Three Pieces in Closed Time Path

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Mr. McGill going home after a hard day's work.

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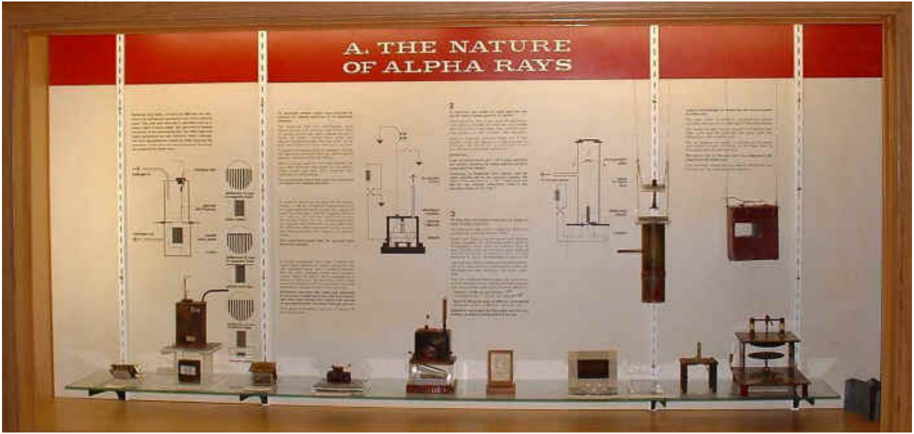


Rutherford carried out his Nobel Prize (1908) winning work at McGill (1898-1907).
His *original* equipment on display

McGill Team

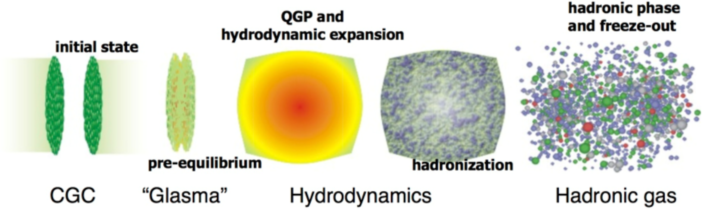
- Charles Gale
- Sangyong Jeon
- Chun Shen $\rightarrow$ BNL
- Alina Czajka
- Li Yan $\leftarrow$ Saclay
- Sangwook Ryu $\rightarrow$ Frankfurt
- Michael Richard
- Igor Kozlov
- Chanwook Park
- Scott McDonald
- Mayank Singh
- Sigtryggur Hauksson

MUSIC (Hydro), MARTINI (Jets), AdS/QCD, FTFT, Many-body QCD, ...

Former members: Björn Schenke, Clint Young, Gabriel Denicol, Matt Luzum, Gojko Vujanovic, Jean-Francois Paquet, Abhijit Majumder, Mohammed Mia, Abhee Kanti Dutt-Mazumder, Prashanth Jaikumar, Champak B. Das, Thomas Epelbaum, ...

Motivations

Stages in heavy ion collisions



- CGC (Color Glass Condensate) & Glasma stages: Dominated by Classical Yang-Mills field
- What is Classical Yang-Mills?
- Quantum corrections?

Motivations

Want to use Closed Time Path $+$ Keldysh Rotation:

- Physics Motivation: Ultra-relativistic Heavy Ion Collisions
  - *Dense* gluon dynamics (i.e. classical Yang-Mills) dominate the initial state as well as the formation of Quark-Gluon Plasma
  - (Local) Thermalization mechanism – Coherent classical field to the maximum entropy state?
- Technical Motivation: *How to do classical field theory in QFT*
  - CTP $+$ Keldysh Rotation – Clean separation of Q and C d.o.f.

Transition Amplitude vs Expectation Value

- Transition amplitude in QM

$$
\begin{aligned}
T_{fi} &= \langle f|\hat{U}(t_f, t_i)|i\rangle \\
&= \langle f|q_f\rangle\langle q_f|\hat{U}(t_f, t_i)|q_i\rangle\langle q_i|i\rangle \\
&= \langle f|q_f\rangle \int_{q_i}^{q_f} \mathcal{D}q\, e^{i\int_{t_i}^{t_f} dt L(q,\dot{q})}\langle q_i|i\rangle
\end{aligned}
$$

  - If a classical path dominates (for instance, large energy), shift $q \to q_{\rm cl} + q$ to get

$$
\int_{q_i}^{q_f} \mathcal{D}q\, e^{i\int_{t_i}^{t_f} dt L(q,\dot{q})} = e^{iS_{\rm cl}(q_f,q_i)} \int_0^0 \mathcal{D}q\, e^{i\int_{t_i}^{t_f} dt\,\delta L(q,\dot{q}|q_{\rm cl})}
$$

    where $\delta L(q, \dot{q}|q_{\rm cl})$ is at least quadratic in q and $\dot{q}$

- The problem here is one of the boundary value problem
- This is *not* what we want

What we want: Initial value problem

- Expectation value in QM

$$\langle O(t) \rangle = \langle i | \hat{U}(t_i, t) \hat{O}_S \hat{U}(t, t_i) | i \rangle$$

is an initial value problem

- It is going to involve *two* transition amplitudes or two path integrals. One for $\hat{U}(t_f, t_i)$ and another for $\hat{U}(t_i, t_f)$
- How does one formulate classical initial value problem from this?

Expectation Value

- Expectation value

$$\langle O(t)\rangle_i = \langle i|\hat{U}(t_{\rm init}, t)\hat{O}_S\hat{U}(t, t_{\rm init})|i\rangle$$

Time flows from $t_{\rm init}$ to t and then back to $t_{\rm in}$. Only the in-state is specified. $\Longrightarrow$ *Initial value problem*

- Natural to define the initial density operator

$$\hat{\rho}_i = |i\rangle\langle i|$$

or more generally

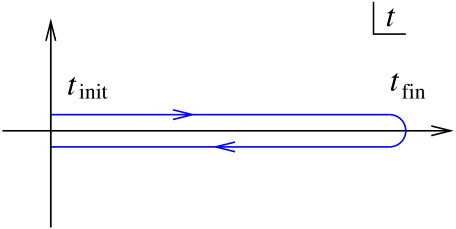
$$\hat{\rho}_{\rm init} = \sum_n P_n|n\rangle\langle n| \quad \text{with} \quad \sum_n P_n = 1$$

so that

$$\langle O(t)\rangle = {\rm Tr}\hat{\rho}(t)\hat{O}_S \;\; \text{with} \;\; \hat{\rho}(t) = \hat{U}(t, t_i)\hat{\rho}_{\rm init}\hat{U}(t_i, t)$$

Formulating the initial value problem

Closed Time Path (Schwinger-Keldysh, Keldysh-Schwinger, in-in . . .)
Using the scalar theory as an example with $V(\phi) = g^2\phi^4/4!$



- Upper branch (ϕ_1) represents $\langle \mathrm{fin}|\hat{U}(t_{\mathrm{fin}}, t_{\mathrm{init}})|\mathrm{init}\rangle$
- Lower branch (ϕ_2) represents $\langle \mathrm{init}|\hat{U}(t_{\mathrm{init}}, t_{\mathrm{fin}})|\mathrm{fin}\rangle$
- Generating functional

$$Z[J_1, J_2]$$
$$= \int [d\phi_f]\langle\phi_f|\hat{\rho}(t_{\mathrm{fin}})|\phi_f\rangle$$
$$= \int [d\phi_f][d\phi_1^i][d\phi_2^i]\,\langle\phi_f|\hat{U}_{J_1}(t_{\mathrm{fin}}, t_{\mathrm{init}})|\phi_1^i\rangle\langle\phi_1^i|\hat{\rho}_{\mathrm{init}}|\phi_2^i\rangle\langle\phi_2^i|\hat{U}_{J_2}(t_{\mathrm{init}}, t_{\mathrm{fin}})|\phi_f\rangle$$

- Generating functional

$$Z[J_1, J_2] = \int [d\phi_f][d\phi_1^i][d\phi_2^i] \int_{\phi_1^i}^{\phi_f} \mathcal{D}\phi_1 \int_{\phi_2^i}^{\phi_f} \mathcal{D}\phi_2 \, \rho_{\rm init}[\phi_i^1, \phi_i^2]$$
$$\exp\left(i \int_{t_i}^{t_f} (L(\phi_1) - L(\phi_2) + J_1\phi_1 - J_2\phi_2) \right)$$

- t_f can be taken to be at $t = \infty$
- Propagator takes on the 2×2 matrix structure

$$\mathbf{G}_{\rm CTP} = \begin{pmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{pmatrix}$$

$$\begin{aligned} G_{11} &= G_{22}^* = G_F = \langle T\phi(x)\phi(y)\rangle \\ G_{12} &= \langle \phi_1(x)\phi_2(y)\rangle = \langle \phi(x)\phi(y)\rangle \\ G_{21} &= \langle \phi_2(x)\phi_1(y)\rangle = \langle \phi(y)\phi(x)\rangle \end{aligned}$$

Keldysh rotation, AKA r-a formalism

- Introduce a change of variables

$$\phi_c\ (=\phi_r) = \frac{\phi_1+\phi_2}{2} \quad \text{and} \quad J_c = J_1 - J_2$$

$$\phi_q\ (=\phi_a) = \phi_1 - \phi_2 \quad \text{and} \quad J_q = \frac{J_1+J_2}{2}$$

- The time derivative terms in the Lagrangian:

$$\frac{\dot\phi_1^2}{2} - \frac{\dot\phi_2^2}{2} = \dot\phi_c\dot\phi_q = \partial_t(\dot\phi_c\phi_q) - \ddot\phi_c\phi_q$$

Upon integrations by part ($\phi_q^f = 0$ because $\phi_1^f = \phi_2^f$)

$$\int (L(\phi_1) - L(\phi_2)) = \int_{t_{\rm init}}^{t_{\rm fin}} \left(\phi_q E[\phi_c] - \frac{g^2}{4!}\phi_q^3\phi_c\right) - \dot\phi_c^i\phi_q^i$$

where $E[\phi_c] = \dfrac{\delta S[\phi_c]}{\delta\phi_c}$ is *the sourceless classical field equation*

Keldysh rotation – Cont.

- Carrying out $[d\phi_q^i]$ integral with the surface term $-\dot{\phi}_c^i\phi_q^i$ transforms the initial density matrix to the Wigner form

$$\int [d\phi_q^i] e^{-i\dot{\phi}_c^i\phi_q^i} \rho[\phi_c^i + \phi_q^i/2, \phi_c^i - \phi_q^i/2] = \rho_W[\phi_c^i, \dot{\phi}_c^i]$$

and the Generating functional now becomes

$$\begin{aligned} & Z[J_c, J_q] \\ = & \int \mathcal{D}\phi_c \int \mathcal{D}\phi_q \, \rho_W[\phi_c^i, \dot{\phi}_c^i] \\ & \exp\left(i \int_{t_i}^{t_f} \left(\phi_q(E[\phi_c] + J_q) - \frac{g^2}{4!}\phi_q^3\phi_c + J_c\phi_c \right) \right) \end{aligned}$$

with no functional integration over ϕ_q at $t_{\rm init}$ and $t_{\rm fin}$

- *If* the $\phi_q^3\phi_c$ term is ignored, the evolution of the field is entirely *classical* with J_q as the source and with the initial data $(\phi_c^i, \dot{\phi}_c^i)$

Free field theory and the Propagators

- Free field theory

$$Z[J_c, J_q] = \int \mathcal{D}\phi_c \int \mathcal{D}\phi_q \, \rho_W[\phi_c^i, \dot{\phi}_c^i] \exp\left(i \int_{t_i}^{t_f} \left(\phi_q(-\partial^2 - m^2)\phi_c + J_q\phi_q + J_c\phi_c \right) \right)$$

- Integrating over ϕ_q yields

$$Z[J_c, J_q] = \int \mathcal{D}\phi_c \rho_W[\phi_c^i, \dot{\phi}_c^i] \, \delta[(\partial^2 + m^2)\phi_c - J_q] \exp\left(i \int_{t_i}^{t_f} J_c\phi_c \right)$$

- Solve the classical EoM in the $(t, \mathbf{k})$ space

$$\phi_c(t, \mathbf{k}) = \phi_h(t, \mathbf{k}) + \int_{t_i}^{t_f} dt' \, G_R(t - t', \mathbf{k}) \, J_q(t', -\mathbf{k})$$

where $G_R(t - t', \mathbf{k}) = \theta(t - t') \sin(E_k(t - t'))/E_k$ and
$\phi_h(t, \mathbf{k}) = \phi_c^i(\mathbf{k}) \cos(E_k(t - t_i)) + \dot{\phi}_c^i(\mathbf{k}) \sin(E_k(t - t_i))/E_k$

Free field theory and the Propagators

$$Z[J_c, J_q] = \int \mathcal{D}\phi_c \rho_W[\phi_c^i, \dot{\phi}_c^i]\, \delta[(\partial^2 + m^2)\phi_c - J_q] \exp\left(i \int_{t_i}^{t_f} J_c \phi_c\right)$$

- Integrating over ϕ_c produces the Jacobian $\left|\mathrm{Det}(\partial^2 + m^2)\right|^{-1}$
- Changing variables from ϕ_f to $\dot{\phi}_i$ produces $\left|\mathrm{Det}\left(\frac{\delta\phi_f}{\delta\dot{\phi}_i}\right)\right|$

  Since $\dot{\phi}_i = \frac{\delta S}{\delta \phi_i}$, this is the inverse of the van Vleck determinant $\left|\mathrm{Det}(\partial^2 + m^2)\right|$ which cancels the Jacobian

$$Z[J_c, J_q] = \int [d\phi_c^i][d\pi_c^i]\, \rho_W[\phi_c^i, \pi_c^i] \exp\left(i \int_{t_i}^{t_f} J_c \phi_c[\phi_c^i, \pi_c^i, J_q]\right)$$

with $\pi_c^i = \dot{\phi}_c^i$

Propagators

Since $\int J\phi_c = \int J_c G_R J_q + \int J_c \phi_h[\phi_c^i, \dot{\phi}_c^i]$,

- These two are obvious
  - $\langle \phi_c(t)\phi_q(t') \rangle = iG_R(t-t')$ (time always flows from ϕ_q to ϕ_c)
  - $\langle \phi_q(t)\phi_q(t') \rangle = 0$
- The symmetric propagator $G_S = \langle \phi_c \phi_c \rangle$ depends on $\rho_W[\phi_c^i, \pi_c^i]$
  - Classical vacuum: $\rho_W = \delta[\phi_c^i]\delta[\pi_c^i]$ which gives $\langle \phi_c(t)\phi_c(t') \rangle = 0$
  - Quantum *perturbative* vacuum:

$$\rho_W[\phi, \pi] = \exp\left(-\int \frac{d^2k}{(2\pi)^3 E_k} \left(E_k^2 \phi(\mathbf{k})\phi(-\mathbf{k}) + \pi(\mathbf{k})\pi(-\mathbf{k}) \right) \right)$$

    which gives $\langle \phi_c(t)\phi_c(t') \rangle = FT\left[\pi\delta(p^2 - m^2)\right]$
  - Thermal medium: $\langle \phi_c(t)\phi_c(t') \rangle = FT\left[(1/2 + n_B(p^0))2\pi\delta(p^2 - m^2)\right]$
- *Quantum effect:* Non-vanishing $G_S = \langle \phi_c \phi_c \rangle$

Almost classical interpretation

$$Z[J_c, J_q] = \int \mathcal{D}\phi_c \int \mathcal{D}\phi_q \, \rho_W[\phi_c^i, \dot{\phi}_c^i] \exp\left(i \int \left(\phi_q(E[\phi_c] + J_q) - \frac{g^2}{4!}\phi_q^3\phi_c + J_c\phi_c \right)\right)$$

- $\rho_W[\phi_c^i, \dot{\phi}_c^i] \sim$ Probability distribution of the initial data (Not strictly, since it's a Wigner transform)
- If for some reason $\phi_q \ll \phi_c$, then drop the ϕ_q^3 term to get

$$\int \mathcal{D}\phi_q e^{i\int \phi_q(E[\phi_c]+J_q)} = \delta[E[\phi_c] + J_q]$$

  which enforces the classical equation of motion
- Origin of quantum effects
  - $\rho_W[\phi_c^i, \dot{\phi}_c^i]$: Includes quantum effects. Especially the zero-point motions.
  - Quantum vertex $g^2\phi_q^3\phi_c/4!$: Provides correlations absent in the classical theory

When the classical field dominates

- Let $V = g^2\phi^4/4!$. The EoM is

$$(\partial^2 + m^2)\phi_c + \frac{g^2}{3!}\phi_c^3 = J_q$$

- Suppose we have a *physical* source $J_q = J_{\rm phys}$.
- If $J_{\rm phys} = O(1/g)$, then

$$\phi_c = O(1/g)$$

and the interaction term is as big as the free field terms.

The Lagrangian

- Let φ be the solution of the classical EoM and let $\phi_c \to \varphi + \phi_c$
- The Lagrangian

$$
\begin{aligned}
L &= \phi_q(E[\varphi + \phi_c] + J_{\rm phys}) + \frac{g^2}{4!}\phi_q^3\varphi + \frac{g^2}{4!}\phi_q^3\phi_c \\
&= \phi_q \left((\partial^2 + m^2 + \underbrace{\frac{g^2}{2}\varphi^2)\phi_c}_{O(1)} + \underbrace{\frac{g^2}{2}\varphi\,\phi_c^2}_{O(g)} + \underbrace{\frac{g^2}{3!}\phi_c^3}_{O(g^2)} \right) \\
&\quad + \underbrace{\frac{g^2}{4!}\varphi\,\phi_q^3}_{O(g)} + \underbrace{\frac{g^2}{4!}\phi_q^3\phi_c}_{O(g^2)}
\end{aligned}
$$

- One can do perturbation theory if one knows $G_R = 1/(\partial^2 + m^2 + g^2\varphi^2/2)$
- If interested in only the leading order corrections, just ignore ϕ_q^3 terms and solve classical field equations with the fluctuating initial condition.

LO + NLO

- Carrying out $\int \mathcal{D}\phi_q$ integrals results in $\delta[E[\phi_c] + J_q]$
- Carrying out $\int \mathcal{D}\phi_c$ integrals results in $\mathrm{Det}^{-1}\left(\frac{\delta E[\phi_c]}{\delta\phi_c}\right)$
- Swapping the boundry value problem (with ϕ_i, ϕ_f) with the initial value problem (with $\phi_i, \dot{\phi}_i$) results in

$$\mathrm{Det}\left(\frac{\delta\phi_f}{\delta\dot{\phi}_i}\right) = \mathrm{Det}\left(\frac{\delta^2 S}{\delta\phi_i\delta\phi_f}\right)^{-1} = \mathrm{Det}\left(\frac{\delta E[\phi_c]}{\delta\phi_c}\right)$$

- Any observable up to LO + NLO (with $\pi = \dot{\phi}$)

$$\langle\mathcal{O}(t)\rangle = \int [d\phi_c^i][d\pi_c^i]\rho_W[\phi_c^i, \pi_c^i]\,\mathcal{O}[\phi_{\rm cl}(t), \pi_{\rm cl}(t)]$$

LO + NLO + NNLO

- In principle

$$
\begin{aligned}
\langle \mathcal{O}(t) \rangle &= \int \mathcal{D}\phi_c \int \mathcal{D}\phi_q \, \rho_W[\phi_c^i, \dot{\phi}_c^i] \, \exp\left(i \int \left(\phi_q (E[\phi_c] + J_q) - \frac{\phi_q^3}{4!} V'''(\phi_c) \right) \right) \mathcal{O}[\phi_c, \pi_c] \\
&= \int \mathcal{D}\phi_c \int \mathcal{D}\phi_q \, \rho_W[\phi_c^i, \dot{\phi}_c^i] \, \exp\left(i \int \left(\phi_q (E[\phi_c] + J_q) \right) \right) \\
&\quad \times \left(1 - i \int d^4x \, \frac{\phi_q^3}{4!} V'''(\phi_c) + \cdots \right) \mathcal{O}[\phi_c, \dot{\phi}_c] \\
&= \int \mathcal{D}\phi_c \, \rho_W[\phi_c^i, \dot{\phi}_c^i] \, \delta[E[\phi_c] + J_q] \mathcal{O}[\phi_c, \dot{\phi}_c] \\
&\quad - i \int d^4x \, \frac{\delta^3}{\delta J_q(x)^3} \int \mathcal{D}\phi_c \, \rho_W[\phi_c^i, \dot{\phi}_c^i] \, \delta[E[\phi_c] + J_q] \mathcal{O}[\phi_c, \dot{\phi}_c] \frac{1}{4!} V'''(\phi_c(x)) + \cdots
\end{aligned}
$$

provide that $\rho_W[\phi_c^i, \dot{\phi}_c^i]$ is also accurate up to the first order quantum correction

- In practice, not so easy

Vacuum Initial State Density

- The vacuum functional satisfies the Schrödinger equation

$$\mathcal{H}|\Psi\rangle = 0$$

where

$$\mathcal{H} = \int d^3x \left(\frac{\pi^2}{2} + \frac{(\nabla\phi)^2}{2} + \frac{m^2}{2}\phi^2 + V(\phi) \right)$$

and

$$\pi(\mathbf{x}) = -i\frac{\delta}{\delta\phi(\mathbf{x})}$$

- Perturbative vacuum (products of SHO ground states)

$$\langle\phi|\Psi\rangle = \exp\left(-\frac{1}{2}\int \frac{d^3k}{(2\pi)^3} E_k \phi(\mathbf{k})\phi(-\mathbf{k})\right)$$

- Vacuum Wigner functional

$$\rho_W[\phi,\pi] = \exp\left(-\int \frac{d^3k}{(2\pi)^3 E_k}\left(E_k^2\phi(\mathbf{k})\phi(-\mathbf{k}) + \pi(\mathbf{k})\pi(-\mathbf{k})\right)\right)$$

Coherent State

Coherent state: Eigenfunctional of the annihilation operator – Minimum uncertainty state $\sim$ Classical field

- Creation operator in the ϕ representation

$$\mathcal{A}^\dagger(\mathbf{k}) = \left(E_k \phi(-\mathbf{k}) - (2\pi)^3 \frac{\delta}{\delta\phi(\mathbf{k})} \right)$$

- Annihilation operator in the ϕ representation

$$\mathcal{A}(\mathbf{k}) = \left(E_k \phi(\mathbf{k}) + (2\pi)^3 \frac{\delta}{\delta\phi(-\mathbf{k})} \right)$$

- Commutator

$$[\mathcal{A}(\mathbf{k}), \mathcal{A}^\dagger(\mathbf{k}')] \quad = \quad 2E_k (2\pi)^3 \delta(\mathbf{k} - \mathbf{k}')$$

Coherent States

- Ground state: Solving

$$\langle \phi | \mathcal{A}(\mathbf{k}) | \Psi \rangle = 0$$

gives

$$\langle \phi | \Psi \rangle = \mathcal{N} \exp\left(- \int \frac{d^3q}{(2\pi)^3} \frac{E_q}{2} \phi(\mathbf{q}) \phi(-\mathbf{q}) \right)$$

- Coherent state: Solving

$$\langle \phi | \mathcal{A}(\mathbf{k}) | \varphi + i\Pi \rangle = \varphi(\mathbf{k}) \langle \phi | \varphi + i\Pi \rangle$$

yields

$$\langle \phi | \varphi + i\Pi \rangle = \exp\left(i \int \frac{d^3k}{(2\pi)^3} \Pi(\mathbf{k}) \phi(-\mathbf{k}) - \frac{1}{2} \int \frac{d^3k}{(2\pi)^3} E_k \left| \phi(\mathbf{k}) - \varphi(\mathbf{k}) \right|^2 \right)$$

Coherent State Initial Density Matrix

- Wigner transform of the coherent state functional

$$\rho_W[\phi,\pi] = \exp\left[-\int \frac{d^2k}{(2\pi)^3 E_k} \left(E_k^2 \left|\phi(\mathbf{k}) - \varphi(\mathbf{k})\right|^2 + \left|\pi(\mathbf{k}) - \Pi(\mathbf{k})\right|^2 \right) \right]$$

- Since this is just a shifted vacuum functional, in practice:

$$\begin{aligned} \phi_c^i &= \varphi + \delta\phi \\ \pi_c^i &= \Pi + \delta\pi \end{aligned}$$

where $\delta\phi$ and $\delta\pi$ follows the usual Gaussian vacuum distributions.

Application 0
Scattering Amplitude
Expansion of Self-Energy

Kadanoff Baym Equation

- Standard application of the CTP formalism
- KB Eq in the more-or-less standard form:

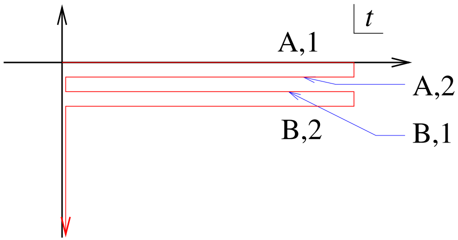
$$(p\cdot\partial)G^{<,>} = \frac{1}{2}\left(\Pi^{>}G^{<} - \Pi^{<}G^{>}\right)$$

- In Quasi-particle approximation with

$$G^{>}(X,p) = 2\pi\delta(p^2 - m^2)\left[\theta(p^0)(1 + f_{+}(X,p)) + \theta(-p_0)f_{-}(X,-p)\right]$$

this can become Kinetic theory equation (e.g. the Boltzmann eq) *provided* that the self-energy is expanded in scattering amplitudes

Two-sweep CTP



Simon Caron-Huot's Masters Thesis
By separating the A fields and B fields, one can show

$$
\begin{aligned}
\Pi^{>}(P) \quad &= \sum_{n,\{Q\}} \frac{1}{n!} \left|\mathcal{M}_{ar\cdots r}(P; Q_1, \cdots, Q_n)\right|^2 \\
&\quad \times G^{>}(Q_1) \cdots G^{>}(Q_n) (2\pi)^4 \delta^{(4)}(Q_1 + \cdots + Q_n - P)
\end{aligned}
$$

where

- r, a index = c, q index
- $\mathcal{M}_{ar\cdots r}(P; Q_1, \cdots, Q_n)$: Fully retarded and 1-PI correlation function
- $G^{>}(Q)$: Full Wightman function

Scattering Matrix Expansion

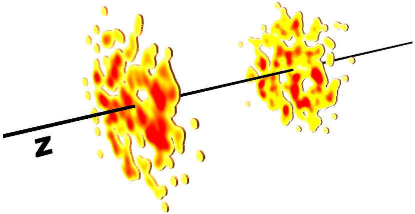
$$\Pi^{>}(P) = \sum_n \frac{1}{n!} |\mathcal{M}_{ar\cdots r}(P; Q_1, \cdots, Q_n)|^2$$
$$\times G^{>}(Q_1) \cdots G^{>}(Q_n)(2\pi)^4 \delta^{(4)}(Q_1 + \cdots + Q_n - P)$$

- This appears in the Kadanoff-Baym equation
- Becomes the collision terms in the kinetic theory
- Tells you *what* to calculate for the *in-medium* scttering amplitude – It's *not* the usual Feynman (time-ordered) amplitude in vacuum

Application 1
Color Glass Condensate and the JIMWLK RG Equation

Application – Color Glass Condensate

[Venugopalan, McLerran, JIMWLK, Gelis, Hatta, Fukushima, Dumitru, Kovchegov, Itakura, Lappi, Nara, . . .]



- Main idea: Highly accelerated hadrons are composed of
  - Large x partons: 2D frozen-in-time (Color Glass) color current
  - Small x gluons: Weizsäcker-Williams field generated by large x partons (Condensate) $\Longrightarrow$ Classical field
- Small x part of the gluon PDF $\sim \langle A_{\rm cl} A_{\rm cl} \rangle$

Yang-Mills with an external color current

Try first

$$L = -\frac{1}{4} G_a^{\mu\nu} G^a_{\mu\nu} - J_a^\mu A^a_\mu$$

- Classical EoM OK: $[D_\mu, G^{\mu\nu}] = J^\nu$ with $[D_\mu, J^\mu] = 0$
- Trouble: Gauge transformation
  - $A' = UAU^\dagger - \frac{1}{ig} U\partial U^\dagger$
  - $J' = UJU^\dagger$
- In QED, $J^\mu A_\mu$ is gauge invariant as long as $\partial_\mu J^\mu = 0$
- In the full QCD, the color current is a part of $\bar{\psi}\gamma^\mu D_\mu \psi$. Without the $\bar{\psi}\gamma^\mu \partial_\mu \psi$ term, however, the L above is *not* gauge invariant even if $[D_\mu, J^\mu] = 0$
- Way out: Non-local interaction. Use $\mathrm{Tr}\ \ln(\gamma^\mu D_\mu)$ or $\mathrm{Tr}\rho W$ or $\mathrm{Tr}\rho \ln W$ where W is the Wilson line along $u^\mu = J^\mu/\rho$ with $U_i = U_f$ [Jalilian-Marian, Jeon, Venugopalan, Phys.Rev.D63:036004,2001]

CTP-YM

- CTP Lagrangian for pure glue with a color current

$$\mathcal{L} = \eta_\nu^a \left([D_\mu, G^{\mu\nu}] - J^\nu\right)_a + \frac{ig}{4}[D_\mu, \eta_\nu]^a[\eta^\mu, \eta^\nu]_a$$

where D_μ and $G^{\mu\nu}$ contains only

$$A_\mu = \frac{A_{1,\mu} + A_{2,\mu}}{2}$$

whereas $\eta_\mu = A_{1,\mu} - A_{2,\mu}$

- The source $J^\mu = J_1^\mu = J_2^\mu$ is the physical external source
- In principle,

$$A'_{1,2} = U_{1,2} A_{1,2} U^\dagger_{1,2} + \frac{1}{ig} U_{1,2} \partial U^\dagger_{1,2}$$

where U_1 and U_2 are not necessarily the same.

CTP-YM

- If $U_1 \neq U_2$, $\mathcal{L} = L_1 - L_2$ is not gauge invariant since neither L_1 nor L_2 is gauge invariant
- If $U_1 = U_2 = U$,

$$
\begin{aligned}
A' &= UAU^\dagger + \frac{1}{ig} U\partial U^\dagger \\
\eta' &= U\eta U^\dagger
\end{aligned}
$$

  and the $J_a^\mu \eta_\mu^a$ term is gauge invariant.
- When one is given a color current J^μ without the corresponding kinetic energy term, this is the results in a gauge invariant *local* theory
- Let

$$A = A_{\rm cl} + a$$

  then systematic perturbative study is possible.

An *exact* solution of Classical YM equation

- There aren't too many exact solution of classical Yang-Mills equation even in static situations
- When $J^\mu = \delta^{\mu\pm}\rho(x^\mp, \mathbf{x}_\perp)$, an *exact* solution of CYM can be found – *MV (McLerran-Venugopalan) model*
- Abelian subgroup solution: Suppose

$$J^\mu = J_3^\mu t_3 + J_8^\mu t^8$$

and

$$A^\mu = A_3^\mu t^3 + A_8^\mu t^8$$

then since $[t^3, t^8] = 0$ (both are diagonal), the classical YM equations reduces to two sets of Maxwell equations – *Of limited utility*.

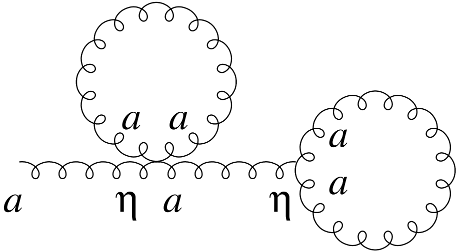
JIMWLK equation

– *Quantum correction (RG equation) on top of the MV solution*

- Formulated in the light-cone coordinate system $x^{\pm} = \dfrac{x^0 \pm x^3}{\sqrt{2}}$
- Main idea: The properties of small x gluons are determined by the underlying color charge distribution.
- Vacuum fluctuation introduces *all* x scales even though the color charge density ρ itself is soft
- Where is the dividing line between small x and large x? $\Longrightarrow$ RG approach is necessary
- Main point: This quantum correction to the MV model can be still treated within the classical theory
- CTP calc: [Jeon, Annals Phys. 340 (2014) 119-170]

JIMWLK = Jalilian-Marian, Iancu, McLerran, Weigert, Leonidov, Kovner

Tadpole diagrams



Caution: *a* here is the *c* field or the "*r*" field fluctuation and η here is the *q* field or the "*a*" field

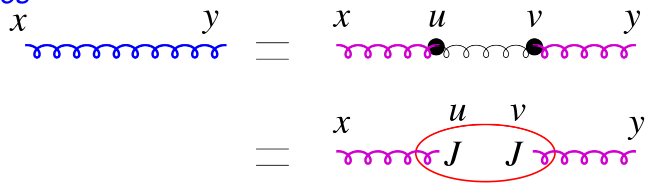
- Leading order quantum corrections
- This is $O(1/g)$ if the UV regulated tadpole contribution is $O(1/g^2)$
- The same as the size of the classical source $J_{\text{phys}} = O(1/g)$
- Large *x* gluons can act like an *additional* $O(1/g)$ source

Tadpole = Source Correlation at the initial time

- Furthermore, since

$$G_S(x,y) \sim \int_{u,v} \partial_{u^+} G_R(x,u)\, G_S^0(u-v)\, \partial_{v^+} G_R(v,y)$$

tadpoles can be generated by *2-point correlation of classical sources*



- Schematically, $\langle J(u)J(v)\rangle \sim \partial_{u^+}\partial_{v^+} G_S^0(u-v)$
- The symmetric propagator is just the transformed Minkowski one

$$G_S^0(x) = \int_k e^{-ik^+x^- - ik^-x^+ + i\mathbf{k}_\perp\cdot\mathbf{x}_\perp}\, \pi\delta(2k^+k^- - \mathbf{k}_\perp^2)$$

Basic Idea

- Construct the source correction $Y[\lambda, \rho]$ so that for any observable $\mathcal{O}[A]$

$$\int \mathcal{D}\rho W_\rho[\rho] \int [da_i]\rho_W[a_i]\mathcal{O}[A[\rho, a_i]]$$
$$= \int \mathcal{D}\rho W_\rho[\rho] \int \mathcal{D}\lambda Y[\lambda, \rho]\mathcal{O}[A[\rho + \lambda]]$$
$$= \int \mathcal{D}\rho W'_\rho[\rho']\mathcal{O}[A[\rho']]$$

  including the leading quantum corrections
- $W[\rho]$: Geometric color charge distribution
- $Y[\lambda, \rho]$: Gives the same $\langle a_i \rangle$ and $\langle a_i(x)a_i(y)\rangle$
  (calculating these correctly in CTP-YM is the main task)
- The combined density $W'_\rho[\rho'] = \int \mathcal{D}\lambda W[\rho - \lambda] Y[\lambda, \rho - \lambda]$

JIMWLK equation

The conbined density can be shown to satisfy

$$\frac{\partial W}{\partial Y} = \mathcal{H} W$$

where

$$\mathcal{H} = \frac{1}{2\pi} \int_{\mathbf{u}_\perp, \mathbf{v}_\perp} \frac{\delta}{\delta \alpha_a(\mathbf{u}_\perp)} \eta^{ab}(\mathbf{u}_\perp | \mathbf{v}_\perp) \frac{\delta}{\delta \alpha_b(\mathbf{v}_\perp)}$$

with

$$\begin{aligned} &\eta(\mathbf{x}_\perp | \mathbf{y}_\perp) \\ &= \; - \int_{\mathbf{u}_\perp} \left(1 - V^\dagger(\mathbf{u}_\perp) V(\mathbf{y}_\perp) - V^\dagger(\mathbf{x}_\perp) V(\mathbf{u}_\perp) + V^\dagger(\mathbf{x}_\perp) V(\mathbf{y}_\perp)\right) \\ &\quad \partial_x^i G_T(\mathbf{x}_\perp - \mathbf{u}_\perp) \partial_i^y G_T(\mathbf{u}_\perp - \mathbf{y}_\perp) \end{aligned}$$

where

$$V(u_1^-, u_2^-; \mathbf{u}_{\perp 1}) = \mathcal{P} \exp\left(ig \int_{u_2^-}^{u_1^-} dz^- \, A_-(z^-, \mathbf{u}_\perp) \right)$$

is the color rotation factor when crossing the current and $\nabla_\perp^2 G_T(\mathbf{x}) = \delta^{(2)}(\mathbf{x})$

JIMWLK equation

To get the rapidity evolution of the gluon PDF,

- Solve

$$\frac{\partial W}{\partial Y} = \mathcal{H} W$$

- At a given Y, sample ρ from W
- Solve CYM with the sampled ρ
- Calculate the gluon number density
- Average over many configurations

Thermalization

Initial Conditions in Heavy Ion Collisions

- Initial condition *before* the collision – Composed of classical particles (source) and the classical field they generate
- Initial moments *right after* the collision – Interaction of two classical Yang-Mills fields from the two nuclei $\Longrightarrow$ Glasma
- Around $\tau \approx 0.5\,\mathrm{fm}$, hydrodynamics starts to apply – Local equilibrium (or some semblance of it) is necessary
- In particular, in the local rest frame

$$T^{\mu\mu} \approx \begin{pmatrix} \varepsilon & 0 & 0 & 0 \\ 0 & P_x & 0 & 0 \\ 0 & 0 & P_y & 0 \\ 0 & 0 & 0 & P_z \end{pmatrix}$$

with $P_x \approx P_y \approx P_z \approx \varepsilon/3$

How do we get here?

Glasma: [Lappi, McLerran, Romatschke, Gelis, Fukushima, Venugopalan, Jeon, Itakura, . . .]

Thermalization in CTP

Scalar theory example

- Simple Simulation: Build up scalar field with

$$-\partial^2\phi - \frac{g^2}{3!}\phi^3 = J$$

  where $J = O(1/g)$ is the source for $t < 0$

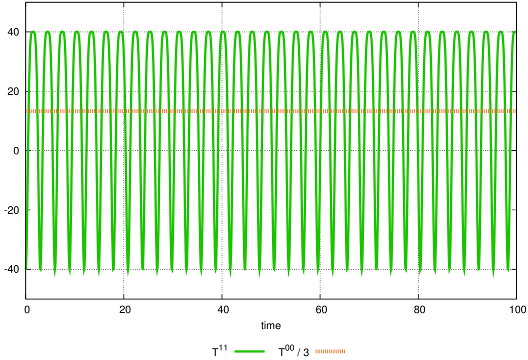
- See what happens at $t > 0$

Thermalization in CTP

Scalar theory example

[Dusling, Epelbaum, Gelis, Venugopalan (DEGS)]

Spatially homogeneous case without vacuum fluctuations:



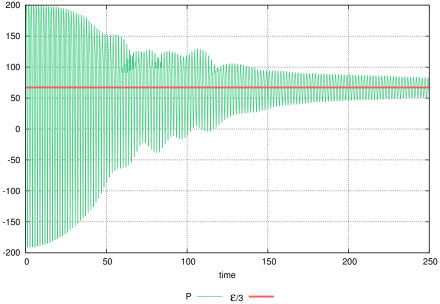
Energy is conserved, but pressure oscillates wildly

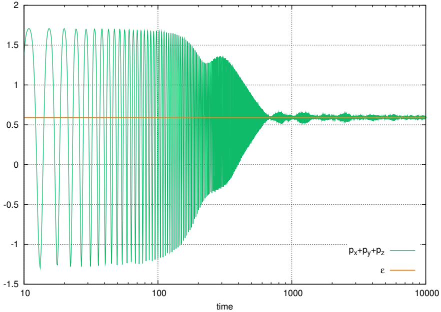
(Analytic solution possible in terms of Jacobi elliptic function)

Scalar theory example

Add vacuum noise

$$\langle O\rangle = \int [d\phi_i][d\pi_i]\,\rho_v[\phi_i,\pi_i]\,\delta[E[\varphi+\phi]]\delta[E[\varpi+\pi]]\,O(\varphi+\phi,\varpi+\pi)$$



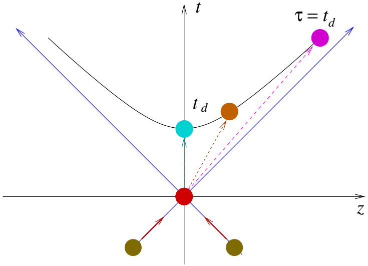


$g = 0.5$, DEGS, NPA 850, 69, 2011 $g = 1.0$, EG, NPA 872, 210, 2011

Miline Space Vacuum Functionals

Initial condition in τ

- Why τ? – Time dilation



  - If the three fireballs all start out from $t = 0, z = 0$ and evolve exactly the same way (e.g. thermalization), the state of the cyan at $t = t_d$ is the same as the state of the brown and magenta at $\tau = t_d$
  - Appropriate "time" variable
    - Relativistic case: $\tau = \sqrt{t^2 - z^2} = t\sqrt{1 - v_z^2}$ is the most natural time variable – Local time at z
    - Non-relativistic case: $z = v_z t \ll t \implies t$ is the the most natural time variable.

Milne space

- Milne space

$$\tau = \sqrt{t^2 - z^2}$$
$$\eta = \tanh^{-1}(z/t)$$

or

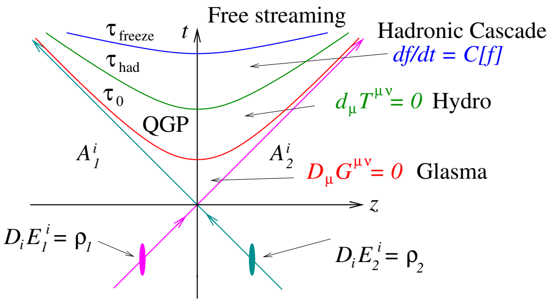
$$t = \tau \cosh \eta$$
$$z = \tau \sinh \eta$$

- Lorentz boost by $v_z = z/t = \tanh \eta$ yields

$$t' = t/\cosh \eta = \tau$$
$$z' = 0$$

- If the collective velocity $v_z = z/t \Longrightarrow$ Boost invariant Bjorken expansion

Glasma Initial Condition



- Glasma initial condition set at $\tau = 0^+$
- Calculate

$$\langle O \rangle = \int [d\phi_i][d\pi_i]\, \rho_v[\phi_i, \pi_i]\, \delta[E[\varphi + \phi]]\delta[E[\varpi + \pi]]\, O(\varphi + \phi, \varpi + \pi)$$

in the $\tau - \eta$ coordinate system $\Longrightarrow$ Need to know the vacuum Wigner functional in the $\tau - \eta$ coordinate system in the forward light cone

Milne space vacuum functional

Scalar theory [Long and Shore, 1996]

- Schrödinger Equation for the perturbative vacuum

$$\int d\eta d^2x_\perp \tau \left(-\frac{1}{2\tau^2}\frac{\delta^2}{\delta\phi^2} - \phi\left(\nabla_\perp^2 - \frac{\partial_\eta^2}{\tau^2}\right)\phi\right) \langle\phi|\mathrm{vac}\rangle = i\frac{\partial}{\partial\tau}\langle\phi|\mathrm{vac}\rangle$$

- Can't let the RHS vanish because of the $\frac{1}{\tau^2}$ term in the LHS
- Gaussian ansatz

$$\langle\phi|\mathrm{vac}\rangle = \mathcal{N}(\tau)\exp\left(-\frac{\tau^2}{2}\int d\eta_x d^2x_\perp \int d\eta_y d^2y_\perp G(\tau, x, y)\phi(x)\phi(y)\right)$$

- In the momentum space, $G_T(\tau, \tilde{k})$ satisfies

$$i\partial_\tau(\tau^2 G_T) = \tau^3 G_T^2 - \left(\tau k_\perp^2 + \frac{k_\eta^2}{\tau}\right)$$

with $\tilde{k} = (\mathbf{k}_\perp, k_\eta)$

Milne space vacuum functional

Scalar theory

- Solution

$$G(\tau, x, y) = \int \frac{d^2 q_\perp dq_\eta}{(2\pi)^3} e^{i\mathbf{q}_\perp\cdot(\mathbf{x}_\perp - \mathbf{y}_\perp) + iq_\eta(\eta_x - \eta_y)} \frac{-i\partial_\tau H^{(1)}_{iq_\eta}(m_T^q \tau)}{\tau H^{(1)}_{iq_\eta}(m_T^q \tau)}$$

- Wigner functional [Jeon & Epelbaum Annals of Phys. 364, 1, 2016]

$$\rho_W[\tau, \phi, \pi] = \mathcal{N} \exp\left(-2 \int \frac{d^3\tilde{k}}{(2\pi)^3} \left| \pi(\tilde{k}) a^*(\tau, \tilde{k}) - \phi(\tilde{k}) e^*(\tau, \tilde{k}) \right|^2 \right)$$

where $\pi = \tau\partial_\tau\phi$ and

$$a(\tau, \tilde{k}) = \sqrt{\frac{\pi}{4}} e^{\pi k_\eta/2} H^{(2)}_{ik_\eta}(k_\perp \tau)$$

$$e(\tau, \tilde{k}) = \tau\partial_\tau a(\tau, \tilde{k})$$

Gauge theory vacuum

- Abelian gauge theory Lagrangian

$$L = \frac{\tau}{2}(\partial_\tau \mathbf{A}_\perp)^2 - \frac{\tau}{2}(\nabla \times \mathbf{A}_\perp)^2 + \frac{1}{2\tau}\mathbf{A}_\perp \cdot \partial_\eta^2 \mathbf{A}_\perp + \frac{1}{2}(\partial_\tau A_\eta)^2 + \frac{1}{2\tau} A_\eta \nabla_\perp^2 A_\eta - \frac{1}{\tau} A_\eta \partial_\eta \nabla_\perp \cdot \mathbf{A}_\perp$$

- Gauss law

$$0 = \frac{\partial_\eta \partial_\tau A_\eta}{\tau^2} + \nabla_\perp \cdot \partial_\tau \mathbf{A}_\perp$$

- Decompose $\mathbf{A}_\perp = \mathbf{A}_T + \mathbf{A}_L$ with

$$\mathbf{A}_T = i \int \frac{d^2 k_\perp}{(2\pi)^2} e^{i\mathbf{k}_\perp \cdot \mathbf{x}_\perp} A_T(\hat{\mathbf{k}}_\perp \times \mathbf{e}_z)$$

and

$$\mathbf{A}_L = \nabla_\perp \varphi$$

Wigner functional

[Jeon & Epelbaum Annals of Phys. 364, 1, 2016]

- Equations for A_T are identical to the scalar case
- Equations for A_η is much more complicated because A_η and φ couple, but solvable.
- Longitudinal Wigner functional

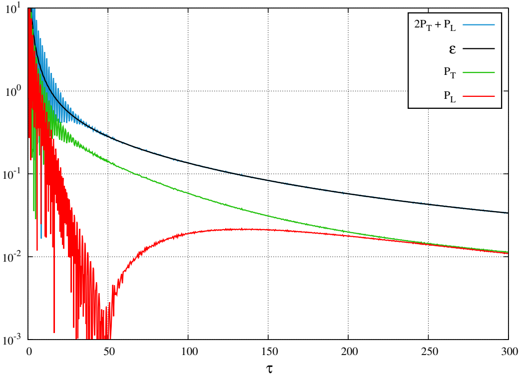
$$\rho_L[\tau, A_\eta, \pi_\eta] = \mathcal{N} \exp\left(-2 \int \frac{d^3\tilde{k}}{(2\pi)^3} \frac{1}{k_\perp^2} \left| e^*(\tau, -\tilde{k})\pi_\eta + k_\perp^2 a^*(\tau, -\tilde{k}) A_\eta \right|^2 \right)$$

with $\pi_\eta = \frac{1}{\tau}\partial_\tau A_\eta$ and

$$a(\tau, \tilde{k}) = \sqrt{\frac{\pi}{4}} e^{\pi k_\eta/2} H^{(2)}_{ik_\eta}(k_\perp \tau)$$

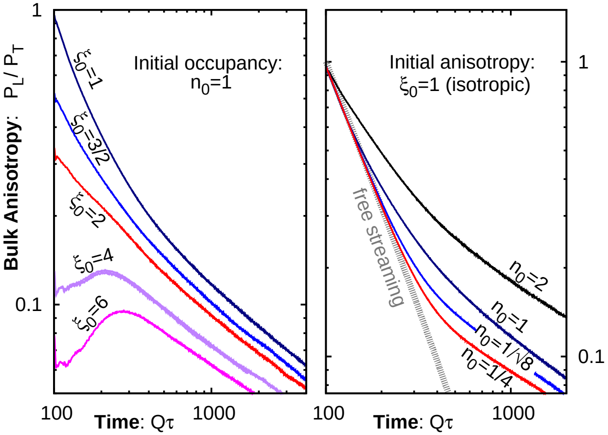
$$e(\tau, \tilde{k}) = \tau \partial_\tau a(\tau, \tilde{k})$$

Scalar theory in Milne space



$g = 4$, DEGV, PRD 86, 085040, 2012

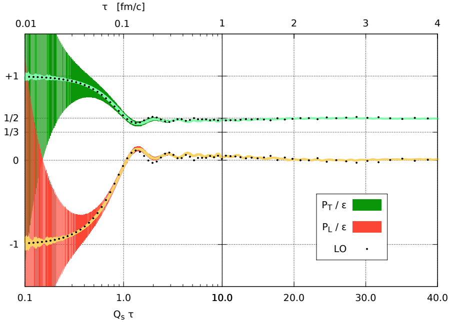
Yang-Mills in Milne space

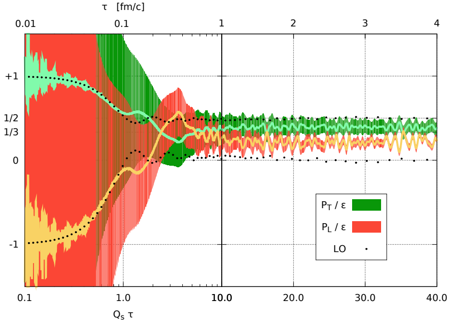


Berges, Boguslavski, Schlichting, Venugopalan, PRD 89, 074011 (2014) (arXiv 1303.5650)

$N_T = 256 - 512$, $N_\eta = 1024 - 4096$

Yang-Mills in Milne space





$g = 0.1$

$g = 0.5$

EG, PRL 111, 232301, 2013 (arXiv 1307.2214)

$N_T = 64$, $N_\eta = 128$

Conclusions, Problems & Perspectives

- CTP is useful in thinking about *initial value problems*
  - Conceptually
  - Practically
- Strong classical + 1st order quantum correction can be done within classical physics – Perfect way to simulate QFT in real time
- Difficulties: Higher order corrections
  - Feynman diagrams are useful conceptually but not computationally – Fourier transform does not produce $\delta(E_i - E_f)$ when $t_i < t < f_f$
  - $\phi_q^3 \phi_c$ terms are intrinsically quantum – Hard to include in purely classical evolution
  - Need to calculate $\delta\rho_W$
- Lots of interesting problems still to be considered:
  - Glasma evolution with quantum corrections
  - Field to Particle transition
  - NNLO JIMWLK
  - Fermions? ...