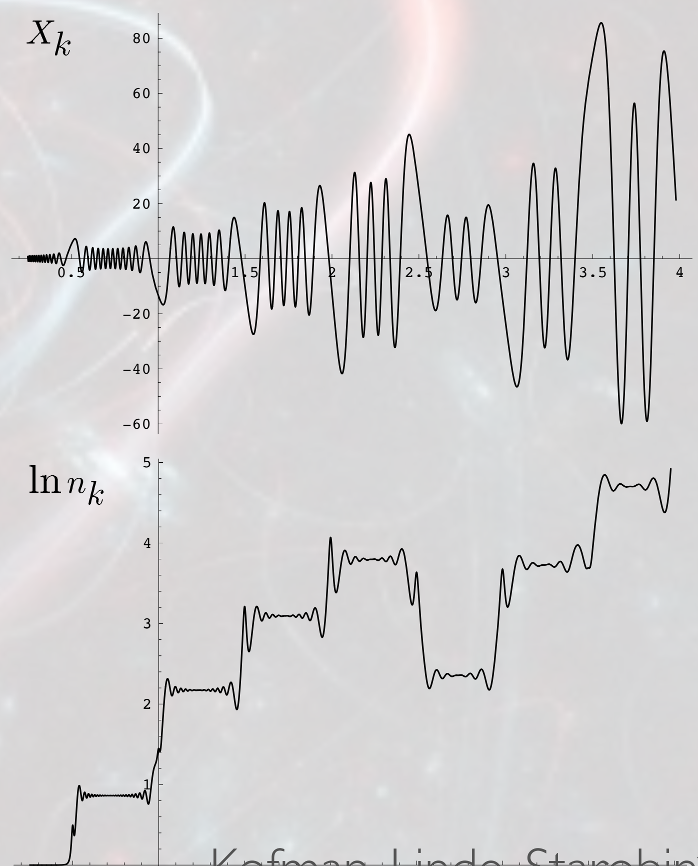


Kofman, Linde, Starobinsky

tachyonic preheating

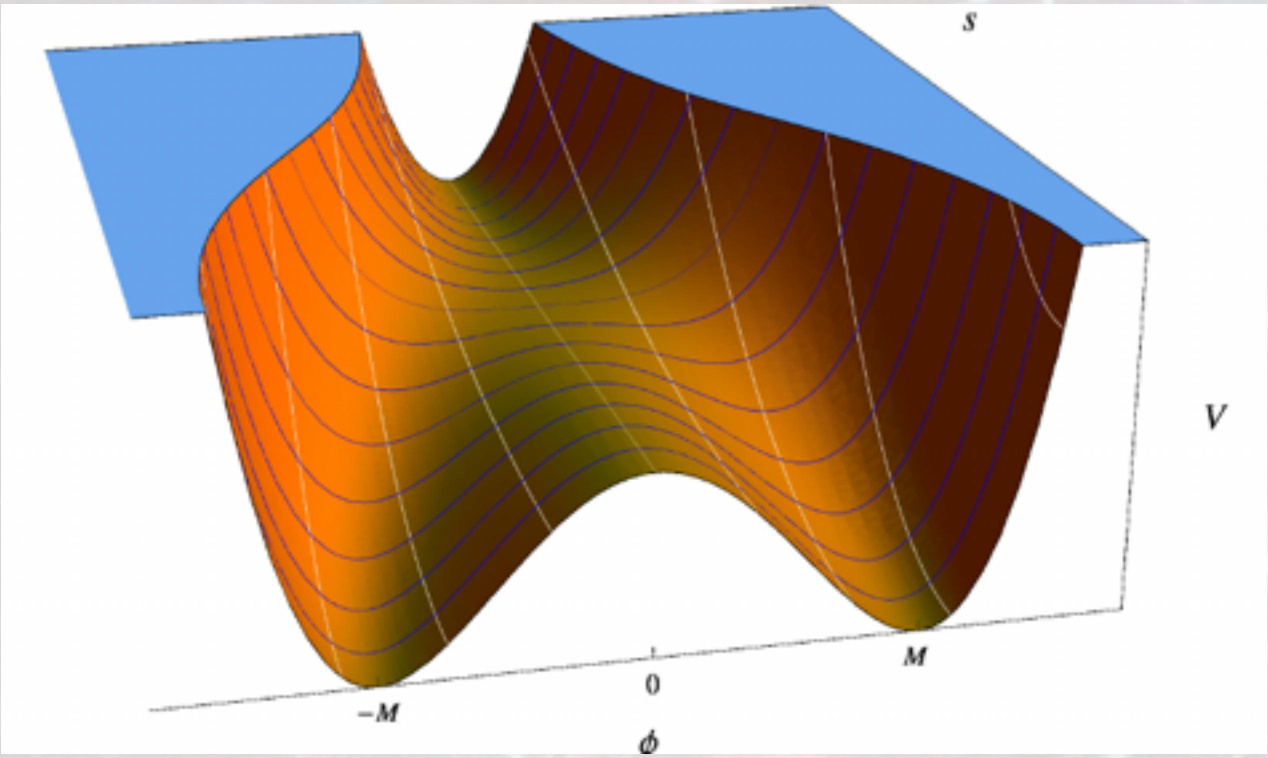
$V = -\frac{1}{2}m^2\sigma^2$ $\qquad$ $\ddot{\sigma}_k = (m^2 - k^2)\sigma_k$ $\qquad$ $k < m$ modes grow

Felder, Garcia-Bellido, Greene, Koeman, Linde, Tkachev, Bastero-Gil, Sanderon, King

tachyonic preheating

$$V = -\frac{1}{2}m^2\sigma^2 \qquad \ddot{\sigma}_k = (m^2 - k^2)\sigma_k \qquad k < m$$ modes grow

hybrid inflation



Felder, Garcia-Bellido, Greene, Kofman, Linde, Tkachev, Bastero-Gil, Sanderon, King

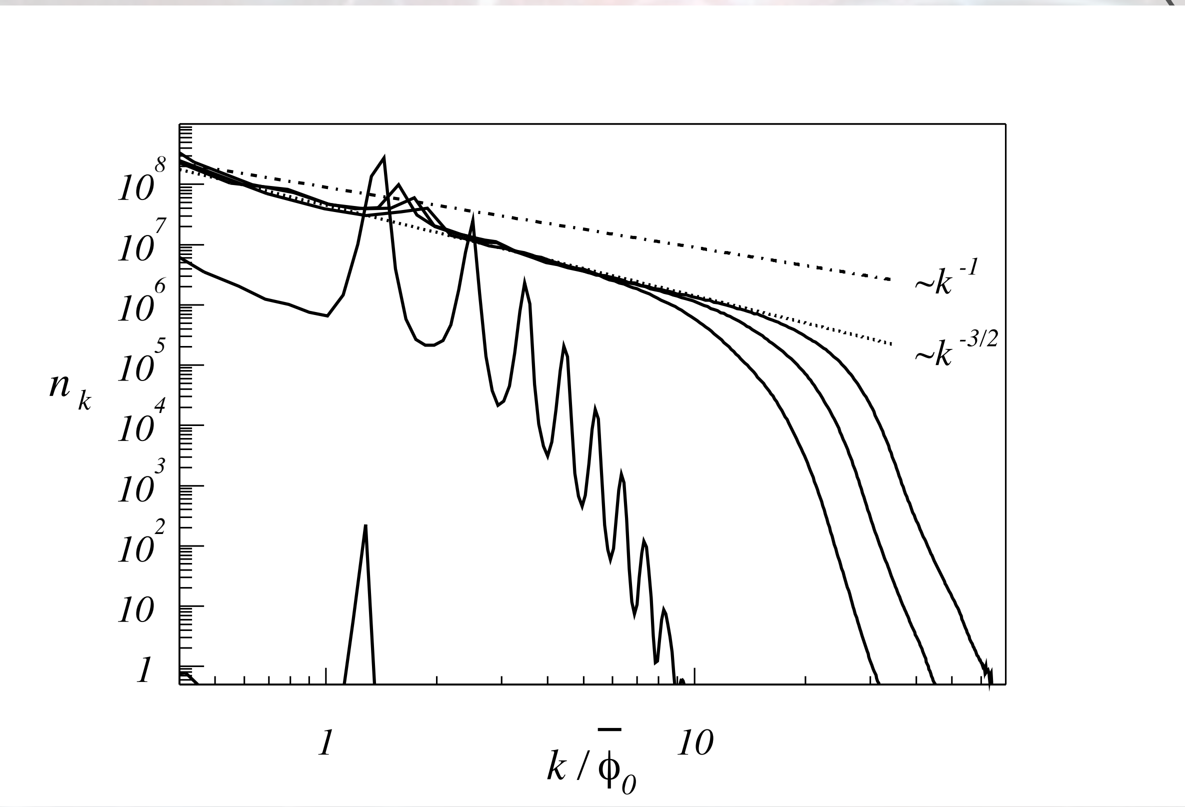
what happens then?

inflation → preheating → ? → thermal state

Khlebnikov, Tkachev, Felder, Koeman, Berges, Serreau, Micha, Rothkopf, Schmidt, Prokopec, Roos…

what happens then?

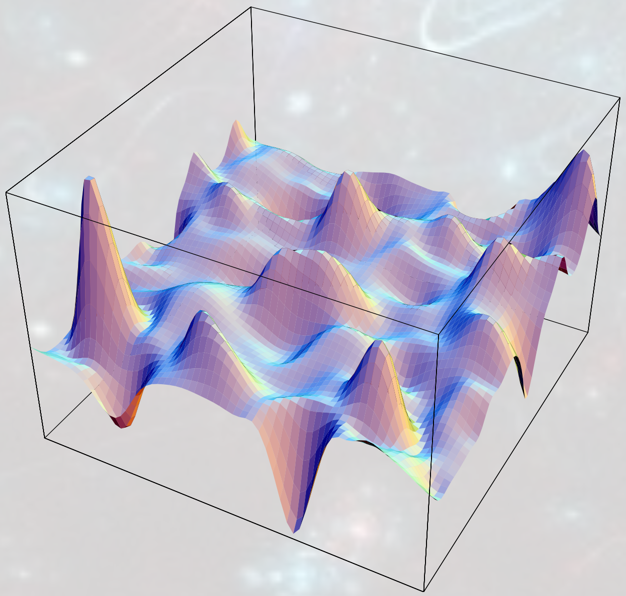
inflation → preheating → non-linear regime (classical) → turbulence (classical) → thermal state



Khlebnikov, Tkachev, Felder, Koeman, Berges, Serreau, Micha, Rothkopf, Schmidt, Prokopec, Roos…

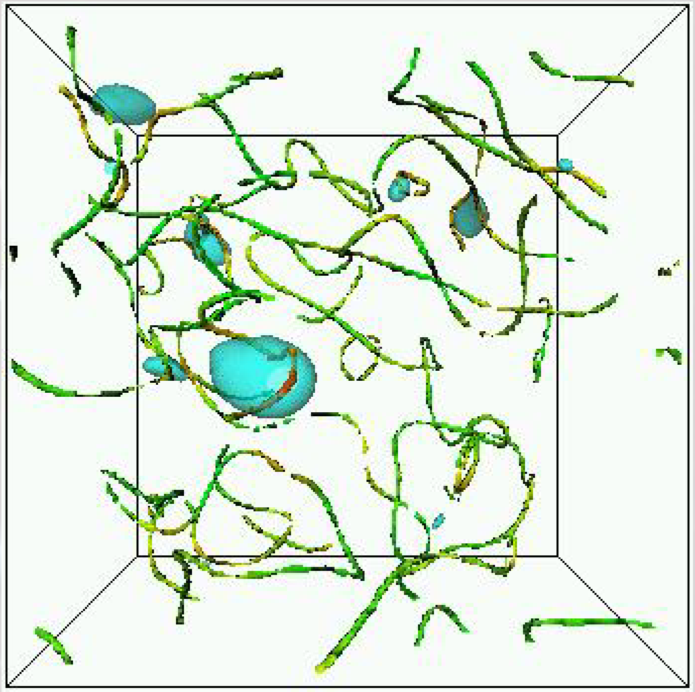
what to do with it?

make things (or make sure that you don't):

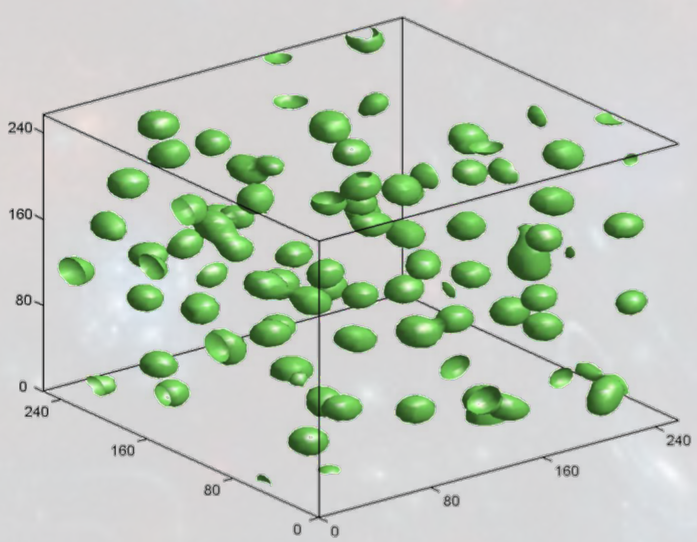


Dufaux, Bergman, Felder, Kofman,Uzan

- gravitational waves
- topological defects
- oscillons
- baryons
- …

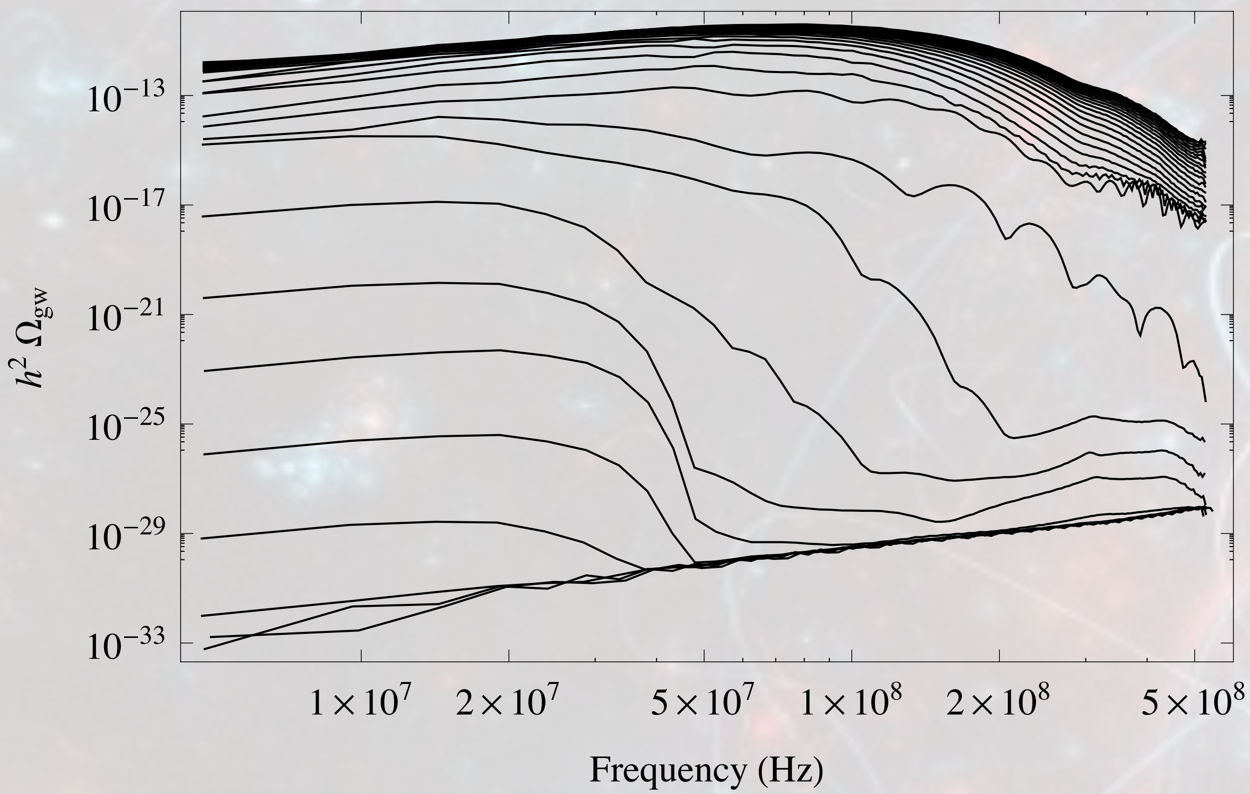


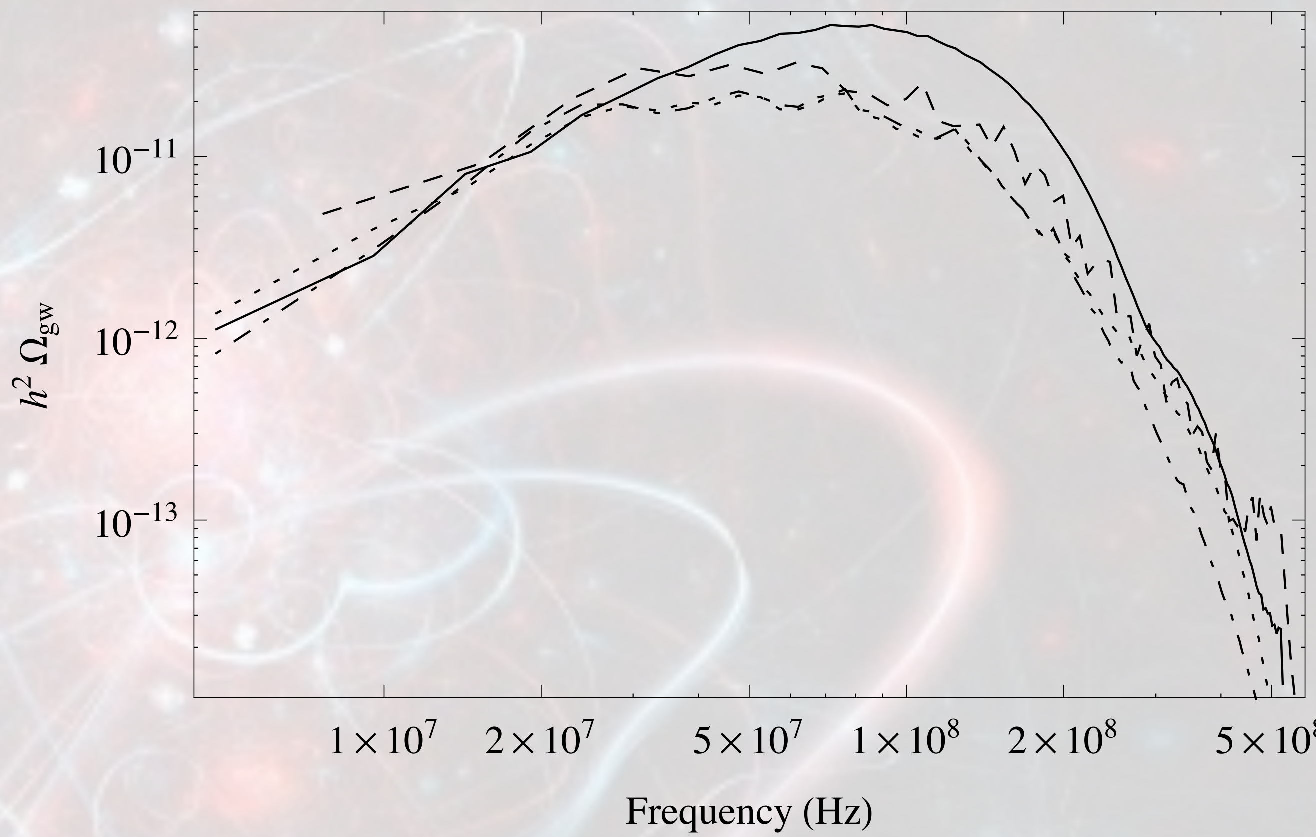
Copeland, Pascoli, Rajantie



Zhou, Copeland, Easther, Finkel, Mou, Saffin

gravitational waves

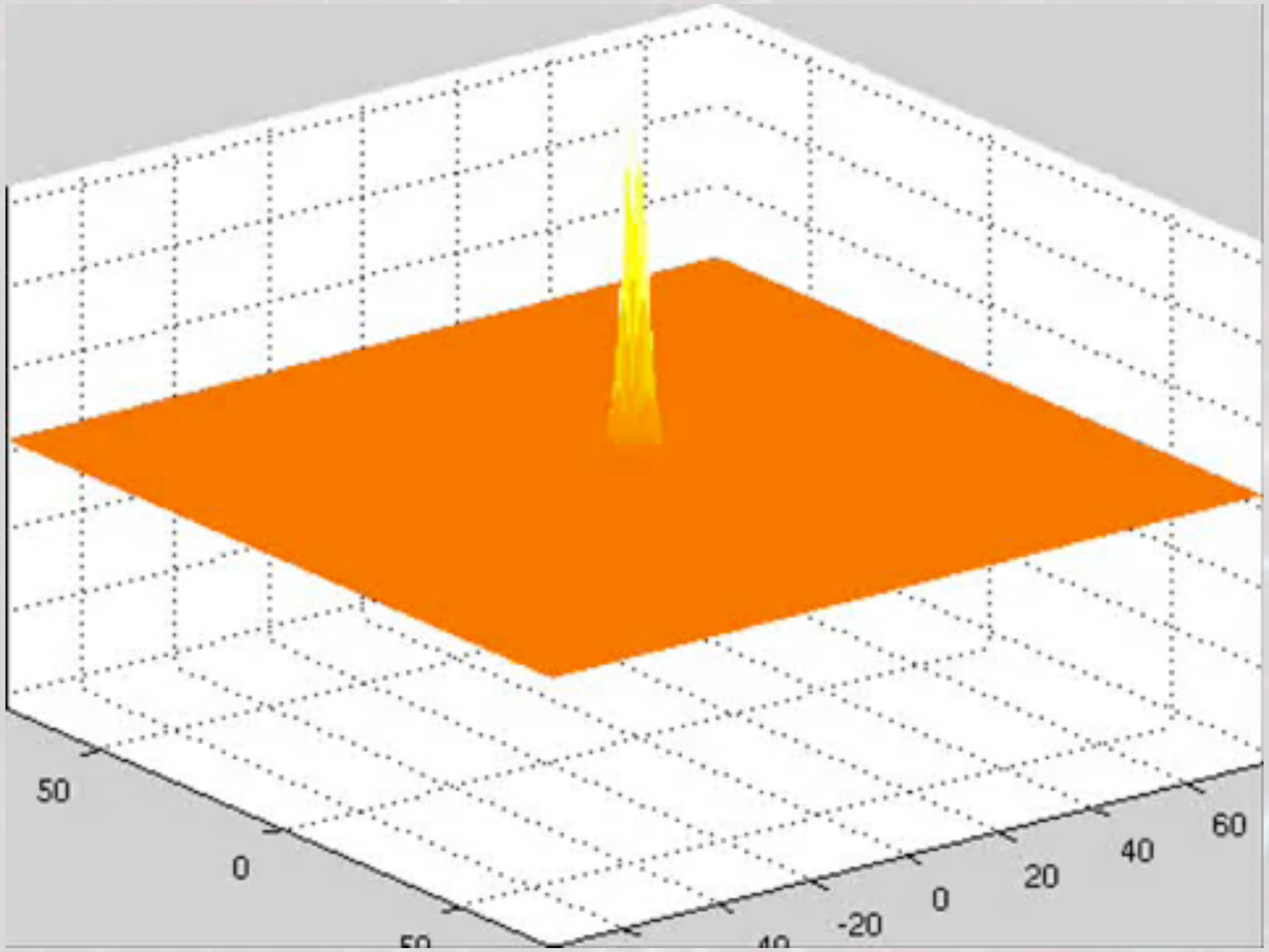




$$\lambda = 10^{-14},\ \phi_0 \sim 0.3 M_P,\ q = \frac{g^2}{\lambda} = 120$$

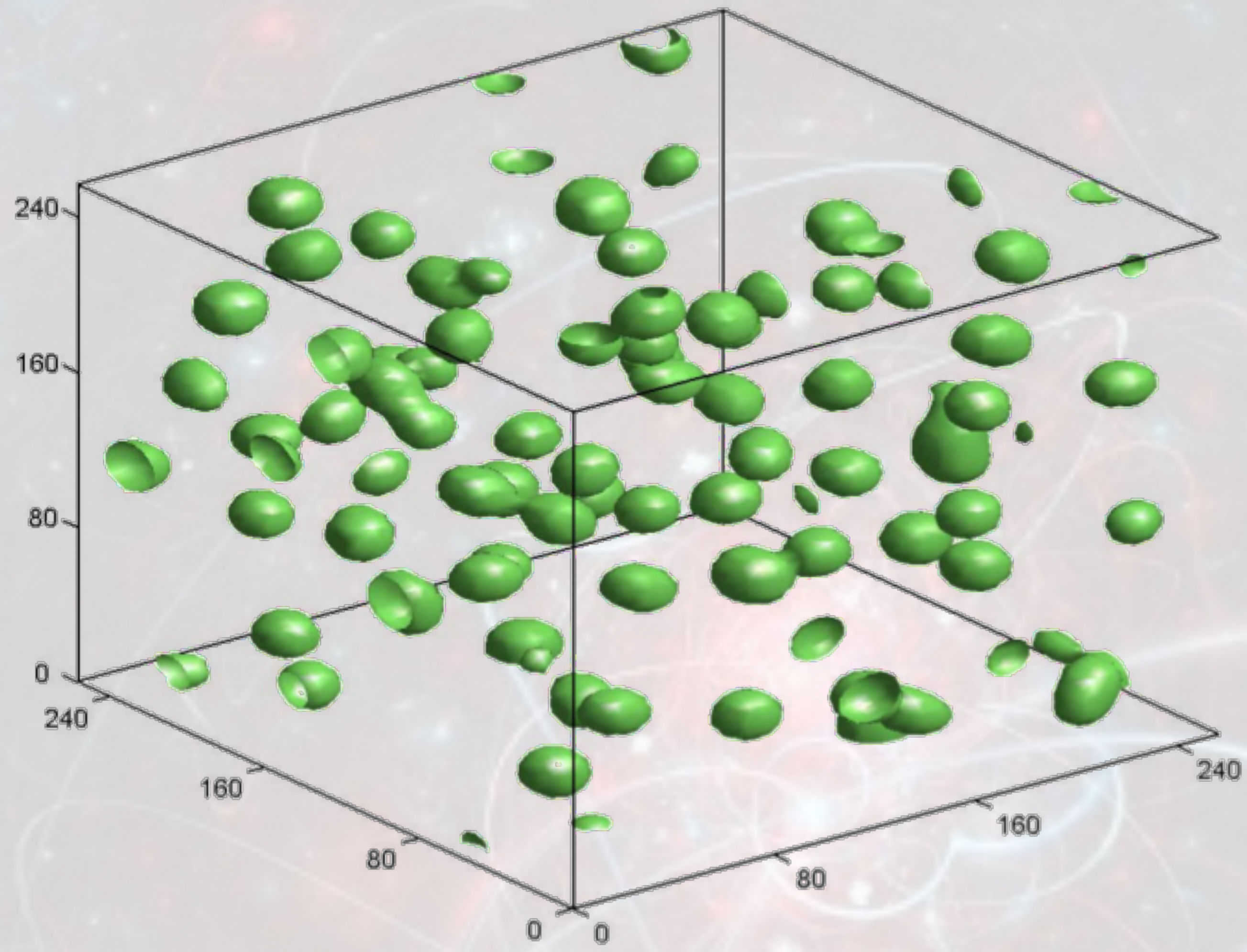
Khlebnikov, Tkachev, Garcia-Bellido, Figueroa, Sastre, Dufaux, Bergman, Felder, Kofman, Uzan, Easther, Giblin, Lim, Bassett, Bastero-Gil…

oscillons



Bogolyubsky, Makhanov, Gleiser,, Copeland, Muller, Honda, Choptuik, Hindmarsh, Salmi, Saffin,Tranberg, Amin, Esther, Finkel, Flau
Hertzberg, Mazumdar, Farhi, Graham, Khemani, Markov, Rosales,Kawasaki,Takahashi,Takeda…

gravitational waves from oscillons

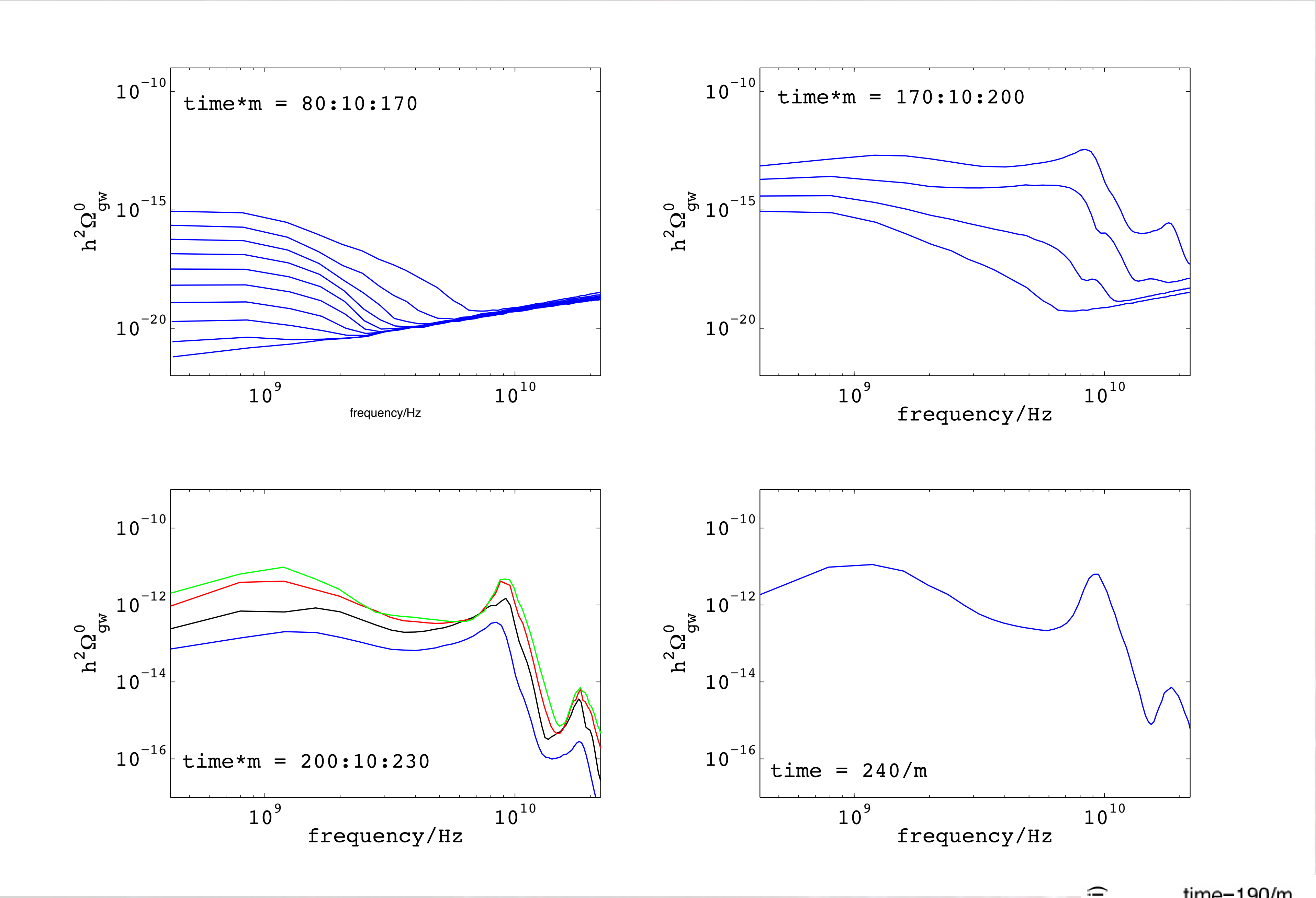


$$V(\phi) = m^2 M^2 \left[\sqrt{1 + \phi^2/M^2} - 1\right]$$

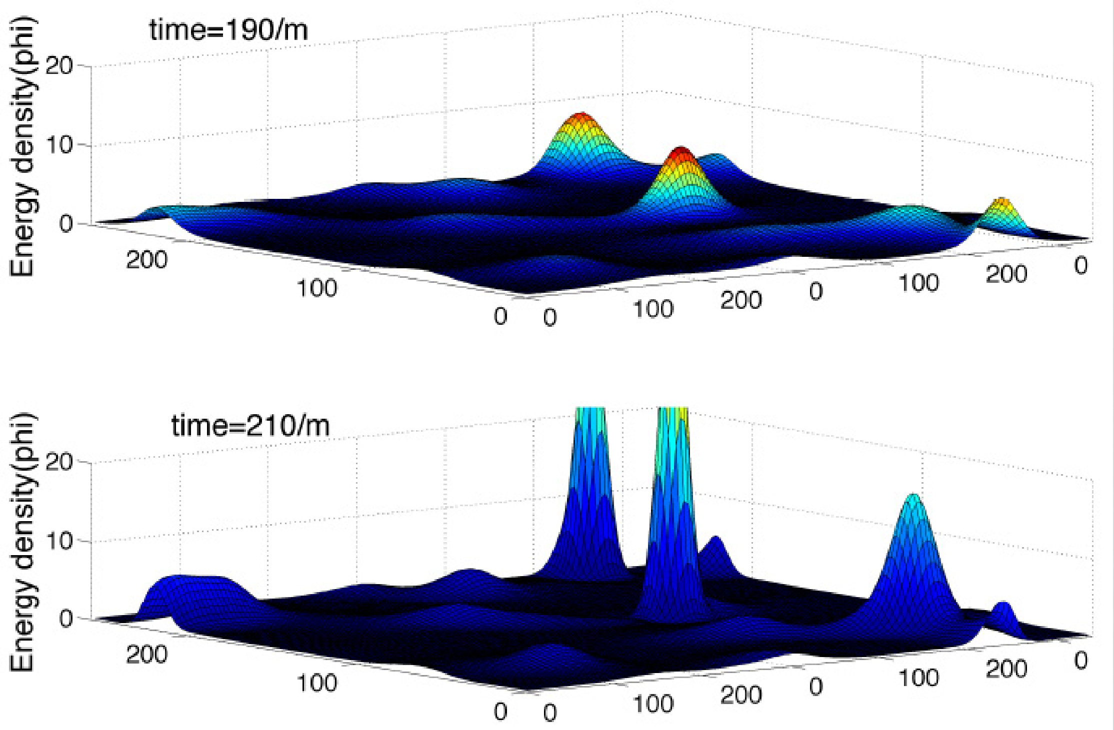
$$V(\phi) : \frac{1}{2} m^2 \phi^2 \to m^2 M \phi$$

$$M = 10^{-2} M_P, \; m = 10^{-5} M_P$$

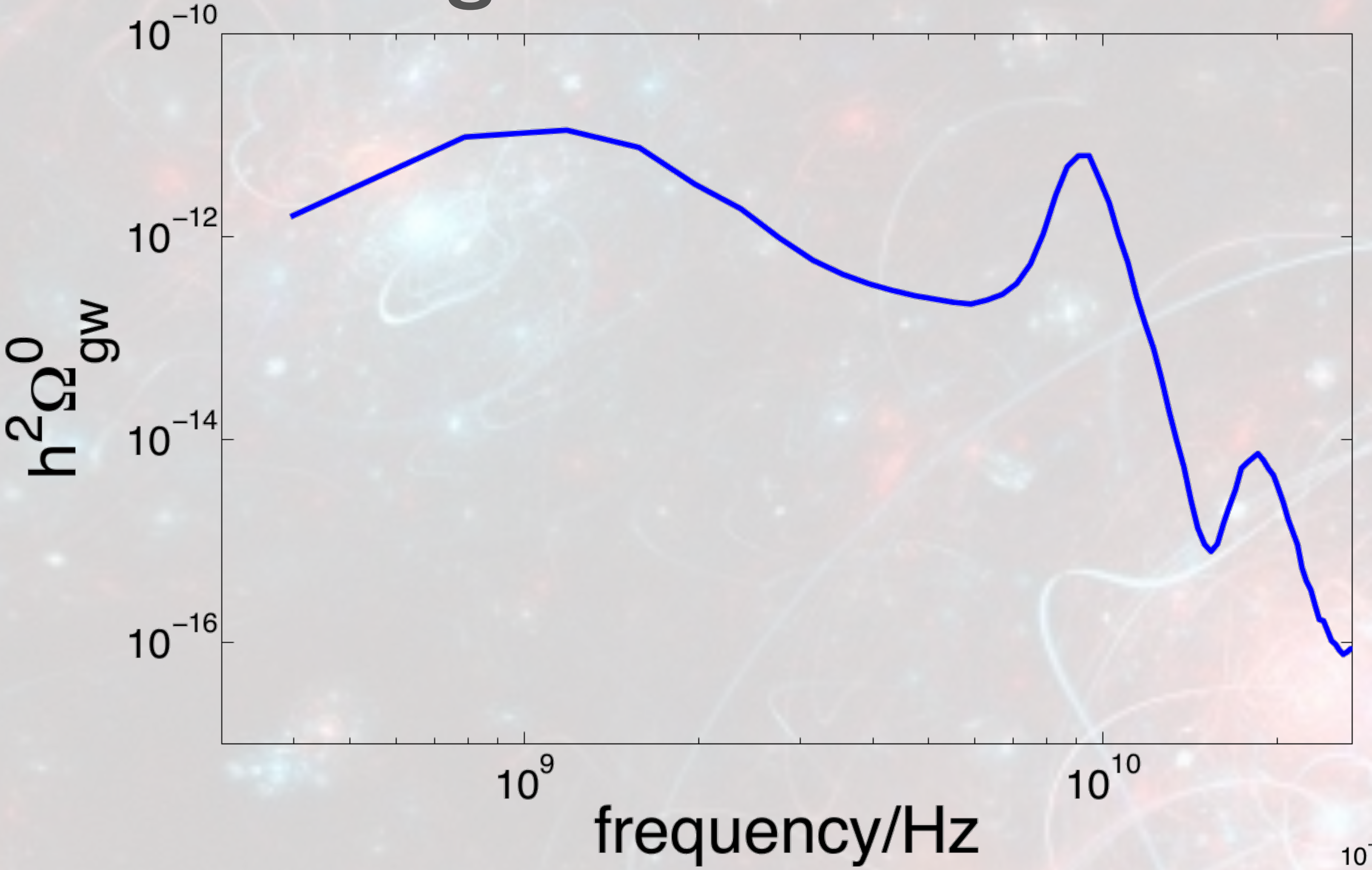
- Zhou, Copeland, Easther, Finkel, Mou, Saffin

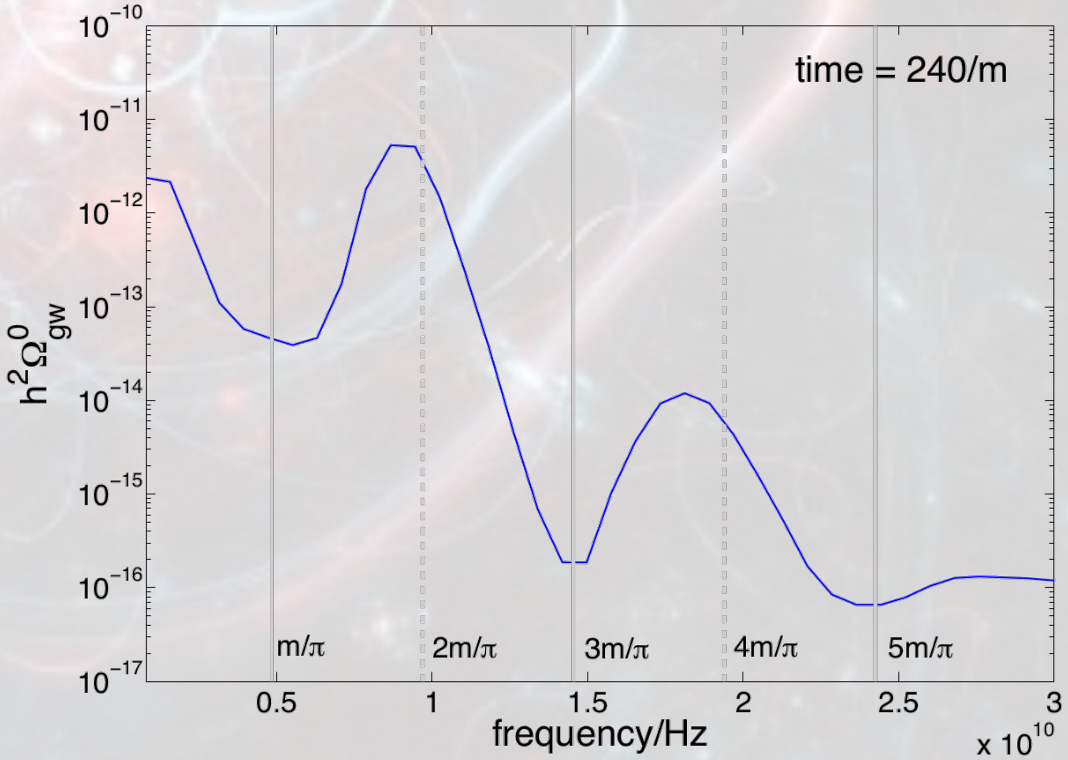


- parametric resonance
- non-linear, oscillons form
- oscillons stabilize
- long stable stage

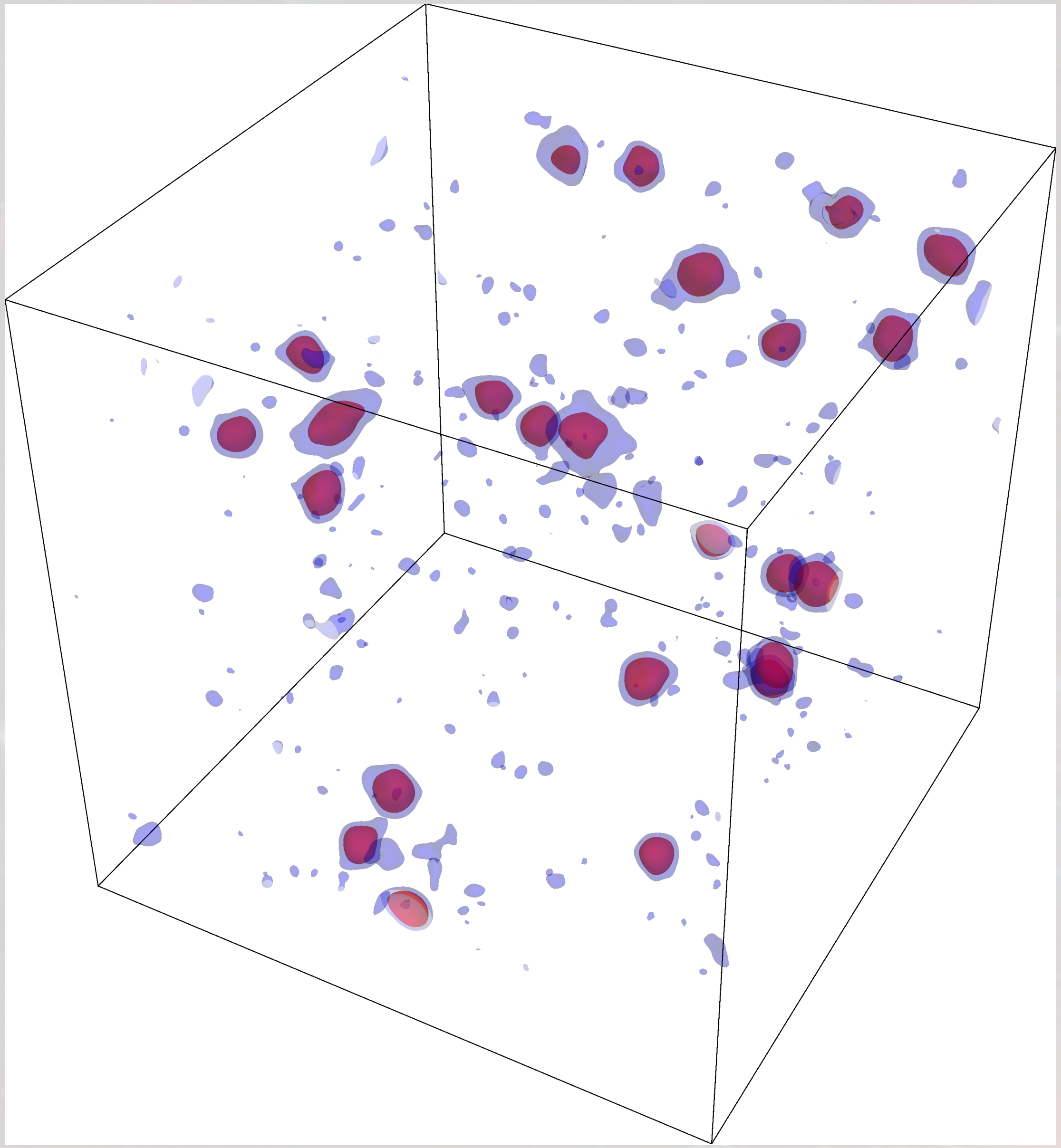


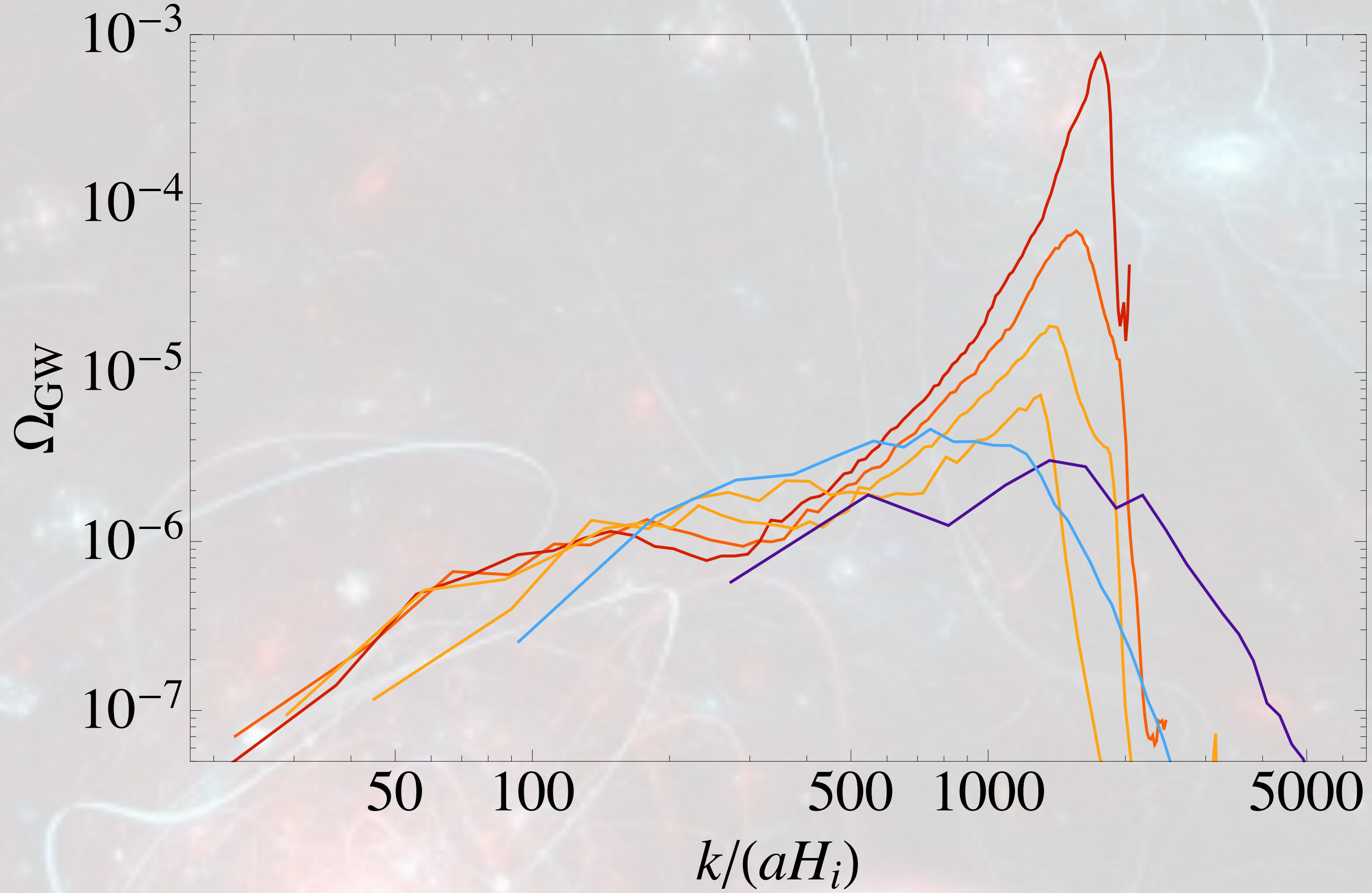
gravitational waves from oscillons





- Zhou, Copeland, Easther, Finkel, Mou, Saffin





$$V(\phi) = V_0\left(1 - \phi^p/v^p\right)^2$$

$$p = 6$$

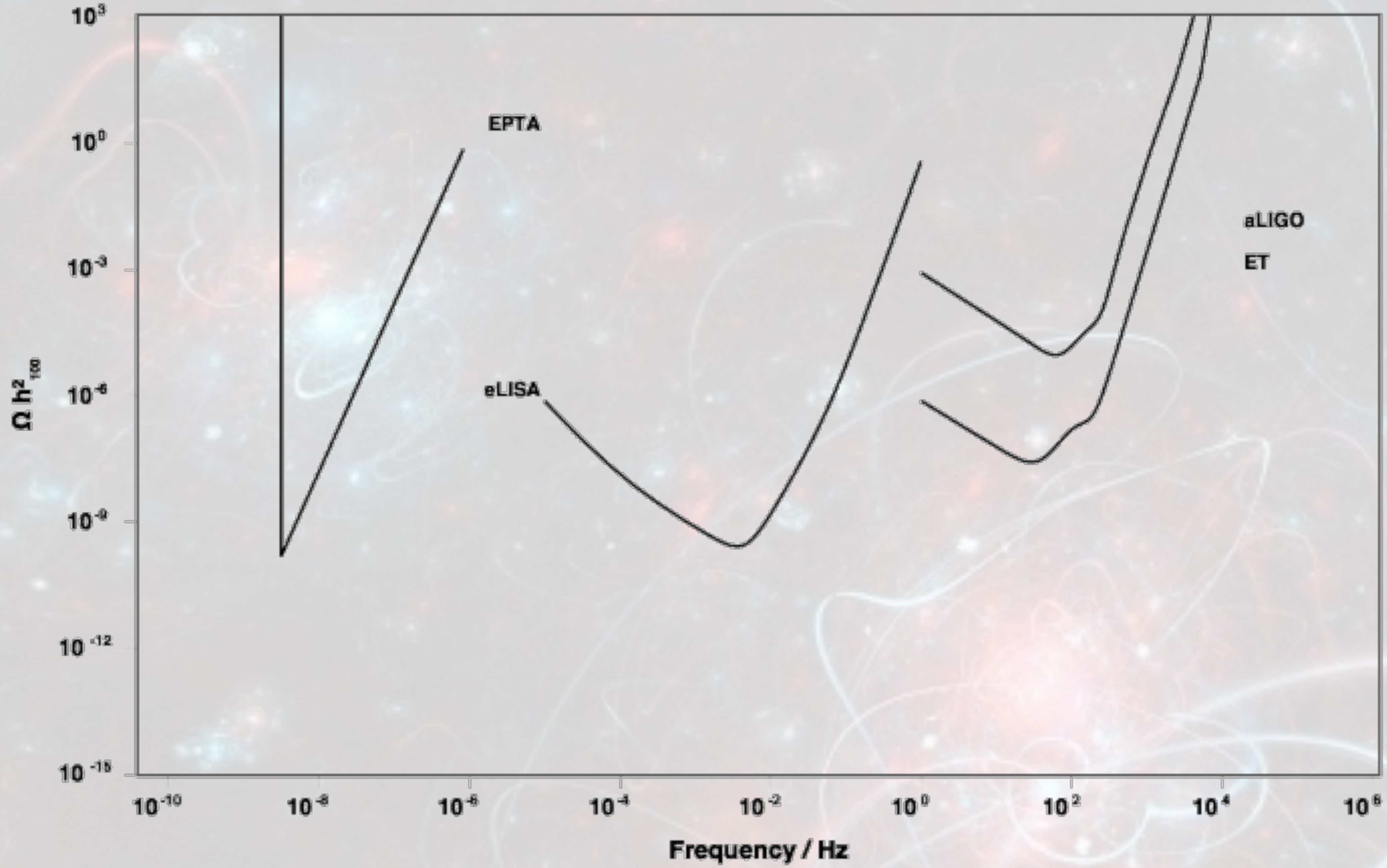
$$V_0 = \left(10^{-5} M_p\right)^4$$

$$v = 10^{-2} M_p$$

$$r = 10^{-12}$$

$$n_s = 0.96$$

Antusch, Cefala,Orani

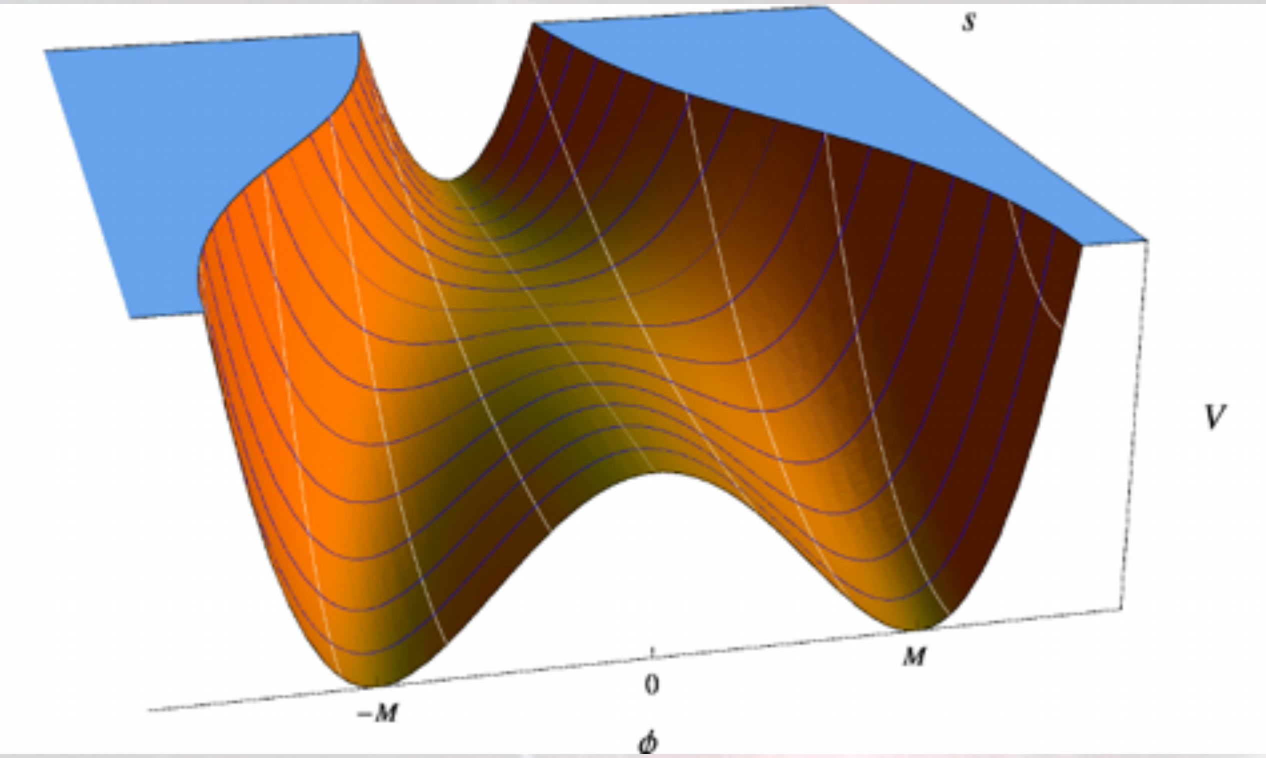


rhcole.com/apps/GWplotter

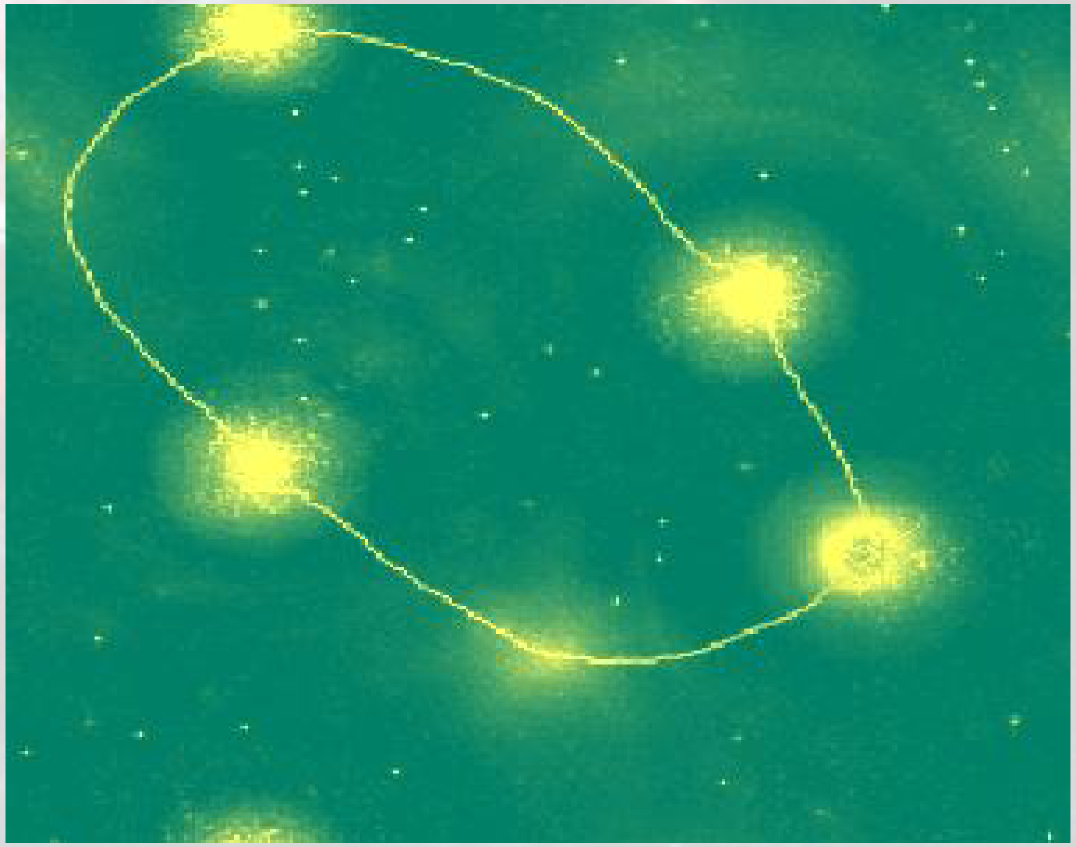
high frequency detectors:

- Goryachev, Tobar - acoustic cavities, MHz-GHz, arXiv:1410.2334
- Arvanitaki, Geraci - optically levitated sensors, MHz, arXiv:1207.5320
- Cruise, Ingley - 100MHz, Class. Quant. Grav. 23
- INFN Genoa -
- Kawamura Japan - 100MHz
- www.GravWave.com

can also form stable topological defects



$$V(\varphi, \chi) = \frac{1}{2}m_\varphi^2\varphi^2 + \frac{1}{2}g^2\varphi^2\chi^2 + \frac{1}{4}\lambda\left(\chi^2 - \nu^2\right)^2$$



$$m = \sqrt{\lambda}\,\nu \qquad \varphi_c = m/g$$

$$\chi^2(t) = [m^2 - g^2\varphi^2(t)]/\lambda$$

$$\omega_\varphi^2 \simeq V''_{eff}(0) = (g^2/\lambda)m^2$$

$$\omega_\varphi^2 \simeq V''_{eff}(\varphi = 0) = (g^2/\lambda)m^2$$

Kasuya, Kawasaki;Tkachev, Khlebnikov, Kofman, Linde, Aska, Buchmuller, Covi, Copeland, Pascoli, Rajantie

electroweak baryogenesis and preheating

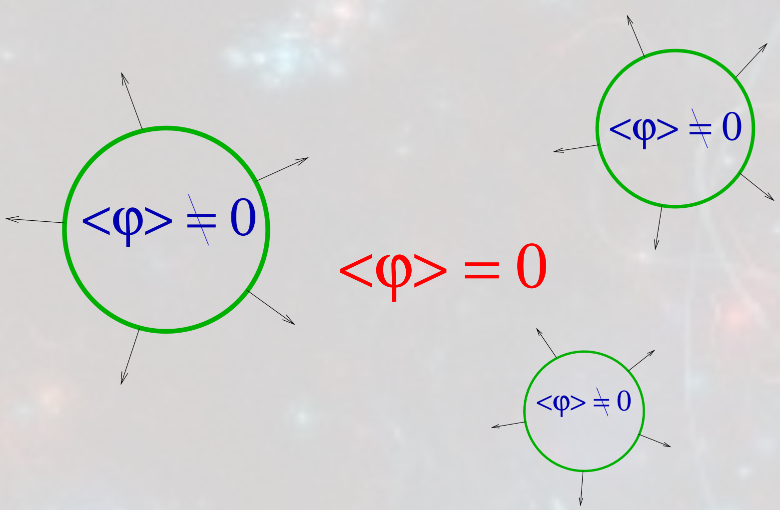
Krauss, Trodden, Garcia-Bellido, Grigoriev, Kusenko, Shaposhnikov, Copeland,,Lyth, Rajantie, Saffin, Tranberg, Mou,Smit,Wu
Konstandin, Servant, Brauner,Enqvist, Stephens, Taanila,Hernandez,Hindmarsh

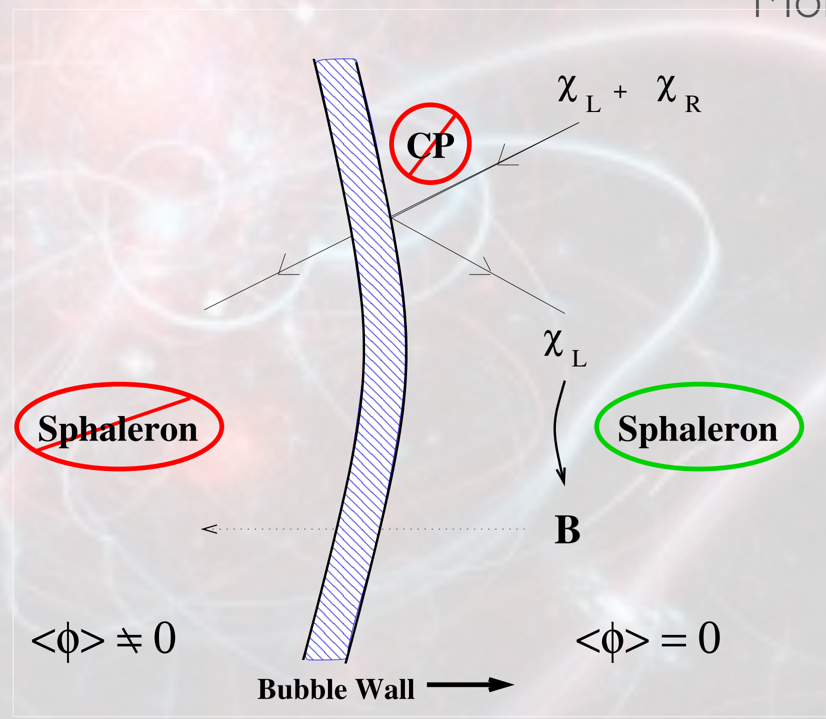
electroweak baryogenesis

$$\eta = \frac{n_B - n_{\bar{B}}}{n_\gamma} \simeq 10^{-10}$$

- violate Baryon number conservation
- violate C, CP
- depart from thermal equilibrium

Morrissey, Ramsey-Musolf





Sakharov,Kizmin, Rubakov, Shaposhnikov,Krauss,Trodden, Garcia-Bellido, Grigoriev, Kusenko, Shaposhnikov, Bochkarev, Kajantie,Laine,Rummukainen,

electroweak baryogenesis and preheating

$$S_H = -\int d^4x \left[D_\mu \phi^\dagger D^\mu \phi + \lambda(\phi^\dagger \phi - \nu^2/2)\right]$$

$$S_W = -\int d^4x \frac{1}{4} W^a_{\mu\nu} W^{a,\mu\nu}$$

$$S_f = -\int d^4x \left[\bar{q}_L \gamma^\mu D_\mu q_L + \bar{u}_R \gamma^\mu D_\mu u_R + \bar{d}_R \gamma^\mu D_\mu d_R \right.$$
$$\left. + \bar{l}_L \gamma^\mu D_\mu l_L + \bar{\nu}_R \gamma^\mu D_\mu \nu_R + \bar{e}_R \gamma^\mu D_\mu e_R\right]$$

$$S_Y = -\int d^4x \left[G^u \bar{q}_L \phi u_R + G^d \bar{q}_L \phi d_R + G^e \bar{l}_L \phi e_R + G^\nu \bar{l}_L \phi \nu_R \right.$$
$$\left. + \hat{G}^u \bar{q}_L \tilde{\phi} u_R + \hat{G}^d \bar{q}_L \tilde{\phi} d_R + \hat{G}^e \bar{l}_L \tilde{\phi} e_R + \hat{G}^\nu \bar{l}_L \tilde{\phi} \nu_R + h.c.\right]$$

$$D_\mu \phi = \left(\partial_\mu - \frac{ig}{2}\sigma^a W^a_\mu\right)\phi \qquad D_\mu q_L = \left(\partial_\mu - \frac{ig}{2}\sigma^a W^a_\mu\right) q_L \qquad D_\mu l_L = \left(\partial_\mu - \frac{ig}{2}\sigma^a W^a_\mu\right) l_L$$

$$D_\mu u_R = \partial_\mu u_R, \ldots \qquad [D_\mu, D_\nu]\,\phi = -\frac{ig}{2}\sigma^a W^a_{\mu\nu}\phi$$

Saffin, Tranberg, Mou

electroweak baryogenesis and preheating

$$q \to e^{i\alpha\gamma^5} q,\ l \to e^{i\alpha\gamma^5} l$$

$$j^\mu_{(b)} = i\left[\bar{q}_L\gamma^\mu q_L + \bar{u}_R\gamma^\mu u_R + \bar{d}_R\gamma^\mu d_R\right]$$

$$j^\mu_{(l)} = i\left[\bar{l}_L\gamma^\mu l_L + \bar{\nu}_R\gamma^\mu \nu_R + \bar{e}_R\gamma^\mu e_R\right]$$

electroweak baryogenesis and preheating

$$j^{\mu}_{(b)} = i\left[\bar{q}_L\gamma^{\mu}q_L + \bar{u}_R\gamma^{\mu}u_R + \bar{d}_R\gamma^{\mu}d_R\right]$$

$$q \to e^{i\alpha\gamma^5}q,\ l \to e^{i\alpha\gamma^5}l$$

$$j^{\mu}_{(l)} = i\left[\bar{l}_L\gamma^{\mu}l_L + \bar{\nu}_R\gamma^{\mu}\nu_R + \bar{e}_R\gamma^{\mu}e_R\right]$$

$$\partial_{\mu}j^{\mu}_{(b)} = \partial_{\mu}j^{\mu}_{(l)} = \frac{n_f}{32\pi^2}\left[\frac{1}{2}\epsilon_{\mu\nu\rho\sigma}W^{a,\mu\nu}W^{a,\rho\sigma}\right] = \partial_{\mu}K^{\mu}$$

$$K^{\mu} = \frac{n_f}{16\pi^2}\epsilon^{\mu\nu\sigma\rho}\left[W^a_{\nu\rho}W^a_{\sigma} - \frac{2}{3}\epsilon_{abc}W^a_{\nu}W^b_{\rho}W^c_{\sigma}\right]$$

electroweak baryogenesis and preheating

$$N_B = \int d^3x \; j_B^0 = n_f \int d^3x \; K^0 = n_f N_{CS}$$

$$B(t) - B(0) = 3[N_{CS}(t) - N_{CS}(0)]$$

Tranberg, Smit Hernandez, Konstandin,Schmidt,Turok, Zadrozny

electroweak baryogenesis and preheating

$$N_B = \int d^3x \; j_B^0 = n_f \int d^3x \; K^0 = n_f N_{CS}$$

$$B(t) - B(0) = 3[N_{CS}(t) - N_{CS}(0)]$$

why bother with fermions if $N_f = N_{CS}$?

$$\frac{K_{CP}}{M_W^2} \phi^\dagger \phi \epsilon^{\mu\nu\sigma\rho} W^a_{\mu\nu} W^a_{\rho\sigma}$$

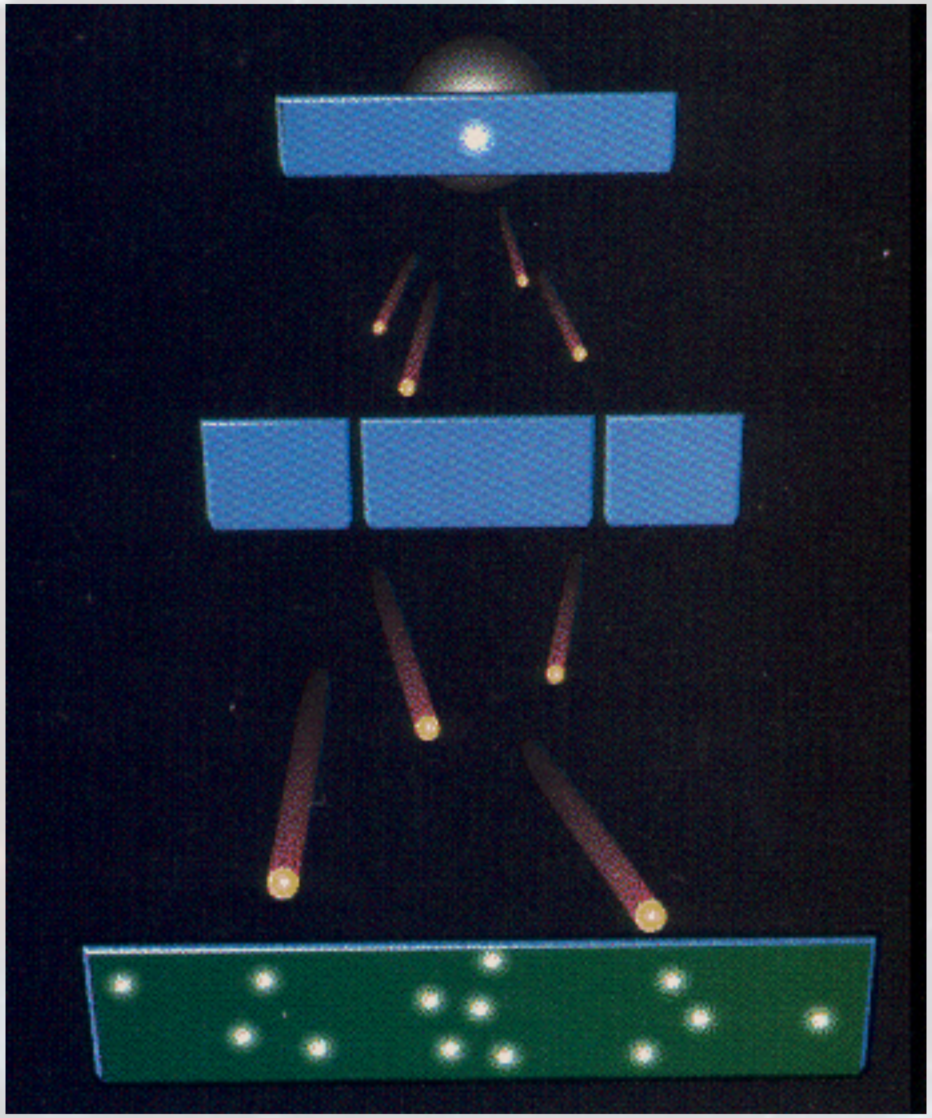
$$\frac{\delta_{CP}}{16\pi^2 m_W^2} i \left(\phi_1^\dagger \phi_2 - \phi_2^\dagger \phi_1 \right) \epsilon^{\mu\nu\sigma\rho} W^a_{\mu\nu} W^a_{\sigma\rho}$$

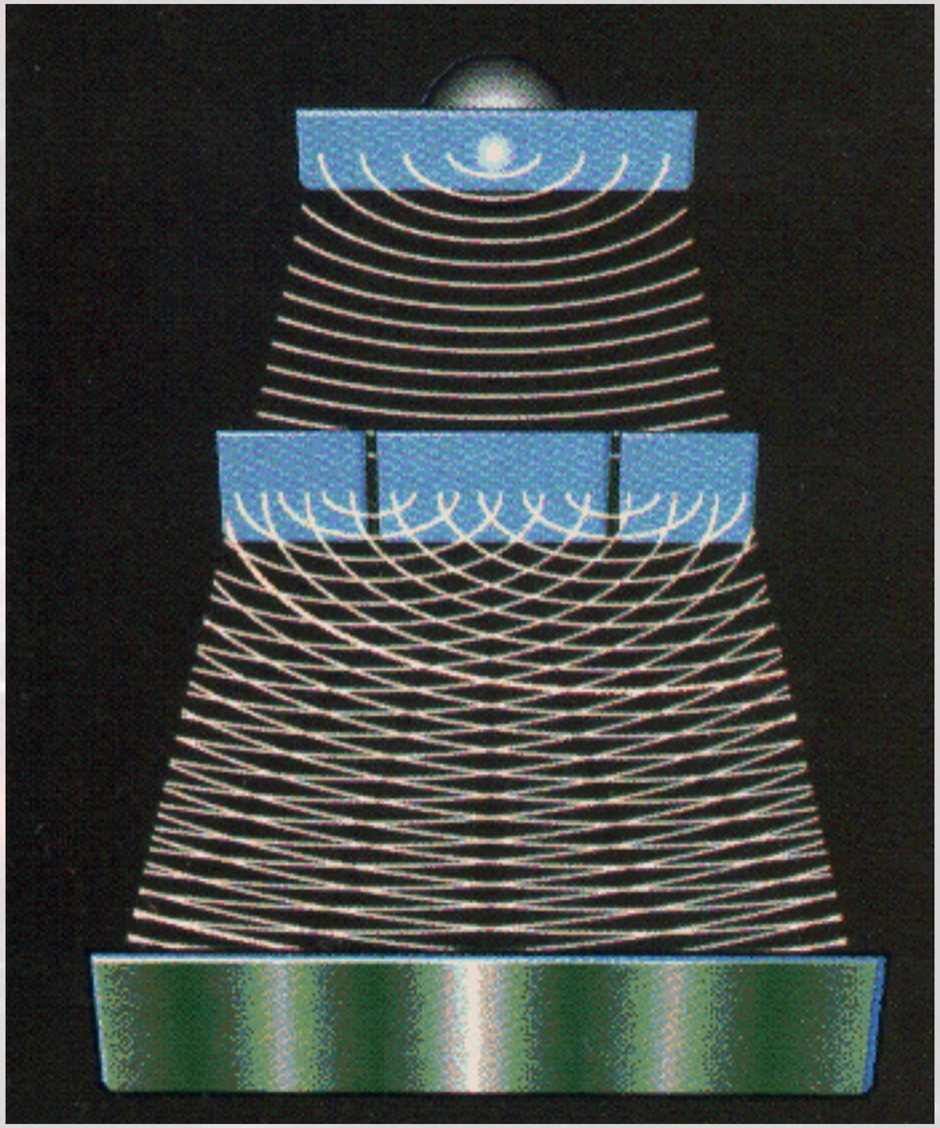
Tranberg, Smit Hernandez, Konstandin,Schmidt,Turok, Zadrozny

classical bosons, quantum fermions

$$\Box\phi - m^2\phi - \lambda\phi^3 = 0$$

$$\gamma^\mu\partial_\mu\psi + m\psi = 0$$





fermion evolution

$$\hat{\psi}(t,\underline{x}) = \sum_s \int \frac{d^3p}{(2\pi)^3}\frac{1}{2\omega_p}\left[\hat{b}_s(\underline{p})U_s(\underline{p})e^{ip.x} + \hat{d}_s(\underline{p})V_s(\underline{p})e^{-ip.x}\right]$$

$$\left\{\hat{b}_r^\dagger(\underline{p}),\hat{b}_s(\underline{p}')\right\} = 2\omega_p(2\pi)^3\delta(\underline{p}-\underline{p}')\delta_{rs}$$

$$\left\{\hat{d}_r^\dagger(\underline{p}),\hat{d}_s(\underline{p}')\right\} = 2\omega_p(2\pi)^3\delta(\underline{p}-\underline{p}')\delta_{rs}$$

ensemble fermion method

$$\hat{\psi}(t,\underline{x}) = \sum_s \int \frac{d^3p}{(2\pi)^3}\frac{1}{2\omega_p}\left[\hat{b}_s(\underline{p})U_s(\underline{p})e^{ip.x} + \hat{d}_s(\underline{p})V_s(\underline{p})e^{-ip.x}\right]$$

$$\left\{\hat{b}_r^\dagger(\underline{p}),\hat{b}_s(\underline{p}')\right\} = 2\omega_p(2\pi)^3\delta(\underline{p}-\underline{p}')\delta_{rs}$$

$$\left\{\hat{d}_r^\dagger(\underline{p}),\hat{d}_s(\underline{p}')\right\} = 2\omega_p(2\pi)^3\delta(\underline{p}-\underline{p}')\delta_{rs}$$

↓

$$\psi_{M,F}(t,\underline{x}) = \frac{1}{\sqrt{2}}\sum_s \int \frac{d^3p}{(2\pi)^3}\frac{1}{2\omega_p}\left[\xi_s(\underline{p})U_s(\underline{p})e^{ip.x} \pm \eta_s(\underline{p})V_s(\underline{p})e^{-ip.x}\right]$$

$$\langle\xi_r(\underline{p})\xi_s^\star(\underline{p}')\rangle_e = 2\omega_p(2\pi)^3\delta(\underline{p}-\underline{p}')\delta_{rs}$$

$$\langle\eta_r(\underline{p})\eta_s^\star(\underline{p}')\rangle_e = 2\omega_p(2\pi)^3\delta(\underline{p}-\underline{p}')\delta_{rs}$$

Borsanyi, Hindmarsh

doublers, Wilson term, anomaly

doublers, Wilson term, anomaly

$$\frac{r_W}{2}\ dx\ \bar{\psi} D_i D_i \psi$$

doublers, Wilson term, anomaly

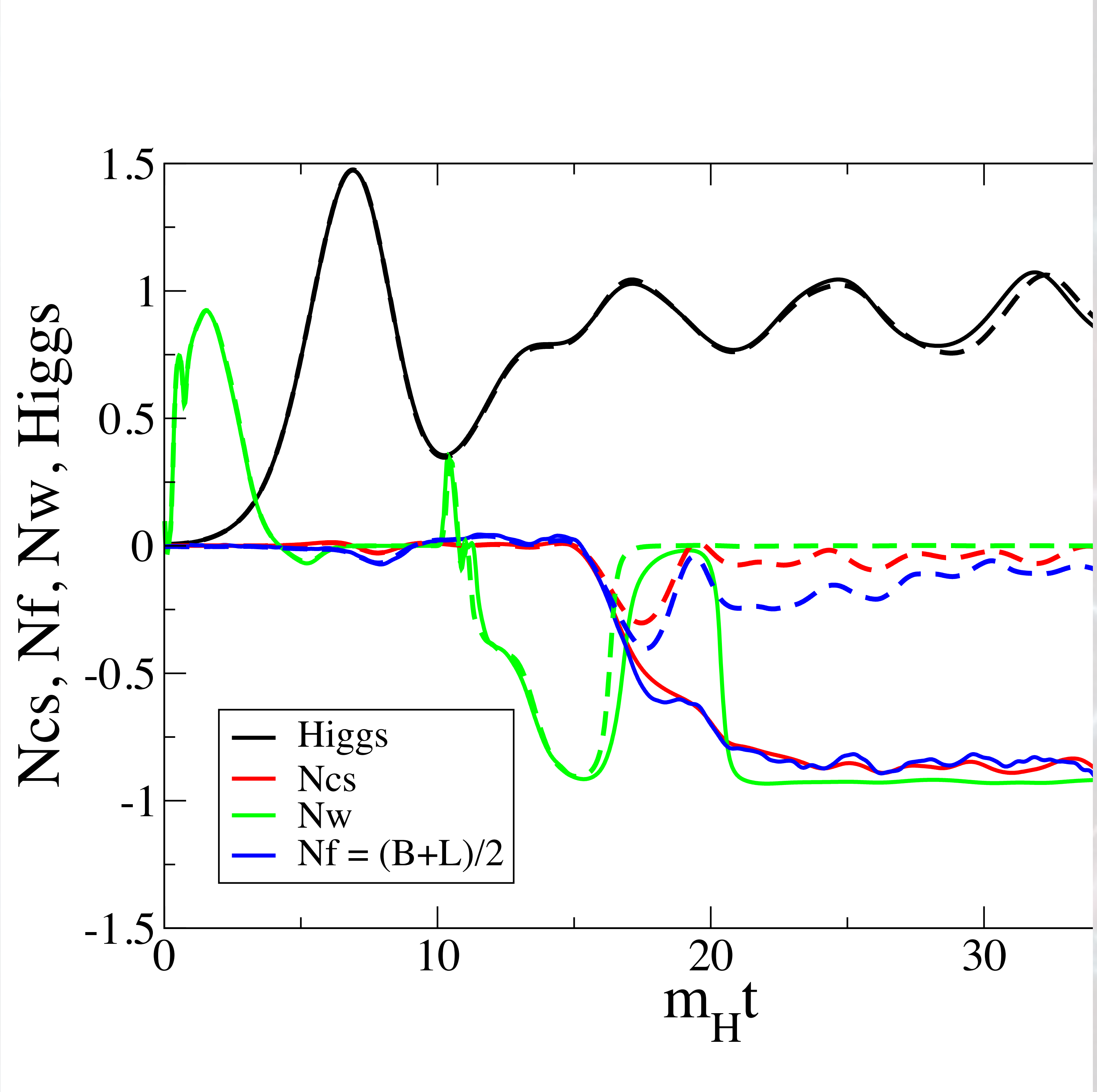
$$\frac{r_W}{2}\ dx\ \bar{\psi} D_i D_i \psi$$

$$\partial_\mu j_5^\mu = r_w \frac{dt}{dx} \bar{\psi} \gamma^5 D_i D_i \psi$$

$$\partial_\mu j_5^\mu = \frac{1}{32\pi^2} W^a_{\mu\nu} \tilde{W}^{a\mu\nu}$$

Karsten, Smit

electroweak baryogenesis and preheating



Saffin, Tranberg, Mou

electroweak baryogenesis and preheating

we still need to generate the asymmetry: C, P, CP

standard model: left-handed fermions gauged - $\not{C}$, $\not{P}$, CP

CKM matrix - $\not{C}$, P, $\not{CP}$

Tranberg, Smit Hernandez, Konstandin, Schmidt, Turok, Zadrozny

electroweak baryogenesis and preheating

$$V(\phi_1, \phi_2) = ... - \frac{1}{2}\mu_{12}^2\left(\phi_1^\dagger\phi_2\right) + \frac{1}{2}\lambda_5\left(\phi_1^\dagger\phi_2\right)^2 + c.c$$

introduce another Higgs doublet

Saffin, Tranberg, Mou

electroweak baryogenesis and preheating

$$V(\phi_1, \phi_2) = ... - \frac{1}{2}\mu_{12}^2\left(\phi_1^\dagger\phi_2\right) + \frac{1}{2}\lambda_5\left(\phi_1^\dagger\phi_2\right)^2 + c.c$$

$$m_1 = 125\ GeV, \quad m_2 = 300\ GeV, \quad m_3 = 350\ GeV$$

$$m_\pm = 400\ GeV$$

$$|\nu_1| = 110\ GeV, \quad |\nu_2| = 220\ GeV, \quad |\nu_1|/|\nu_2| = \tan\beta = 2$$

$$|\nu_1|^2 + |\nu_2|^2 = (246\ GeV)^2$$

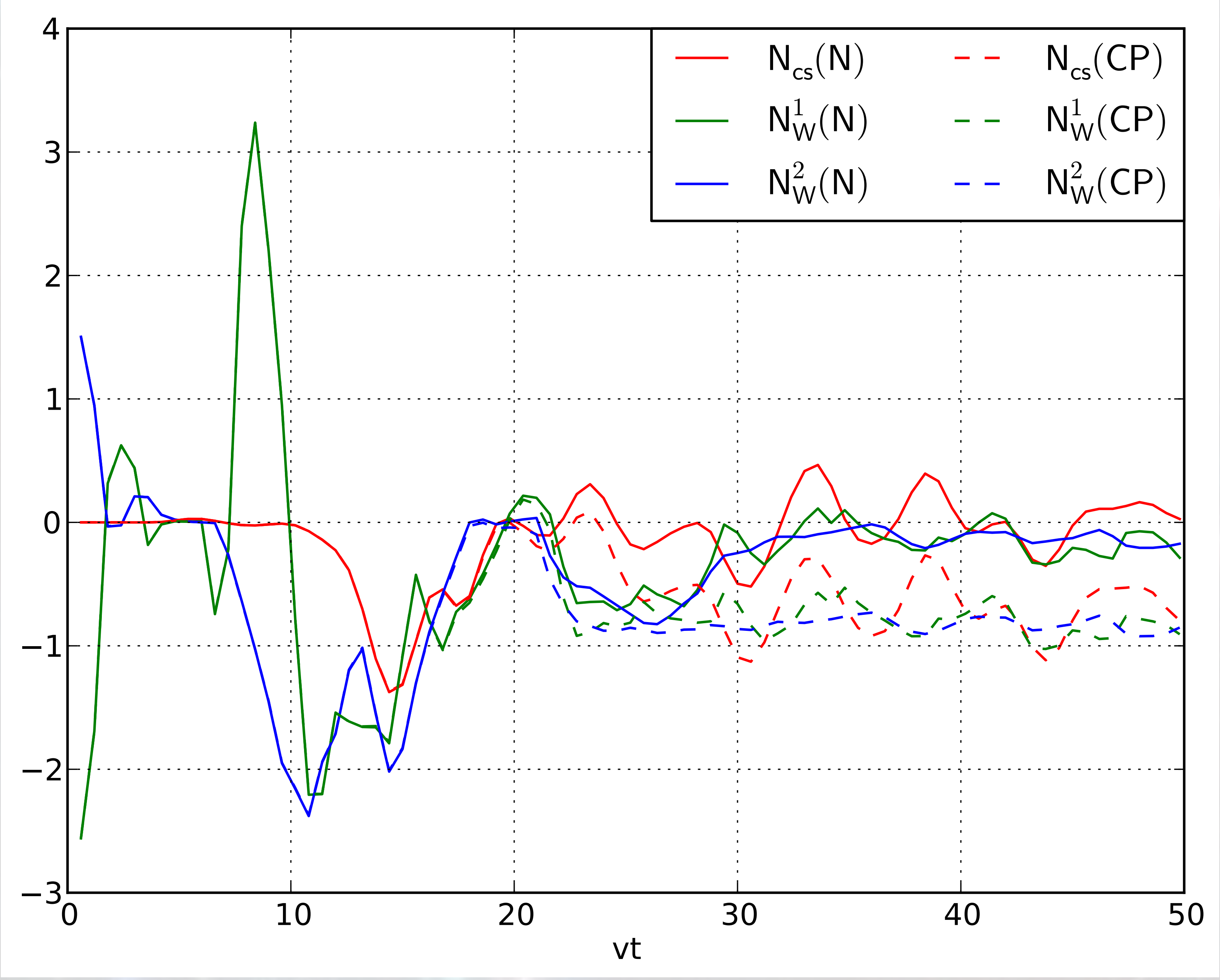
$$arg\left(\nu_2^\dagger\nu_1\right) = \pi/2$$

Saffin, Tranberg, Mou

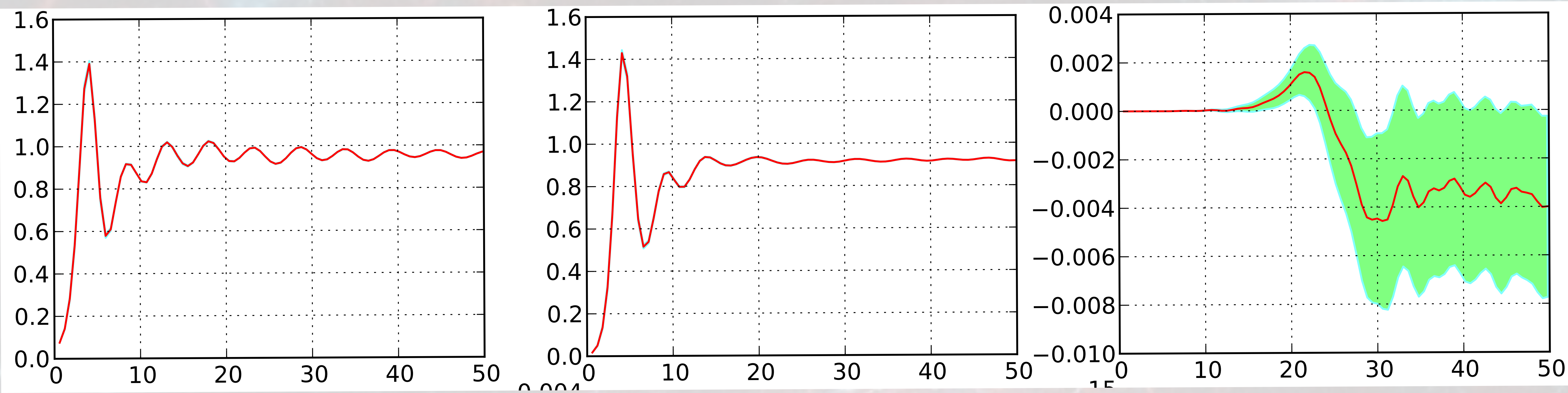
simulate in CP pairs:

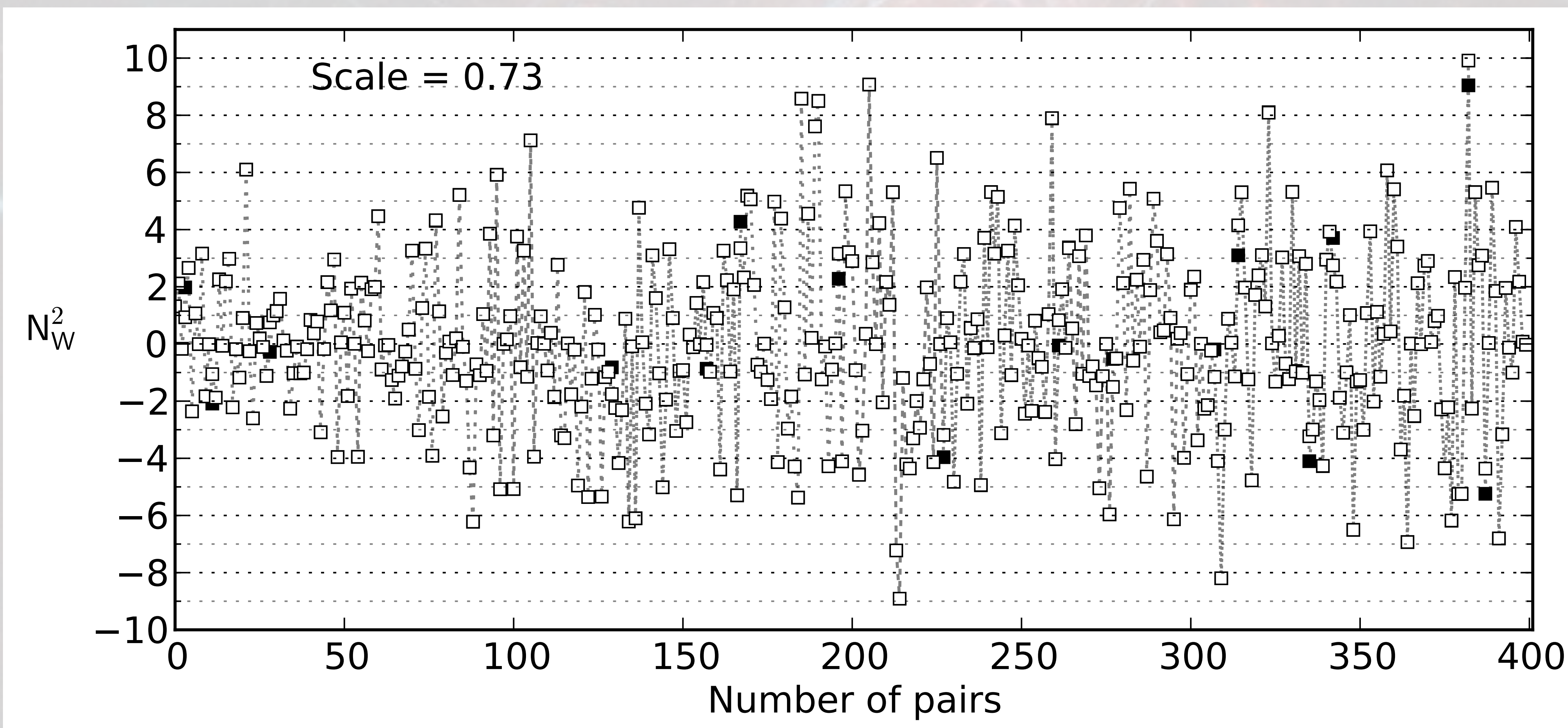
$$\psi_{CP}(t,\underline{x}) = i\gamma^0\gamma^2\psi^\star(t,-\underline{x})$$

$$\phi_{CP}(t,\underline{x}) = \phi^\star(t,-\underline{x}) \qquad W_{CP}(t,\underline{x}) = W^T(t,-\underline{x})$$



$|\phi_1|^2$ $|\phi_2|^2$ N_{CS}





$$N_+ = 6, \quad N_- = 10$$

final baryon asymmetry

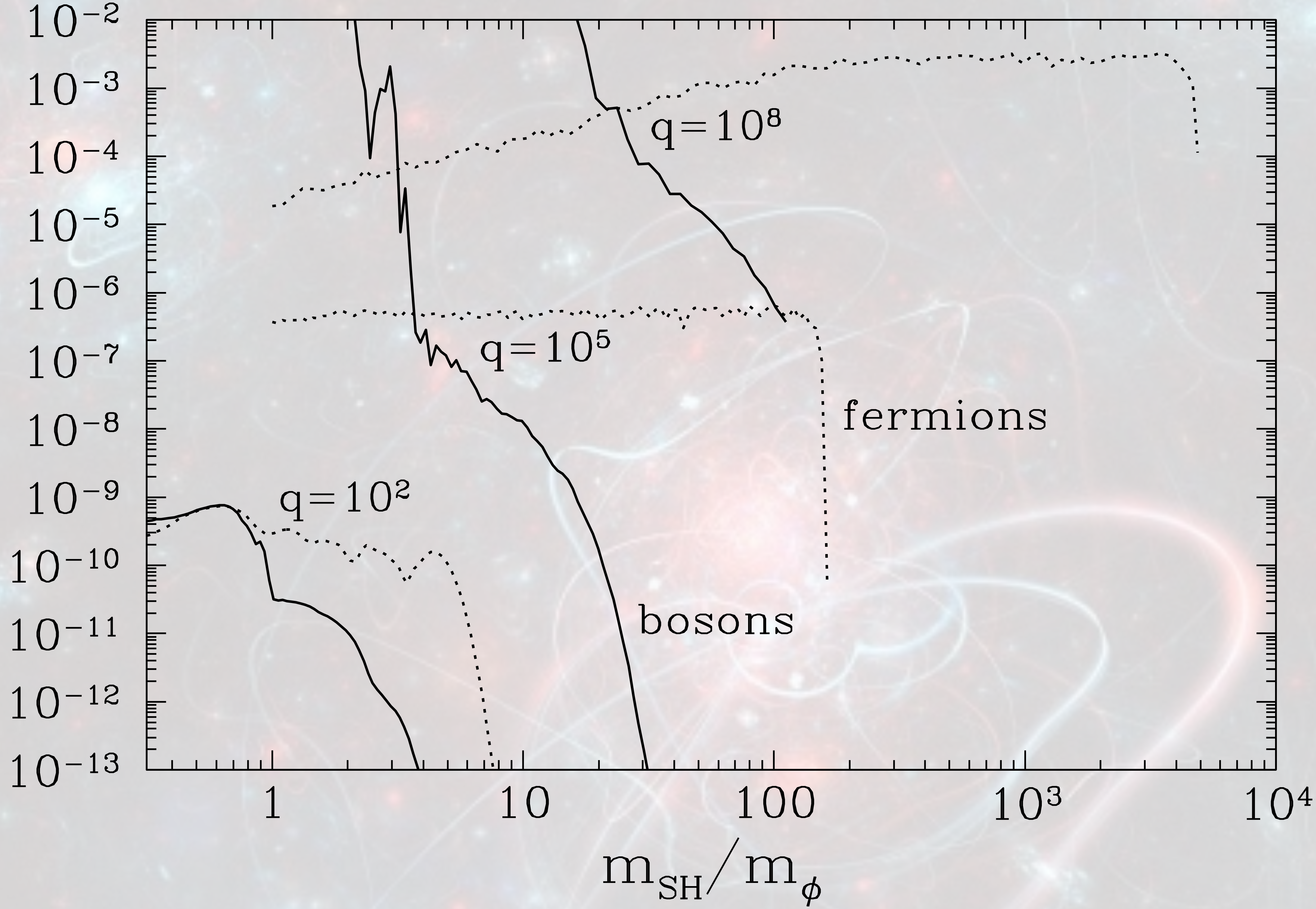
$$n_B = \frac{1}{Vol}\frac{N_+ - N_-}{2N_{pairs}}$$

$$7.04 n_\gamma = s = \frac{2\pi^2}{45} g_\star T^3 \qquad \mathcal{V}_0 = \frac{\pi^2}{30} g_\star T^4$$

$$\eta = \frac{n_B}{n_\gamma} \simeq -3.5 \times 10^{-7} \times (1 \pm 1)$$

conclusions

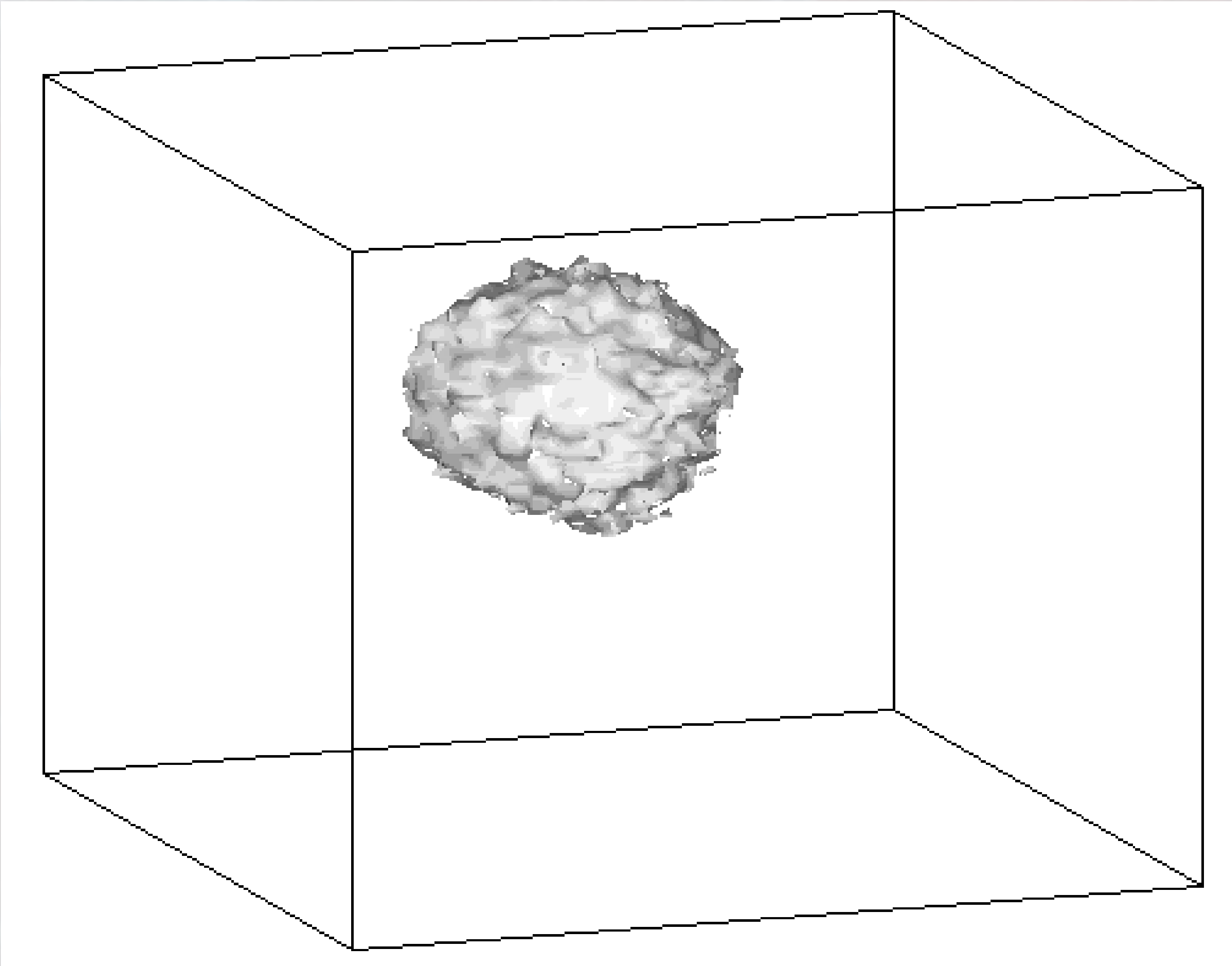
- end of inflation can be a busy period
  - defects - topological and nontopological
  - gravitational waves
  - baryogenesis - GUT and electroweak
- numerical methods are available
  - classical for highly populated bose fields
  - quantum for fermions



Micha, Tkachev, Giudice, Riotto, Greene, Koffman, Berges, Gelfand, Pruschke, Sexty, Kolb

first order phase transitions

$$\mathcal{L} = -\frac{1}{2}(\partial\varphi)^2 - \frac{1}{2}(\partial\chi)^2 - \frac{\lambda}{4}\left(\varphi^2 - \nu^2\right)^2 - \frac{g^2}{2}\varphi^2\chi^2$$



bubble of broken phase

Khlebnikov, Kofman,Linde,Tkachev

the numbers

$$\mathcal{V}_0 = (226\ GeV)^4$$

$$g_\star = 3\times 3 + 5 + \frac{7}{8}\times 4\times 4 = 28$$

$$\rho_f = \int \frac{d^3p}{(2\pi)^3}\frac{p}{e^{p/T}+1} = \frac{7}{8}\frac{\pi^2}{30}T^4$$

$$\rho_b = \int \frac{d^3p}{(2\pi)^3}\frac{p}{e^{p/T}-1} = \frac{\pi^2}{30}T^4$$

$$s = g_\star\frac{2\pi^3}{45}T^3$$

$$n_\gamma = 2\int \frac{d^3p}{(2\pi)^3}\frac{1}{e^{p/T}-1} = \frac{2\zeta(3)}{\pi^2}T^3$$

$$s = 7.04 n_\gamma$$

comoving Hubble radius $\frac{1}{\mathcal{H}} = \frac{1}{aH}$ increases for standard matter

$$p = w\rho$$

$$\frac{d}{dt}\left[\frac{1}{aH}\right] = -\frac{\ddot{a}}{(aH)^2}$$

$$\frac{\ddot{a}}{a} = -\frac{1}{6}(1+3w)\rho$$

$$\frac{1}{aH} = \frac{1}{H_0} a^{(1+3w)/2}$$