On chemical equilibration of long-lived heavy particles (in kinetic equilibrium)[1,2]

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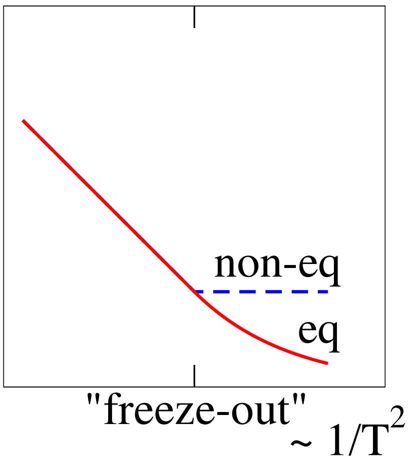
[1] Seyong Kim and ML, 1602.08105; work in progress.

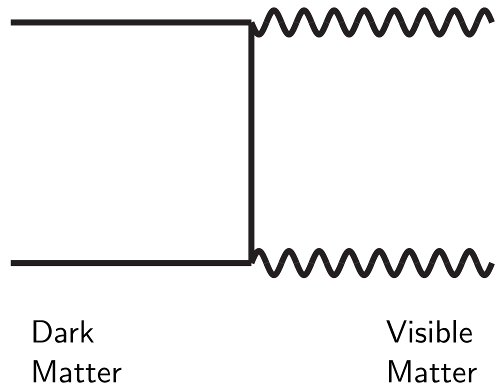
[2] Supported by the SNF under grant 200020-155935.

Motivation

(i) Cosmology: Could WIMPs/SIMPs be dark matter?

An initially thermal system chemically decouples when pair annihilation is not fast enough to track the equilibrium distribution, which is $n_{\rm eq} \sim (\frac{MT}{2\pi})^{3/2} e^{-M/T}$ at $T \ll M$.



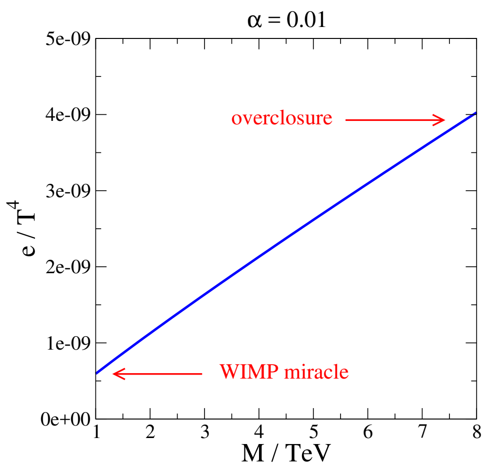


Back of the envelope estimate

Equate the Hubble rate with the co-annihilation rate:

$$H \sim n\langle \sigma v \rangle \;\Leftrightarrow\; \frac{T^2}{m_{\rm Pl}} \sim \Big(\frac{MT}{2\pi}\Big)^{3/2} e^{-M/T} \frac{\alpha^2}{M^2} \overset{\alpha\sim 0.01}{\Rightarrow} T \sim \frac{M}{25} \; .$$

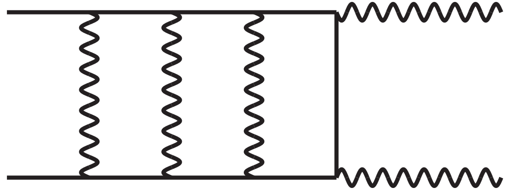
Compare $e \equiv M n$ at the freeze-out with radiation $\sim T^4$:



LHC pushes up lower bound on M, so there is a danger "overclosure".

Could efficient decays help to avoid overclosure?

Indeed co-annihilating particles with $v \ll 1$ interact "strongly".



In particular the "Sommerfeld effect"[3] has been widely discussed.[4] It is an $\gtrsim O(1)$ correction for $T \lesssim \alpha^2 M$.

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[3] L.D. Landau and E.M. Lifshitz, *Quantum Mechanics, Non-Relativistic Theory*, Third Edition, §136; V. Fadin, V. Khoze and T. Sjöstrand, *On the threshold behavior of heavy top production*, Z. Phys. C 48 (1990) 613.

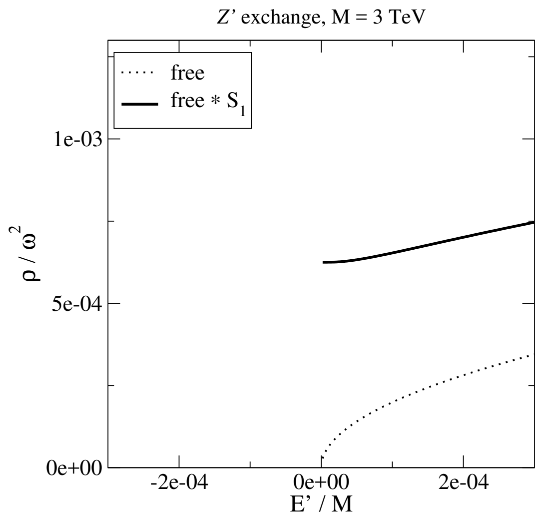
[4] e.g. J. Hisano, S. Matsumoto, M. Nagai, O. Saito and M. Senami, *Non-perturbative effect on thermal relic abundance of dark matter*, hep-ph/0610249; J.L. Feng, M. Kaplinghat and H.-B. Yu, *Sommerfeld Enhancements for Thermal Relic Dark Matter*, 1005.4678.

Rapid summary of the Sommerfeld effect

For attractive s-wave interaction:

$$S_1 = \frac{X_1}{1 - e^{-X_1}} \; , \quad X_1 = \frac{g^2 C_\mathrm{F}}{4v} \; .$$

Corresponding "spectral function" ($E' \equiv \omega - 2M \equiv Mv^2$):



What happens below the threshold?

Perhaps there could be bound states?[5]

This sounds exotic, but we are interested in **rare processes** where two dilute particles come together, i.e. $|\partial_t n| \sim e^{-2M/T}$. In bound states they are "already" together, with a less suppressed Boltzmann weight, because of a binding energy $\Delta E > 0$:

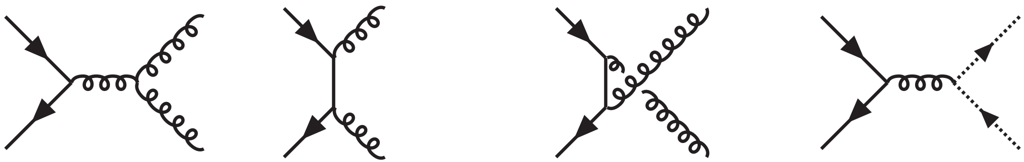
$$|\partial_t n_{\text{bound}}| \sim e^{-(2M-\Delta E)/T} \ .$$

If $T \lesssim \Delta E$, this contribution dominates the co-annihilation rate.

[5] e.g. B. von Harling and K. Petraki, *Bound-state formation for thermal relic dark matter and unitarity*, 1407.7874.

(ii) Heavy ion collision experiments

Could charm chemically equilibrate at Future Circular Collider?



(If so, thermodynamic functions change from normal ones, e.g. charm quarks would boost the bulk viscosity by α_s^{-4}:[6]

$$\delta\zeta = \frac{1}{18T}\lim_{\omega\to 0^+}\left\{\frac{2M^2\chi_f\Gamma_{\rm chem}}{\omega^2+\Gamma^2_{\rm chem}}\right\} = \frac{M^2\chi_f}{9T\Gamma_{\rm chem}}\,.$$
)

[6] ML and K. Sohrabi, *Charm contribution to bulk viscosity*, 1410.6583.

A formalism

Comments on Boltzmann equations

Classic Boltzmann for WIMP abundance:

$$(\partial_t + 3H)\, n \approx -\langle \sigma v \rangle \,(n^2 - n_{\text{eq}}^2)\ .$$

Problem: by construction n contains only scattering states.

Boltzmann boosted by on-shell bound states.

Problem: How many? Width? Melting? Matrix elements?

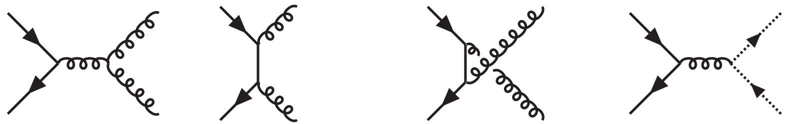
General "linear response" formulation:[7]

$$(\partial_t + 3H)\, n = -\Gamma_{\text{chem}}(n - n_{\text{eq}}) + \mathcal{O}(n - n_{\text{eq}})^2\ .$$

$\Gamma_{\text{chem}} = 2n_{\text{eq}}\langle \sigma v \rangle$ is a "transport coefficient", and the total density $n \equiv e/M$ includes the contribution of bound states.

[7] D. Bödeker and ML, *Heavy quark chemical equilibration rate as a transport coefficient*, 1205.4987; *Sommerfeld effect in heavy quark chemical equilibration*, 1210.6153.

Physical picture of the co-annihilation process



The energy released in the inelastic reaction is $2M \gg T \Rightarrow$ the "hard" process is effectively **local:**

Initial state: $E_{\text{rest}} \sim 2M$, $E_{\text{kin}} \sim \frac{k^2}{2M} \sim T$.

Final state: $E_{\text{kin}} \sim 2p \sim 2M$, $\Delta x \sim \frac{1}{p} \sim \frac{1}{M} \ll \frac{1}{\sqrt{MT}}, \frac{1}{T}$.

Soft effects are encoded in the thermal expectation value of a 4-particle operator ("$\mathcal{M}^*\mathcal{M}$") describing the hard process.[8]

[8] e.g. L.S. Brown and R.F. Sawyer, *Nuclear reaction rates in a plasma*, astro-ph/9610256.

This can be implemented with NRQCD [9]

Let θ, η annihilate DM and DM'. Like in the optical theorem, decays are contained in an imaginary part of a 4-particle operator:

$$\mathcal{O} = \frac{ic_1\alpha^2\,\theta^\dagger\eta^\dagger\,\eta\theta}{M^2} + O(\alpha^3(2M), v^2)\ .$$

Through a linear response analysis, this yields

$$n_{\text{eq}}\Gamma_{\text{chem}} = \frac{8c_1\alpha^2}{M^2}\frac{1}{\mathcal{Z}}\sum_m e^{-E_m/T}\langle m|\theta^\dagger\eta^\dagger\,\eta\theta|m\rangle\ .$$

[9] G.T. Bodwin, E. Braaten and G.P. Lepage, *Rigorous QCD analysis of inclusive annihilation and production of heavy quarkonium*, hep-ph/9407339.

Thermal average can be resolved into a Wightman fcn

$$\begin{aligned}
\gamma &\equiv \frac{1}{\mathcal{Z}} \sum_m e^{-E_m/T} \langle m | \theta^\dagger \eta^\dagger \, \eta\theta | m \rangle \\
&= \Big\langle (\theta^\dagger \eta^\dagger)(0, \mathbf{0}) (\eta\theta)(0, \mathbf{0}) \Big\rangle_T \\
&= \int_{\omega, \mathbf{k}} \underbrace{\int_{t, \mathbf{x}} e^{i(\omega t - \mathbf{k}\cdot\mathbf{x})} \Big\langle (\theta^\dagger \eta^\dagger)(0, \mathbf{0}) (\eta\theta)(t, \mathbf{x}) \Big\rangle_T}_{\Pi_<(\omega, \mathbf{k})}
\end{aligned}$$

Wightman fcn can be expressed through a spectral fcn

$$\Pi_<(\omega,\mathbf{k}) \;=\; 2n_{\mathsf{B}}(\omega)\rho(\omega,\mathbf{k}) \stackrel{\omega\gg T}{\approx} 2e^{-\omega/T}\rho(\omega,\mathbf{k}) \; .$$

Moreover, in a non-relativistic 2-body problem, the dependence on the center-of-mass momentum $\mathbf{k}$ can be factored out:

$$\omega \;=\; 2M + \frac{k^2}{4M} + E' \; , \quad \rho(\omega,\mathbf{k}) \approx \rho(E') \; ,$$

$$\Rightarrow \gamma \;\approx\; \Big(\frac{MT}{\pi}\Big)^{3/2} e^{-2M/T} \int_{-\Lambda}^{\infty} \frac{\mathrm{d}E'}{\pi} \, e^{-E'/T} \rho(E') \; .$$

Spectral fcn is a cut of a Green's function

$$\left[H - i\,\Gamma(r) - E'\right]G(E';\mathbf{r},\mathbf{r}') \quad = \quad \delta^{(3)}(\mathbf{r}-\mathbf{r}')\ ,$$

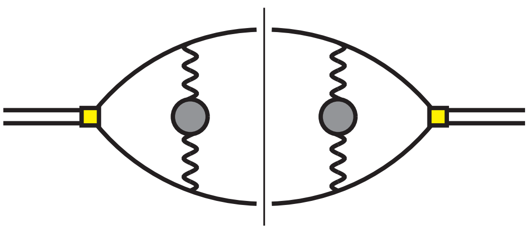
$$\lim_{\mathbf{r},\mathbf{r}'\to\mathbf{0}} \mathrm{Im}\,G(E';\mathbf{r},\mathbf{r}') \quad = \quad \rho(E')\ .$$

$$H = -\frac{\nabla_r^2}{M} + V(r)\ .$$

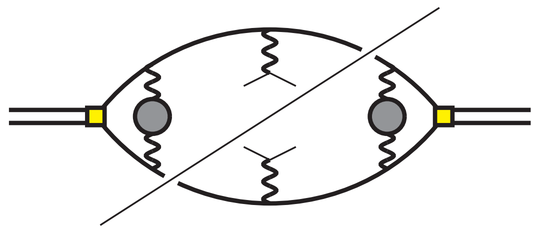
$V(r)$ and $\Gamma(r)$ emerge from gauge exchange

$$V(r) - i\,\Gamma(r) = g^2 \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3} \left(1 - e^{i\mathbf{k}\cdot\mathbf{r}}\right) i\Delta_{00\mathrm{T}}(0,k) \ .$$

The width represents real scatterings, present in a plasma:



$\sim V(r)$



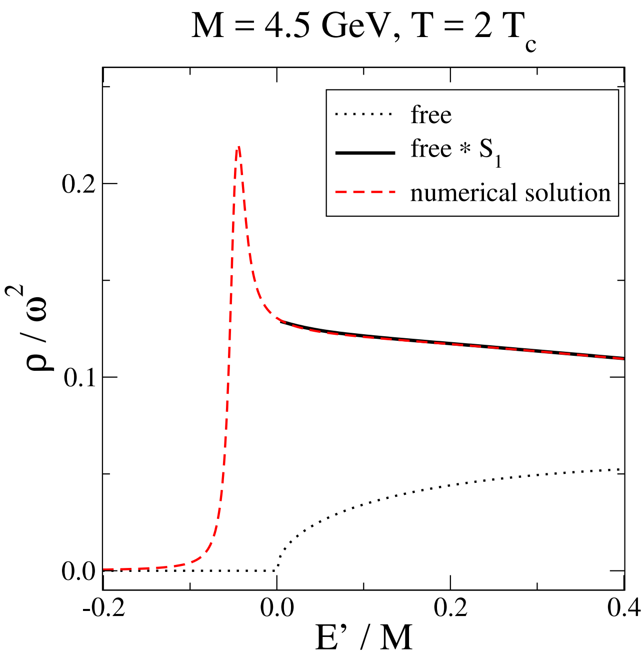
$\sim \Gamma(r)$

In a nutshell

- Compute thermal (full or HTL) gauge field self-energy
- Determine corresponding time-ordered propagator
- Fourier-transform for potential and width
- Solve for $\rho(E') = \operatorname{Im} G(E'; \mathbf{0}, \mathbf{0})$
- Laplace-transform with weight $e^{-E'/T}$ for γ

Results for QCD

Perturbative side [10]

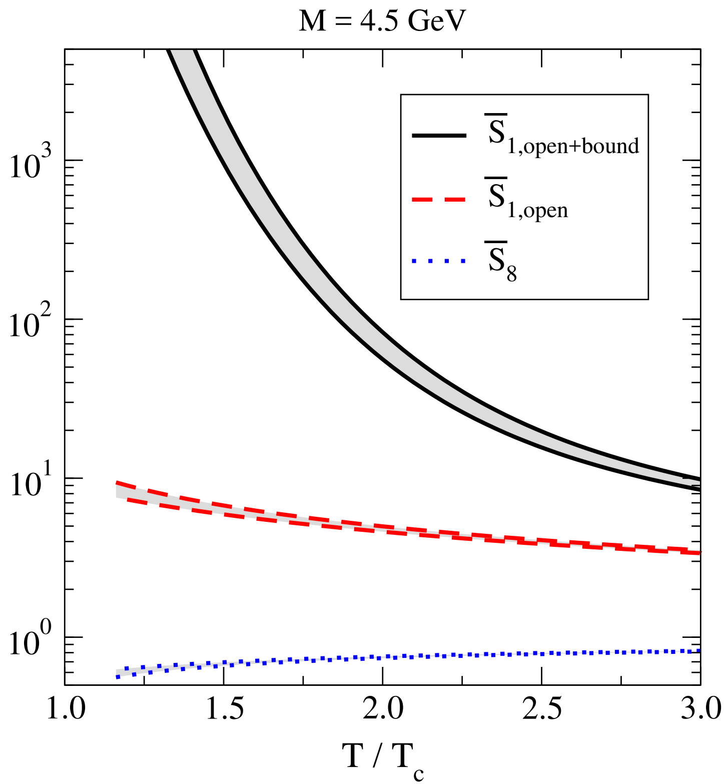


$$V(r) = -\alpha_s \left[m_\mathrm{D} + \frac{\exp(-m_\mathrm{D} r)}{r} \right] ,$$

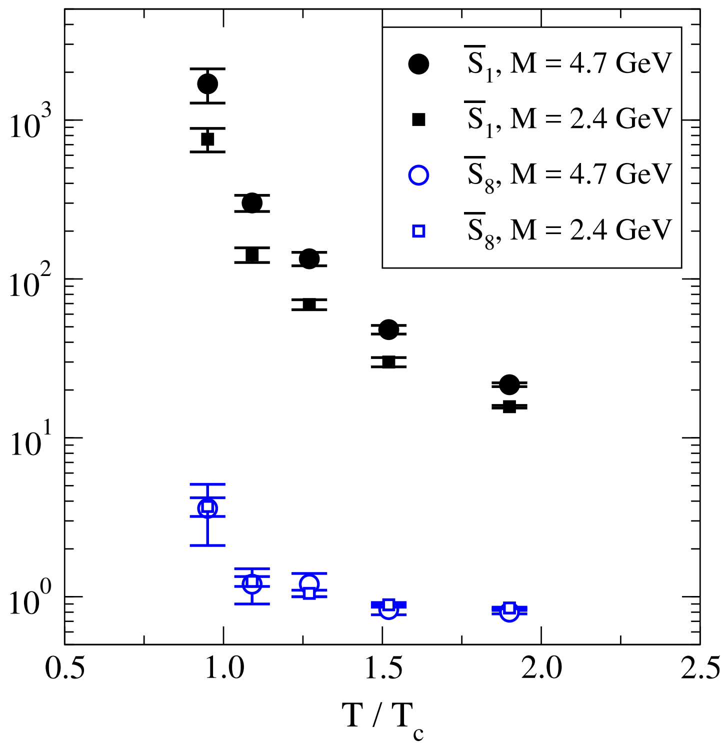
$$\Gamma(r) = 2\alpha_s T \int_0^\infty \frac{\mathrm{d}x\, x}{(x^2+1)^2} \times \left[1 - \frac{\sin(x m_\mathrm{D} r)}{x m_\mathrm{D} r} \right] .$$

[10] ML, O. Philipsen, P. Romatschke and M. Tassler, *Real-time static potential in hot QCD*, hep-ph/0611300; A. Beraudo, J.-P. Blaizot and C. Ratti, *Real and imaginary-time $Q\overline{Q}$ correlators in a thermal medium*, 0712.4394; N. Brambilla, J. Ghiglieri, A. Vairo and P. Petreczky, *Static quark-antiquark pairs at finite temperature*, 0804.0993.

Thermal average ⇒ bound states dominate singlet decays



Lattice NRQCD confirms this on a qualitative level



Implication for heavy ions

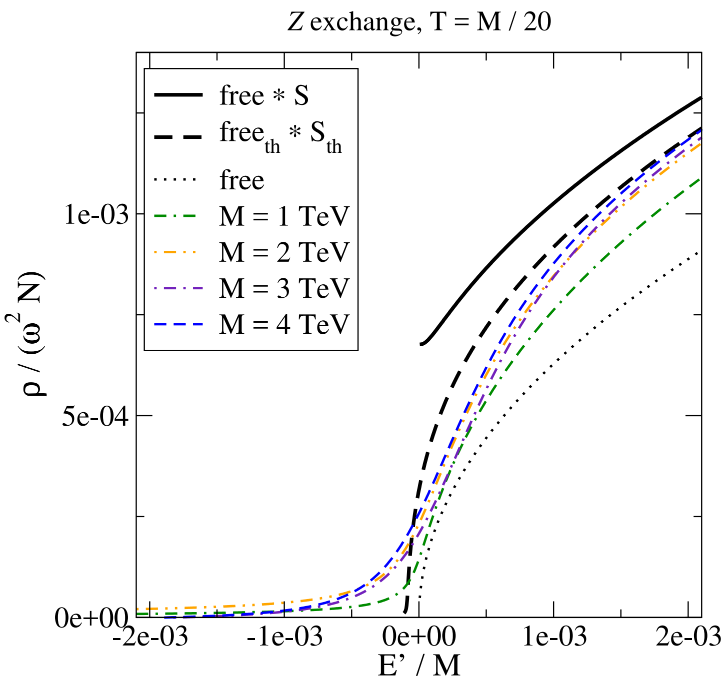
In pQCD the process splits up into two parts, the "colour-singlet" discussed above as well as a "colour-octet" one, in which case the interaction is repulsive and $\bar{S}_8 < 1$.

$$\Gamma_{\text{chem}} = \frac{g^4 C_\mathrm{F}}{8\pi M^2}\left(\frac{MT}{2\pi}\right)^{3/2} e^{-M/T} \times \left[\frac{1}{N_\mathrm{c}}\bar{S}_1 + \left(\frac{N_\mathrm{c}^2-4}{2N_\mathrm{c}} + N_\mathrm{f}\right)\bar{S}_8\right].$$

$\bar{S}_8 \simeq 0.8$ is weighted more than $\bar{S}_1 \simeq 15$ so the numerical effect on charm equilibration in QCD is modest: $\Gamma^{-1}_{\text{chem}} \sim 150$ fm/c at $T \approx 400$ MeV, and $\Gamma^{-1}_{\text{chem}} \sim 40$ fm/c at $T \approx 600$ MeV.

Results for cosmology

Z exchange: no bound states are found



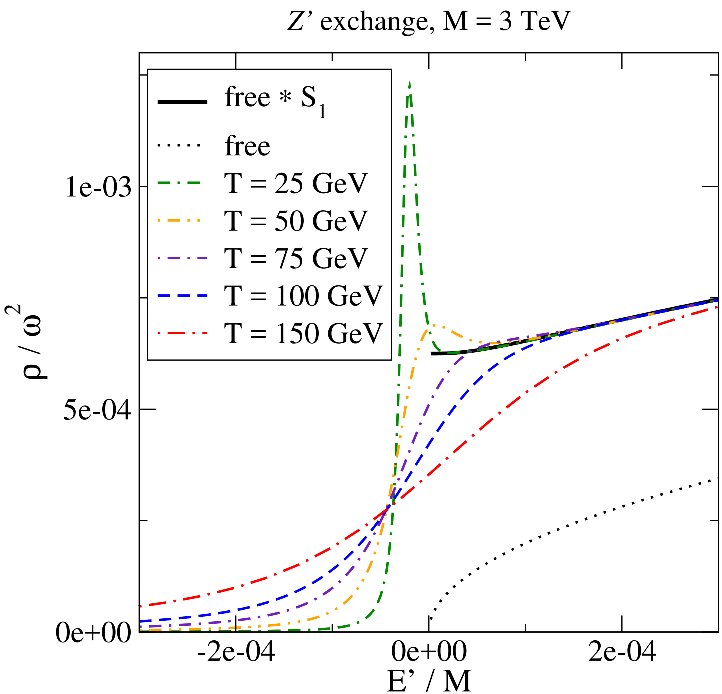
$$\alpha \equiv \frac{g_1^2 + g_2^2}{16\pi} \approx 0.01 \,,$$

screening by m_Z

$\simeq$ 91 GeV + Debye mass .

Average $\int \mathrm{d}E' \, e^{-E'/T} \ldots \Rightarrow S$ works with $\sim 1\%$ errors.

Z' exchange:[11] bound states melt below freeze-out



$$\alpha' \equiv (e')^2/(4\pi) \sim 0.01 \,,$$

screening by $m_{Z'}$

$$\approx \quad 1 \text{ GeV} + \text{Debye mass} \,.$$

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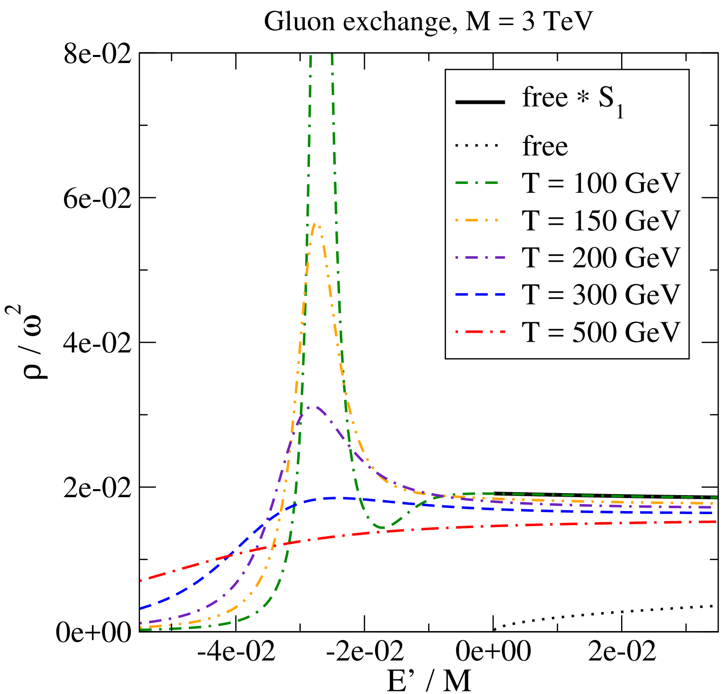
[11] e.g. M. Pospelov, A. Ritz, M.B. Voloshin, *Secluded WIMP Dark Matter*, 0711.4866; W. Shepherd *et al*, *Bound states of weakly interacting dark matter*, 0901.2125.

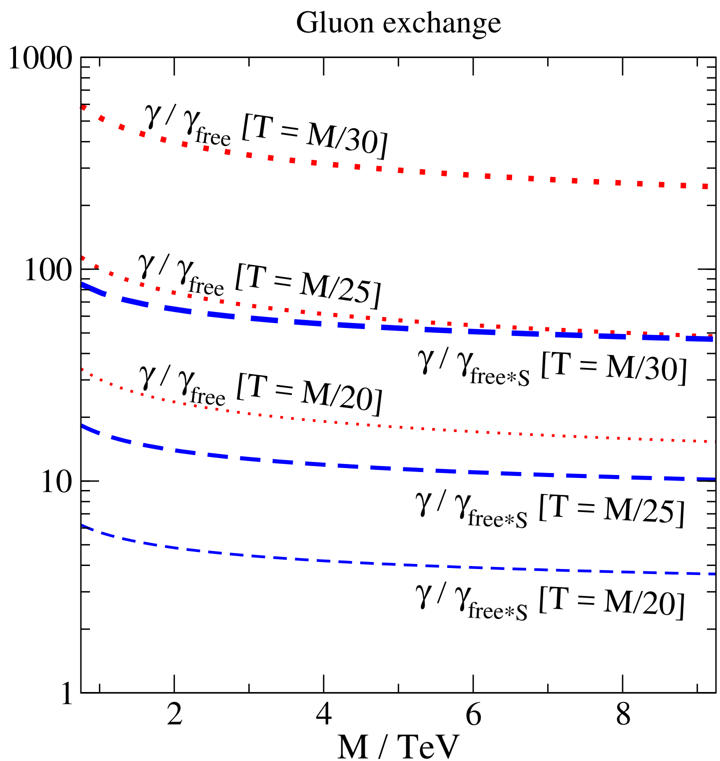
Gluon exchange between gluinos:[12] like in QCD



$$
\begin{aligned}
m_{\rm D}^2 &\equiv 2 g_s^2 T^2 \,, \\
\alpha_3 &\equiv \frac{3 g_s^2}{4\pi} \,.
\end{aligned}
$$

[12] e.g. J. Ellis, F. Luo and K.A. Olive, *Gluino Coannihilation Revisited*, 1503.07142.

Sommerfeld effect is large and bound-state effect equally so



Summary

• If precision is needed (NLO corrections, thermal effects, bound-state contributions, non-perturbative studies), techniques do exist for non-relativistic scenarios.

• Weak interactions: we confirm the presence of the Sommerfeld effect. For practical purposes it seems to be all there is to it.

• Strong interactions: the co-annihilation rate is much enhanced because of bound states. This may help to avoid overclosure in cosmology, and could find applications for heavy quarks at FCC.

• Model-specific studies are needed for definite conclusions.