Effective Description of Dark Matter as a Viscous Fluid

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Distribution of dark and baryonic matter in the Universe

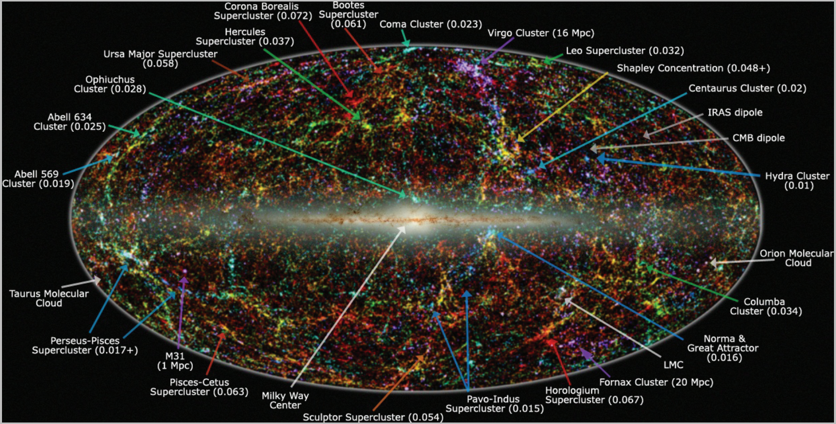


Figure: *2MASS Galaxy Catalog (more than 1.5 million galaxies).*

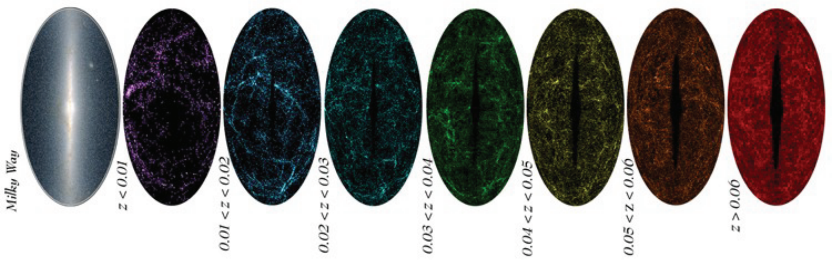


Figure: *2MASS Galaxy Catalog: Various redshifts.*

Background

- Homogeneity and isotropy of the background:

$$ds^2 = -dt^2 + a^2(t)\delta_{ij}dx^i dx^j$$

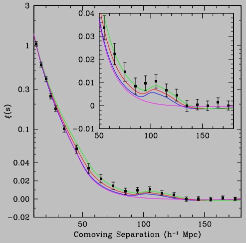
$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\sum_i \rho_i \qquad \frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\sum_i(\rho_i + 3p_i)$$

- For acceleration: $p < -\rho/3$
- Dominant contributions today (not observed directly) to the energy content of the Universe:
  1. Dark matter: Perfect fluid with $p = 0$ ($\sim 25\%$)
     Usual assumption: Weakly interacting massive particles
  2. Cosmological constant or dark energy with $p < 0$ ($\sim 70\%$)
- Other contributions: baryons ($\sim 5\%$), photons, neutrinos ...

Inhomogeneities

- Inhomogeneities are treated as perturbations on top of the homogeneous background.
- Under gravitational attraction, the matter overdensities grow and produce the observed large-scale structure.
- The distribution of matter at various redshifts reflects the detailed structure of the cosmological model.
- Define the density field $\delta = \delta\rho/\rho_0$ and its spectrum

$$\langle \delta(\mathbf{k})\delta(\mathbf{q})\rangle \equiv \delta_D(\mathbf{k}+\mathbf{q})P(\mathbf{k}).$$



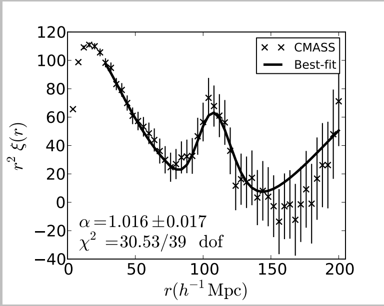


Figure: Galaxy correlation function (Sloan Digital Sky Survey 2005 (left) and 2012 (right)).

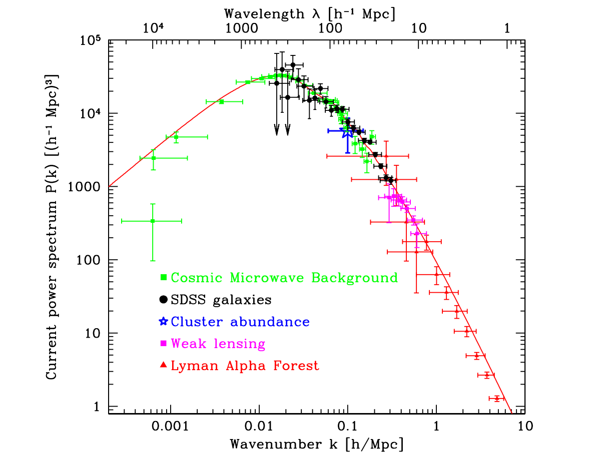


Figure: Matter power spectrum (Tegmark *et al.* 2003).

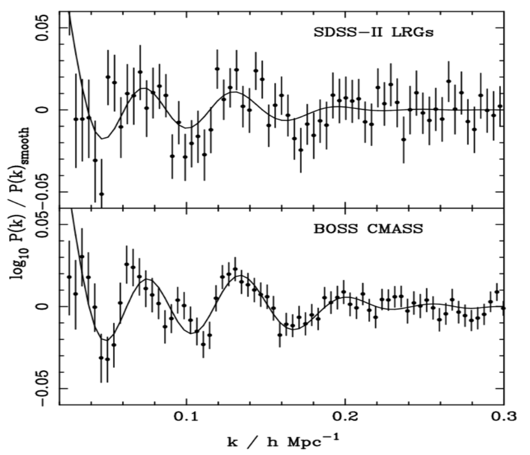


Figure: Matter power spectrum in the range of baryon acoustic oscillations.

A scale from the early Universe

- The characteristic scale of the baryon acoustic oscillations is approximately 150 Mpc (490 million light-years) today.
- It corresponds to the wavelength of sound waves (the sound horizon) in the baryon-photon plasma at the time of recombination, which took place at a redshift $z = a_{\rm today}/a - 1$=1100, when the Universe was 380,000 years old and had a temperature of 3000 K.
- It is also imprinted on the spectrum of the photons of the cosmic microwave background.
- Comparing the measured with the theoretically calculated spectra constrains the cosmological model.
- The aim is to achieve a 1% precision both for the measured and calculated spectra.
- Galaxy surveys: Euclid, DES, LSST, SDSS ...

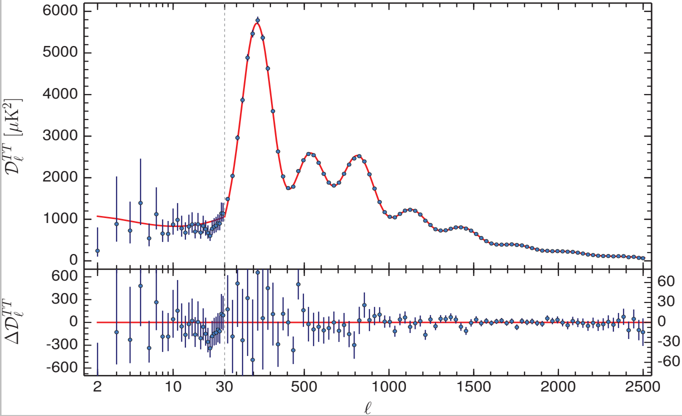


Figure: The power spectrum of the cosmic microwave background (Planck 2015).

Problems with standard perturbation theory

- In the linearized hydrodynamic equations each mode evolves independently. Higher-order corrections take into account mode-mode coupling.
- Analytical calculation of the matter spectrum beyond the linear level. (Crocce, Scoccimarro 2005)
- Baryon acoustic oscillations ($k \simeq 0.05 - 0.2\, h/\text{Mpc}$): Mildly nonlinear regime of perturbation theory.
- Higher-order corrections dominate for $k \simeq 0.3 - 0.5\, h/\text{Mpc}$.
- The theory becomes strongly coupled for $k \gtrsim 1\, h/\text{Mpc}$.
- The deep UV region is out of the reach of perturbation theory.
- Way out: Introduce an effective low-energy description in terms of an imperfect fluid (Baumann, Nicolis, Senatore, Zaldarriaga 2010, Carrasco, Foreman, Green, Senatore 2014)

Lessons from the Wilsonian renormalization group (Wilson 1971)

- Coarse-graining: Integrate out the modes with $k > k_m$ and replace them with effective couplings in the low-k theory.
- Example: Consider a quantum scalar field χ with action $S[\chi]$.
- Add a regulating piece $R_k(q)$ that cuts off modes with $q^2 < k^2$.
- Take the Lagrange transform and remove the regulator to obtain the coarse-grained effective action $\Gamma_k[\phi]$.
- $\Gamma_k[\phi]$ satisfies the exact equation ($t = \ln k$) (Wetterich 1993)

$$\frac{\partial \Gamma_k[\phi]}{\partial t} = \frac{1}{2}\mathrm{Tr}\left[\left(\Gamma_k^{(2)}[\phi] + \hat{R}_k\right)^{-1} \frac{\partial \hat{R}_k}{\partial \ln k}\right].$$

- For a standard kinetic term and potential $U_k[\phi]$, with a sharp cutoff, the first step of an iterative solution gives

$$U_{k_m}(\phi) = V(\phi) + \frac{1}{2}\int_{k_m}^{\Lambda} \frac{d^d q}{(2\pi)^d} \ln\left(q^2 + V''(\phi)\right). \tag{1}$$

- The low-energy theory contains new couplings, not present in the tree-level action. It comes with a UV cutoff k_m.

- Why is this intuition relevant for the problem of classical cosmological perturbations?
- The primordial Universe is a stochastic medium.
- The fluctuating fields (density, velocity) at early times are Gaussian random variables with an almost scale-invariant spectrum.
- The generation of this spectrum is usually attributed to inflation.
- During the quasi-de Sitter phase, quantum fluctuations of the inflaton are stretched by the very fast expansion. When their wavelength becomes larger than the Hubble radius, they stop propagating and their amplitude is frozen. When the inflaton decays, density inhomogeneities are generated. After the end of inflation, these inhomogeneities reenter the horizon and, after matter-radiation equality, grow (dark matter) or oscillate (baryons, up to recombination).
- The coarse graining can be implemented formally on the initial condition for the spectrum at recombination.

The question

- Question: Is dark matter at large scales (in the BAO range) best described as a perfect fluid?
- I shall argue that there is a better description in the context of the effective theory.
- Going beyond the perfect-fluid approximation, the description must include effective (shear and bulk) viscosity and nonzero speed of sound.
- Formulate the perturbative approach for viscous dark matter.
- I shall focus on effective viscosity, but I shall also consider briefly the case of fundamental viscosity.

Estimates

- $k_\Lambda \sim 1 - 3\ h/\mathrm{Mpc}$ (length $\sim 3 - 10$ Mpc):
  The fluid description becomes feasible.
- $k_m \sim 0.5 - 1\ h/\mathrm{Mpc}$ (length $\sim 10 - 20$ Mpc):
  The fluid parameters have a simple form.
- The description includes viscosity terms arising either through coarse-graining (effective viscosity), or fundamental dark matter interactions (fundamental viscosity).
- At k_m there is effective viscosity resulting from the integration of the modes $k > k_m$. The form of the power spectrum $\sim k^{-3}$ means that the effective viscosity is dominated by $k \simeq k_m$.
- Scales $k > k_\Lambda$ correspond to virialized structures, which are essentially decoupled.
- k_m acts as an UV cutoff for perturbative corrections in the large-scale theory.
- Good convergence, in contrast to standard perturbation theory.

- Dark matter can be treated as a fluid because of its small velocity and the finite age of the Universe. Dark matter particles drift over a finite distance, much smaller than the Hubble radius. (Baumann, Nicolis, Senatore, Zaldarriaga 2010)
- The phase space density $f(\mathbf{x},\mathbf{p},\tau)=f_0(p)[1+\delta_f(\mathbf{x},p,\hat{\mathbf{p}},\tau)]$ can be expanded in Legendre polynomials:

$$\delta_f(\mathbf{k},p,\hat{\mathbf{p}},\tau)=\sum_{n=0}^{\infty}(-i)^n(2n+1)\delta_f^{[n]}(\mathbf{k},p,\tau)P_n(\hat{\mathbf{k}}\cdot\hat{\mathbf{p}}).$$

For the moments $\delta_f^{[n]}$, the Vlasov equation leads to:

$$\frac{d\delta_f^{[n]}}{d\tau}=kv_p\left[\frac{n+1}{2n+1}\delta_f^{[n+1]}-\frac{n}{2n+1}\delta_f^{[n-1]}\right], \qquad n\geq 2,$$

with $v_p=p/am$ the particle velocity.
- The time τ available for the higher moments to grow is $\sim 1/\mathcal{H}$.
- A fluid description is possible for $kv_p/\mathcal{H}\lesssim 1$.
- Estimate the particle velocity from the fluid velocity v at small length scales.

- At the comoving scale k, the linear evolution indicates that $(\theta/\mathcal{H})^2 \sim k^3 P^L(k)$,
  with $\theta = \vec{k}\vec{v}$ and $P^L(k)$ the linear power spectrum.
- The linear power spectrum scales roughly as k^{-3} above $\sim k_m$.
  $k^3 P^L(k)$ is rougly constant, with a value of order 1 today.
  Its time dependence is given by D_L^2, with D_L the linear growth factor.
- If the maximal particle velocity is identified with the fluid velocity at the scale k_m, we have

$$v_p \sim \frac{\mathcal{H}}{k_m} D_L.$$

- The dimensionless factor characterizing the growth of higher moments is $kv_p/\mathcal{H} \sim D_L k/k_m$.
- Scales with $k \gg k_m/D_L$ require the use of the whole Boltzmann hierarchy.
- In practice, the validity of the fluid description extends beyond k_m, as the initial values of the various moments are constrained by the CMB to be much smaller than 1.

- The shear viscosity is estimated as $\eta/(\rho+p) \sim l_{\rm free} v_p$, with $l_{\rm free}$ the mean free path.
- For the effective viscosity we can estimate $l_{\rm free} \sim v_p/H$, with $H = \mathcal{H}/a$. In this way we obtain

$$\nu_{\rm eff}\mathcal{H} = \frac{\eta_{\rm eff}}{(\rho+p)a}\mathcal{H} \sim l_{\rm free} v_p H \sim \frac{\mathcal{H}^2}{k_m^2} D_L^2,$$

- For interacting dark matter, with number density n, mass m and cross section σ, we have $l_{\rm free} \sim 1/(n\sigma)$. We expect then

$$\nu_{\rm fund}\mathcal{H} = \frac{\eta_{\rm fund}}{(\rho+p)a}\mathcal{H} \sim l_{\rm free} v_p H \sim \frac{1}{n\sigma}\frac{H^2}{k_m} a D_L \sim \frac{m}{\sigma}\frac{8\pi G_N}{3k_m}\frac{aD_L}{\Omega_m}.$$

  The fundamental viscosity drops very quickly at early times.
- $\nu_{\rm fund}$ does not grow indefinitely for decreasing σ. The mean free path cannot exceed $v_p/H \sim aD_L/k_m$. Otherwise, the particle scattering has no effect.
- The fundamental viscosity cannot exceed $\sim (\mathcal{H}^2/k_m^2)D_L^2$, and is alway subleading or comparable to the effective viscosity.

Plan of the talk

- Basic formalism
- Backreaction of perturbations on the average expansion
- Determination of the effective viscosity
- Calculation of the spectrum
- Conclusions

- S. Floerchinger, N. T., U. Wiedemann
  arXiv:1411.3280[gr-qc], Phys. Rev. Lett. 114: 9, 091301 (2015)
- D. Blas, S. Floerchinger, M. Garny, N. T., U. Wiedemann
  arXiv:1507.06665[astro-ph.CO], JCAP 1511, 049 (2015)
- S. Floerchinger, M. Garny, N. T., U. Wiedemann
  arXiv:1607.03453[astro-ph.CO]

Covariant hydrodynamic description of a viscous fluid

- Work within the first-order formalism.
- Energy-momentum tensor:

$$T^{\mu\nu} = \rho u^\mu u^\nu + (p + \pi_b)\Delta^{\mu\nu} + \pi^{\mu\nu}.$$

ρ: energy density
p: pressure in the fluid rest frame
π_b: bulk viscosity
$\pi^{\mu\nu}$: shear viscosity, satisfying: $u_\mu \pi^{\mu\nu} = \pi^\mu_{\ \mu} = 0$
$\Delta^{\mu\nu}$ projector orthogonal to the fluid velocity: $\Delta^{\mu\nu} = g^{\mu\nu} + u^\mu u^\nu$

- New elements:
  Bulk viscosity: $\pi_b = -\zeta \nabla_\rho u^\rho$
  Shear viscosity tensor:

$$\pi^{\mu\nu} = -2\eta\sigma^{\mu\nu} = -2\eta\left(\frac{1}{2}\left(\Delta^{\mu\rho}\nabla_\rho u^\nu + \Delta^{\nu\rho}\nabla_\rho u^\mu\right) - \frac{1}{3}\Delta^{\mu\nu}(\nabla_\rho u^\rho)\right).$$

Dynamical equations

- Einstein equations:

$$G_{\mu\nu} = 8\pi G_N T_{\mu\nu},$$

- Conservation of the energy momentum tensor ($\nabla_\nu T^{\mu\nu} = 0$):

$$\begin{aligned} u^\mu \nabla_\mu \rho + (\rho + p)\nabla_\mu u^\mu - \zeta \left(\nabla_\mu u^\mu\right)^2 - 2\eta\sigma^{\mu\nu}\sigma_{\mu\nu} &= 0 \\ (\rho + p + \pi_b)u^\mu \nabla_\mu u^\rho + \Delta^{\rho\mu}\nabla_\mu (p + \pi_b) + \Delta^\rho{}_\nu \nabla_\mu \pi^{\mu\nu} &= 0. \end{aligned}$$

Subhorizon scales

- Ansatz for the metric:

$$ds^2 = a^2(\tau)\left[-(1+2\Psi(\tau,\mathbf{x}))\,d\tau^2 + (1-2\Phi(\tau,\mathbf{x}))\,d\mathbf{x}\,d\mathbf{x}\right].$$

- The potentials Φ and Ψ are weak. Their difference is governed by the shear viscosity. We can take $\Phi \simeq \Psi \ll 1$.
- The four-velocity $u^\mu = dx^\mu/\sqrt{-ds^2}$ can be expressed through the coordinate velocity $v^i = dx^i/d\tau$ and the potentials Φ and Ψ:

$$u^\mu = \frac{1}{a\sqrt{1+2\Psi-(1-2\Phi)\vec{v}^2}}(1,\vec{v}).$$

- Neglect vorticity and consider the density $\delta = \frac{\delta\rho}{\rho_0}$ and velocity $\theta = \vec{\nabla}\vec{v}$ fields. Combine them in a doublet $\varphi_{1,2} = (\delta, -\theta/\mathcal{H})$, where $\mathcal{H} = \dot{a}/a$ is the Hubble parameter.
- The spectrum is

$$\langle \varphi_a(\mathbf{k},\tau)\varphi_b(\mathbf{q},\tau)\rangle \equiv \delta_D(\mathbf{k}+\mathbf{q})P_{ab}(\mathbf{k},\tau).$$

- $P(\mathbf{k},\tau) \equiv P_{11}(\mathbf{k},\tau)$.

- Continuity equation:

$$\dot{\rho} + \vec{v}\cdot\vec{\nabla}\rho + (\rho + p)\left(3\frac{\dot{a}}{a} + \vec{\nabla}\cdot\vec{v}\right)$$
$$= \frac{\zeta}{a}\left[3\frac{\dot{a}}{a} + \vec{\nabla}\cdot\vec{v}\right]^2 + \frac{\eta}{a}\left[\partial_i v_j \partial_i v_j + \partial_i v_j \partial_j v_i - \frac{2}{3}(\vec{\nabla}\cdot\vec{v})^2\right]$$

- Take the spatial average:

$$\frac{1}{a}\dot{\bar{\rho}} + 3H\left(\bar{\rho} + \bar{p} - 3\bar{\zeta}H\right) = D\,,$$

with Hubble parameter $H = \dot{a}/a^2$ and

$$D = \frac{1}{a^2}\langle \eta\left[\partial_i v_j \partial_i v_j + \partial_i v_j \partial_j v_i - \frac{2}{3}\partial_i v_i \partial_j v_j\right]\rangle$$
$$+ \frac{1}{a^2}\langle \zeta[\vec{\nabla}\cdot\vec{v}]^2\rangle + \frac{1}{a}\langle \vec{v}\cdot\vec{\nabla}\left(p - 6\zeta H\right)\rangle\,.$$

- The average of the trace of Einstein's equations $\langle R\rangle = 8\pi G_{\rm N}\,\langle T^{\mu}{}_{\mu}\rangle$ reads:

$$\frac{\ddot{a}}{a^3} = \frac{1}{a}\dot{H} + 2H^2 = \frac{4\pi G_{\rm N}}{3}\left(\bar{\rho} - 3\bar{p} + 9\bar{\zeta}H\right)\,.$$

Backreaction

S. Floerchinger, N. T., U. Wiedemann
arXiv:1411.3280[gr-qc], Phys. Rev. Lett. 114: 9, 091301 (2015)

- Ignore bulk viscosity: $\zeta = 0$.
- Define $\bar{p} = \hat{w}\,\bar{\rho}$ and the deceleration parameter $q = -1 - \dot{H}/(aH^2)$.
- One finds:

$$-\frac{dq}{d\ln a} + 2(q-1)\left(q - \tfrac{1}{2}(1+3\hat{w})\right) = \frac{4\pi G_N D(1-3\hat{w})}{3H^3}.$$

- For

$$\frac{4\pi G_N D}{3H^3} > \frac{1+3\hat{w}}{1-3\hat{w}}$$

there is a fixed point with acceleration: $q < 0$.

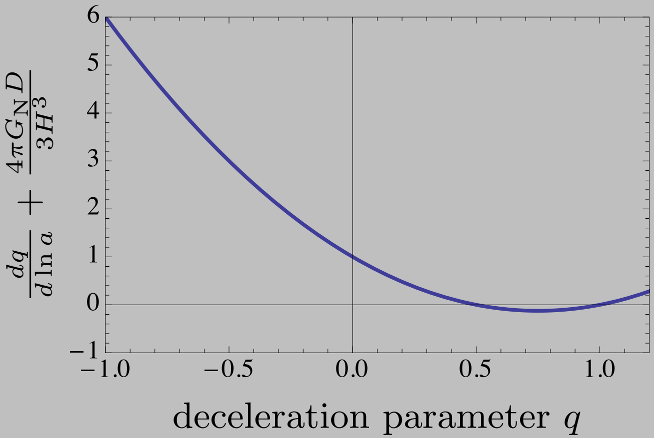


Figure: Graphical representation of the evolution equation for the deceleration parameter q for vanishing pressure, $\hat{w} = 0$.

Estimate of backreaction

- Assume that typical fluid-velocity gradients are of order H:

$$\bar{\eta}\langle\partial_i v_j \partial_i v_j + \partial_i v_j \partial_j v_i - \frac{2}{3}\partial_i v_i \partial_j v_j\rangle / a^2 \sim \bar{\eta} H^2 .$$

  This corresponds to realistic peculiar velocities of the order of 100 km$/s$ on distances of 1 Mpc.
- A large shear viscosity arises in systems of particles with a long mean free path (e.g. an additional component to CDM):

$$\eta \sim \rho_\eta \tau_\eta$$

  ρ_η: energy density of these particles, τ_η: mean free time.
- Accelerating expansion would result for

$$\frac{4\pi G_N D}{3H^3} \sim \frac{\rho_\eta \tau_\eta H}{2\rho_{\rm cr}} \sim 1$$

  For $\tau_\eta \lesssim 1/H$ one needs $\rho_\eta \sim \rho_{\rm cr}$.
- Realistic model, consistent with large-scale structure? Difficult, but worth pursuing.

Convergence of perturbation theory (no viscosity)

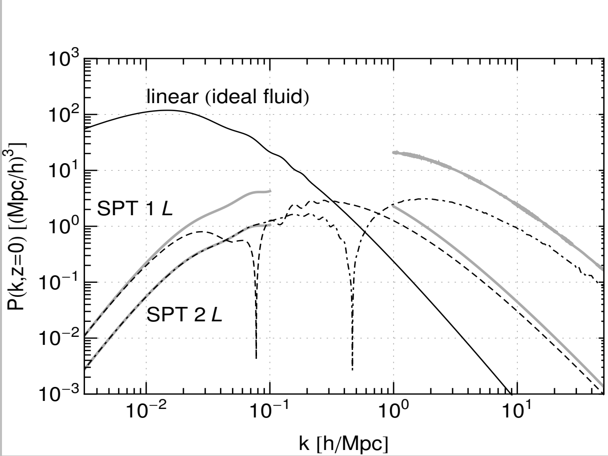


Figure: Linear power spectrum and the one- and two-loop corrections in standard perturbation theory (SPT) at $z = 0$.

- Higher-order (loop) corrections dominate for $k \gtrsim 0.3 - 0.5\ h/\text{Mpc}$.
- The theory becomes strongly coupled for $k \simeq 1\ h/\text{Mpc}$.
- The deep UV region is out of the reach of perturbation theory.
- Higher-order corrections are increasingly more sensitive to the UV.
- For small k, the one-loop correction depends on the dimensionful scale

$$\sigma_d^2(\eta) = \frac{4\pi}{3} \int_0^\infty dq\, P^L(q,\eta) = \frac{4\pi}{3} D_L^2(\eta) \int_0^\infty dq\, P^L(q,0),$$

  with $\eta = \ln a = -\ln(1+z)$ and $D_L(\eta)$ the linear growth factor: $\delta(\mathbf{k},\eta) = D_L(\eta)\delta(\mathbf{k},0)$ on the growing mode.
- For the spectrum, the complete expression is (Blas, Garny, Konstandin 2013)

$$P_{ab}^{1-\text{loop}}(k,\eta) = -\begin{pmatrix} \frac{61}{105} & \frac{25}{21} \\ \frac{25}{21} & \frac{9}{5} \end{pmatrix} k^2 \sigma_d^2 P^L(k,\eta).$$

Effective description (in terms of pressure and viscosity)

D. Blas, S. Floerchinger, M. Garny, N. T., U. Wiedemann
arXiv:1507.06665[astro-ph.CO], JCAP 1511, 049 (2015)

- Introduce an effective low-energy description in terms of an imperfect fluid (Baumann, Nicolis, Senatore, Zaldarriaga 2010, Carrasco, Foreman, Green, Senatore 2014)
- Use the Wilsonian approach: Integrate out the modes with $k \gtrsim k_m$ and replace them with effective couplings (viscosity, pressure) in the low-k theory.
- Determine the effective couplings in terms of

$$\sigma^2_{dk}(\eta) = \frac{4\pi}{3}\int_{k_m}^{\infty} dq\, P^L(q,\eta) = \frac{4\pi}{3} D_L^2(\eta)\int_{k_m}^{\infty} dq\, P^L(q,0).$$

- For a spectrum that scales as $1/k^3$, the integral is dominated by the region near k_m. The deep UV does not contribute significantly.

Parameters of an effective viscous theory

$$\rho(\mathbf{x},\tau) = \rho_0(\tau) + \delta\rho(\mathbf{x},\tau) \qquad p(\mathbf{x},\tau) = 0 + \delta p(\mathbf{x},\tau)$$

$$\delta = \frac{\delta\rho}{\rho_0} \qquad \theta = \vec{\nabla}\vec{v}$$

$$\mathcal{H} = \frac{\dot{a}}{a} \qquad H = \frac{1}{a}\mathcal{H}$$

$$c_s^2(\tau) = \frac{\delta p}{\delta\rho} = \alpha_s(\tau)\frac{\mathcal{H}^2}{k_m^2} \qquad \nu(\tau)\mathcal{H} = \frac{\eta\mathcal{H}}{\rho_0 a} = \frac{3}{4}\alpha_\nu(\tau)\frac{\mathcal{H}^2}{k_m^2}.$$

with $\alpha_s, \alpha_\nu = \mathcal{O}(1)$ today.

- We rely on a hierarchy supported by linear perturbation theory for subhorizon perturbations.
  1. We treat δ and $\theta/\mathcal{H}$ as quantities of order 1, $\vec{v}$ as a quantity of order $\mathcal{H}/k$ and Ψ, Φ as quantities of order $\mathcal{H}^2/k^2$.
  2. We assume that a time derivative is equivalent to a factor of $\mathcal{H}$, while a spatial derivative to a factor of k.
  3. We assume that $c_s^2, \nu\mathcal{H}$ are of order $\mathcal{H}^2/k_m^2$.
- Keeping the dominant terms, we obtain

$$\begin{aligned}
\dot{\delta} + \vec{\nabla}\vec{v} + (\vec{v}\vec{\nabla})\delta + \delta\vec{\nabla}\vec{v} &= 0 \\
\dot{\vec{v}} + \mathcal{H}\vec{v} + (\vec{v}\vec{\nabla})\vec{v} + \vec{\nabla}\Phi + c_s^2(1-\delta)\vec{\nabla}\delta & \\
-\nu(1-\delta)\left(\nabla^2\vec{v} + \frac{1}{3}\vec{\nabla}(\vec{\nabla}\vec{v})\right) &= 0. \\
\nabla^2\Phi &= \frac{3}{2}\Omega_m\mathcal{H}^2\delta.
\end{aligned}$$

Use Fourier-transformed quantities to obtain

$$\dot{\delta}_{\mathbf{k}} + \theta_{\mathbf{k}} + \int d^3\mathbf{p}\, d^3\mathbf{q}\, \delta(\mathbf{k}-\mathbf{p}-\mathbf{q})\, \alpha_1(\mathbf{p},\mathbf{q})\, \delta_{\mathbf{p}}\, \theta_{\mathbf{q}} = 0$$

$$\dot{\theta}_{\mathbf{k}} + \left(\mathcal{H} + \frac{4}{3}\nu k^2\right) \theta_{\mathbf{k}} + \left(\frac{3}{2}\Omega_m \mathcal{H}^2 - c_s^2 k^2\right) \delta_{\mathbf{k}} + \int d^3\mathbf{p}\, d^3\mathbf{q}\, \delta(\mathbf{k}-\mathbf{p}-\mathbf{q}) \left(\beta_1(\mathbf{p},\mathbf{q})\, \delta_{\mathbf{p}}\, \delta_{\mathbf{q}} + \beta_2(\mathbf{p},\mathbf{q})\, \theta_{\mathbf{p}}\, \theta_{\mathbf{q}} + \beta_3(\mathbf{p},\mathbf{q})\, \delta_{\mathbf{p}}\, \theta_{\mathbf{q}}\right) = 0,$$

with

$$\alpha_1(\mathbf{p},\mathbf{q}) = \frac{(\mathbf{p}+\mathbf{q})\mathbf{q}}{q^2}$$

$$\beta_1(\mathbf{p},\mathbf{q}) = c_s^2(\mathbf{p}+\mathbf{q})\mathbf{q}$$

$$\beta_2(\mathbf{p},\mathbf{q}) = \frac{(\mathbf{p}+\mathbf{q})^2\,\mathbf{p}\cdot\mathbf{q}}{2p^2q^2}$$

$$\beta_3(\mathbf{p},\mathbf{q}) = -\frac{4}{3}\nu(\mathbf{p}+\mathbf{q})\mathbf{q}.$$

Define the doublet

$$\begin{pmatrix} \varphi_1(\mathbf{k},\eta) \\ \\ \varphi_2(\mathbf{k},\eta) \end{pmatrix} = \begin{pmatrix} \delta_{\mathbf{k}}(\tau) \\ \\ -\dfrac{\theta_{\mathbf{k}}(\tau)}{\mathcal{H}} \end{pmatrix},$$

where $\eta = \ln a(\tau)$. The evolution equations take the form

$$\partial_\eta \phi_a(\mathbf{k}) = -\Omega_{ab}(\mathbf{k},\eta)\varphi_b(\mathbf{k}) + \int d^3p\, d^3q\, \delta(\mathbf{k}-\mathbf{p}-\mathbf{q})\gamma_{abc}(\mathbf{p},\mathbf{q},\eta)\,\varphi_b(\mathbf{p})\,\varphi_c(\mathbf{q}),$$

where

$$\Omega(\mathbf{k},\eta) = \begin{pmatrix} 0 & -1 \\ -\frac{3}{2}\Omega_m + \alpha_s \frac{k^2}{k_m^2} & 1 + \frac{\mathcal{H}'}{\mathcal{H}} + \alpha_\nu \frac{k^2}{k_m^2} \end{pmatrix}.$$

and a prime denotes a derivative with respect to η. The nonzero elements of γ_{abc} are expressed in terms of $\alpha_1(\mathbf{p},\mathbf{q})$, $\beta_1(\mathbf{p},\mathbf{q})$, $\beta_2(\mathbf{p},\mathbf{q})$, $\beta_3(\mathbf{p},\mathbf{q})$.

The evolution of the spectrum

- Define the spectra, bispectra and trispectra as

$$
\begin{aligned}
\langle \varphi_a(\mathbf{k},\eta)\varphi_b(\mathbf{q},\eta)\rangle \equiv& \delta_D(\mathbf{k}+\mathbf{q})P_{ab}(\mathbf{k},\eta)\\
\langle \varphi_a(\mathbf{k},\eta)\varphi_b(\mathbf{q},\eta)\varphi_c(\mathbf{p},\eta)\rangle \equiv& \delta_D(\mathbf{k}+\mathbf{q}+\mathbf{p})B_{abc}(\mathbf{k},\mathbf{q},\mathbf{p},\eta)\\
\langle \varphi_a(\mathbf{k},\eta)\varphi_b(\mathbf{q},\eta)\varphi_c(\mathbf{p},\eta)\varphi_d(\mathbf{r},\eta)\rangle \equiv& \delta_D(\mathbf{k}+\mathbf{q})\delta_D(\mathbf{p}+\mathbf{r})P_{ab}(\mathbf{k},\eta)P_{cd}(\mathbf{p},\eta)\\
&+\delta_D(\mathbf{k}+\mathbf{p})\delta_D(\mathbf{q}+\mathbf{r})P_{ac}(\mathbf{k},\eta)P_{bd}(\mathbf{q},\eta)\\
&+\delta_D(\mathbf{k}+\mathbf{r})\delta_D(\mathbf{q}+\mathbf{p})P_{ad}(\mathbf{k},\eta)P_{bc}(\mathbf{q},\eta)\\
&+\delta_D(\mathbf{k}+\mathbf{p}+\mathbf{q}+\mathbf{r})Q_{abcd}(\mathbf{k},\mathbf{p},\mathbf{q},\mathbf{r},\eta).
\end{aligned}
$$

- Bogoliubov-Born-Green-Kirkwood-Yvon (BBGKY) hierarchy.
- Essential (rather crude) approximation: Neglect the effect of the trispectrum on the evolution of the bispectrum (Pietroni 2008).
- In this way we obtain

$$
\begin{aligned}
\partial_\eta P_{ab}(\mathbf{k},\eta) = & -\Omega_{ac}P_{cb}(\mathbf{k},\eta) - \Omega_{bc}P_{ac}(\mathbf{k},\eta) \\
& + \int d^3q \big[\gamma_{acd}(\mathbf{k},-\mathbf{q},\mathbf{q}-\mathbf{k})B_{bcd}(\mathbf{k},-\mathbf{q},\mathbf{q}-\mathbf{k}) \\
& + \gamma_{bcd}(\mathbf{k},-\mathbf{q},\mathbf{q}-\mathbf{k})B_{acd}(\mathbf{k},-\mathbf{q},\mathbf{q}-\mathbf{k})\big], \\
\partial_\eta B_{abc}(\mathbf{k},-\mathbf{q},\mathbf{q}-\mathbf{k}) = & -\Omega_{ad}B_{dbc}(\mathbf{k},-\mathbf{q},\mathbf{q}-\mathbf{k}) - \Omega_{bd}B_{adc}(\mathbf{k},-\mathbf{q},\mathbf{q}-\mathbf{k}) \\
& -\Omega_{cd}B_{abd}(\mathbf{k},-\mathbf{q},\mathbf{q}-\mathbf{k}) \\
& +2\int d^3q \big[\gamma_{ade}(\mathbf{k},-\mathbf{q},\mathbf{q}-\mathbf{k})P_{db}(\mathbf{q},\eta)P_{ec}(\mathbf{k}-\mathbf{q},\eta) \\
& +\gamma_{bde}(-\mathbf{q},\mathbf{q}-\mathbf{k},\mathbf{k})P_{dc}(\mathbf{k}-\mathbf{q},\eta)P_{ea}(\mathbf{k},\eta) \\
& +\gamma_{cde}(\mathbf{q}-\mathbf{k},\mathbf{k},-\mathbf{q})P_{da}(\mathbf{k},\eta)P_{eb}(\mathbf{q},\eta)\big].
\end{aligned}
$$

Alternative approach

- Expand the fields in powers of the initial perturbations at $\eta = \eta_0$,

$$\phi_a(\mathbf{k},\eta) = \sum_n \int d^3q_1 \cdots d^3q_n \, (2\pi)^3 \delta^{(3)}(\mathbf{k} - \sum_i \mathbf{q}_i) \times F_{n,a}(\mathbf{q}_1, \ldots, \mathbf{q}_n, \eta) \delta_{\mathbf{q}_1}(\eta_0) \cdots \delta_{\mathbf{q}_n}(\eta_0) \, .$$

- From the equation of motion we can get evolution equations for the kernels $F_{n,a}$

$$(\partial_\eta \delta_{ab} + \Omega_{ab}(\mathbf{k},\eta)) F_{n,b}(\mathbf{q}_1, \ldots, \mathbf{q}_n, \eta) = \sum_{m=1}^{n} \gamma_{abc}(\mathbf{q}_1 + \cdots + \mathbf{q}_m, \mathbf{q}_{m+1} + \cdots + \mathbf{q}_n) \times F_{m,b}(\mathbf{q}_1, \ldots, \mathbf{q}_m, \eta) F_{n-m,c}(\mathbf{q}_{m+1}, \ldots, \mathbf{q}_n, \eta) \, .$$

- When neglecting the pressure and viscosity terms, the solution is known analytically for an Einstein-de Sitter Universe.
- For non-zero pressure and viscosity, the time-dependence does not factorize. We solve the differential equations numerically.

Matching the perfect-fluid and viscous theories

- For the matching one could use the propagator

$$G_{ab}(\mathbf{k},\tilde{\eta},\tilde{\eta}')\delta^{(3)}(\mathbf{k}-\mathbf{k}') = \left\langle \frac{\delta\phi_a(\mathbf{k},\tilde{\eta})}{\delta\phi_b(\mathbf{k}',\tilde{\eta}')} \right\rangle ,$$

- Define appropriate fields for the background to be effectively Einstein-de Sitter to a very good approximation: $D_L(\tilde{\eta}) = \exp(\tilde{\eta})$.
- The one-loop propagator of the perfect-fluid theory is

$$G_{ab}(\mathbf{k},\tilde{\eta},\tilde{\eta}') = g_{ab}(\tilde{\eta}-\tilde{\eta}') - k^2 e^{\tilde{\eta}-\tilde{\eta}'}\sigma_d^2(\tilde{\eta}) \begin{pmatrix} \frac{61}{350} & \frac{61}{525} \\ \frac{27}{50} & \frac{9}{25} \end{pmatrix} ,$$

where the linear propagor for the growing mode is

$$g_{ab}(\tilde{\eta}-\tilde{\eta}') = \frac{e^{\tilde{\eta}-\tilde{\eta}'}}{5} \begin{pmatrix} 3 & 2 \\ 3 & 2 \end{pmatrix} .$$

- The contribution from $k > k_m$ can be isolated by taking

$$\sigma_d^2(\tilde{\eta}) \longrightarrow \sigma_{dk}^2(\tilde{\eta}) \equiv \frac{4\pi}{3}\exp(2\tilde{\eta}) \int_{k_m}^{\infty} dq\, P^L(q,0) .$$

- Compute the propagator of the viscous theory at linear order, for kinematic viscosity and sound velocity of the form

$$\nu\mathcal{H} = \frac{\eta}{\rho_0\, a}\mathcal{H} = \beta_\nu\, e^{2\tilde{\eta}}\, \frac{\mathcal{H}^2}{k_m^2}\,, \qquad c_s^2 = \frac{\delta p}{\delta \rho} = \frac{3}{4}\,\beta_s\, e^{2\tilde{\eta}}\, \frac{\mathcal{H}^2}{k_m^2}\,.$$

- The propagator contains a contribution

$$\delta g_{ab}(\mathbf{k},\tilde{\eta}) = -\frac{k^2}{k_m^2}(\beta_\nu + \beta_s)\,\frac{e^{3\tilde{\eta}}}{45}\begin{pmatrix} 3 & 2 \\ 9 & 6 \end{pmatrix}$$

in addition to the perfect-fluid linear contribution (for $\tilde{\eta} \gg \tilde{\eta}'$).
- Identify the linear contribution $\sim k^2 c_s^2$ and $\sim k^2 \nu\mathcal{H}$ with the one-loop correction $\sim k^2 \sigma_{dk}^2$ of the perfect-fluid propagator.
- This can be achieved with 1% accuracy, and gives

$$\beta_s + \beta_\nu = \frac{27}{10}\, k_m^2 \sigma_{dk}^2(0).$$

- Solve for the nonlinear spectrum in the effective viscous theory with an UV cutoff k_m in the momentum integrations.
- There are no free parameters in this approach.

Particular features

- The results are independent of the ratio β_ν/β_s to a good approximation. They depend mainly on the value of $\beta_\nu + \beta_s$.
- They are also insensitive to the (mode-mode) couplings proportional to the viscosity or the shound velocity. The (mode-mode) couplings of the perfect-fluid theory are the dominant ones.

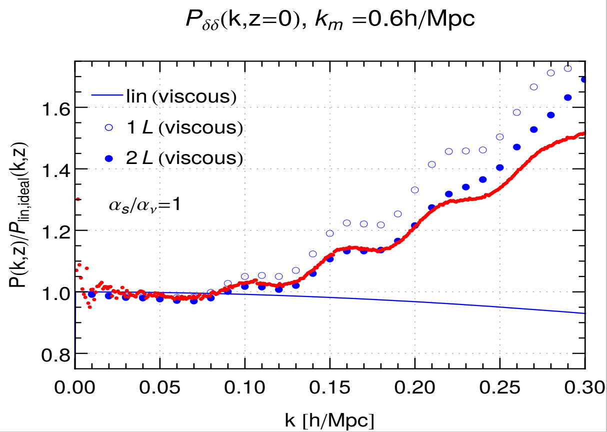


Figure: Power spectrum obtained in the viscous theory for redshift $z = 0$, normalized to $P_{lin,ideal}$. The open (filled) circles show the one-loop (two-loop) result in the viscous theory. The solid blue line is the linear spectrum in the viscous theory, and the red points show results of the Horizon N-body simulation.

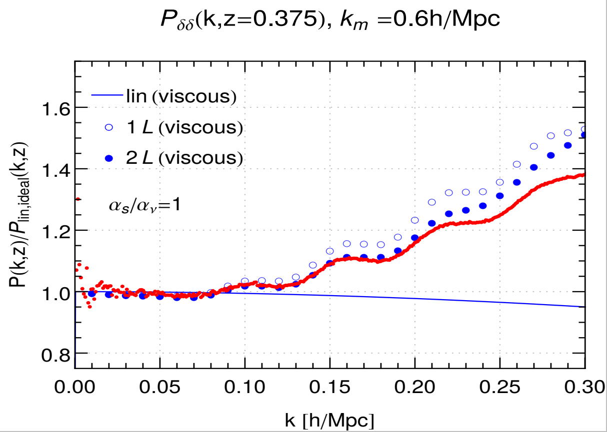


Figure: One-loop spectrum in the effective theory at $z = 0.375$.

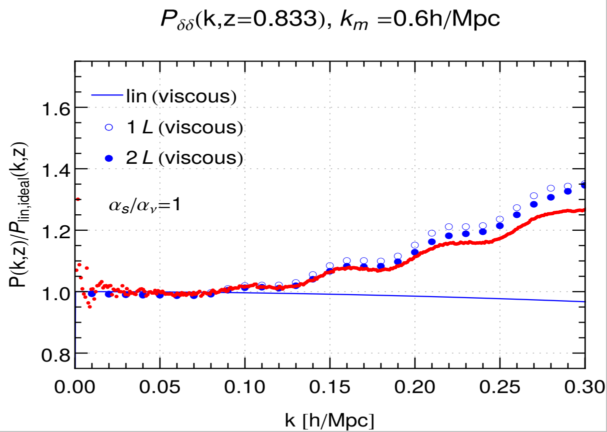


Figure: One-loop spectrum in the effective theory at $z = 0.833$.

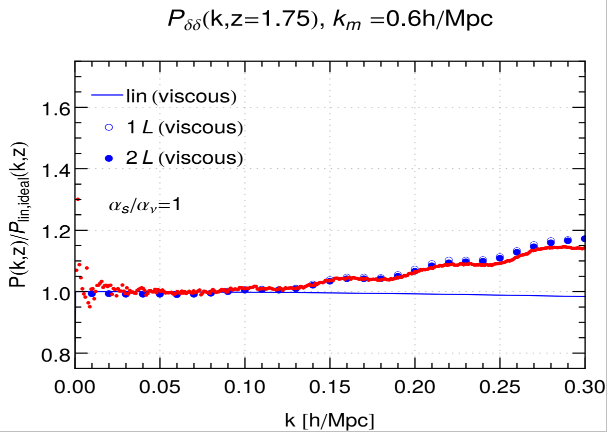


Figure: One-loop spectrum in the effective theory at $z = 1.75$.

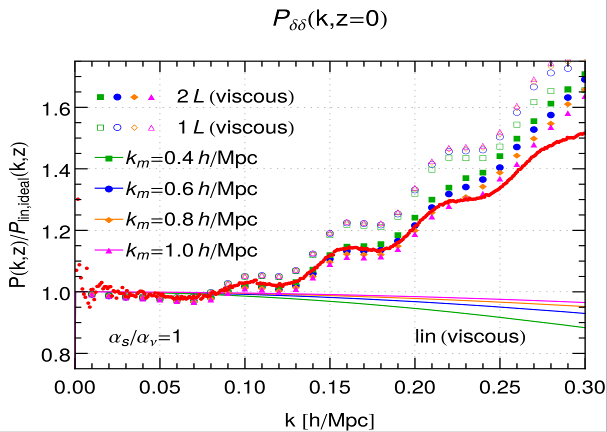


Figure: One-loop spectrum in the effective theory at $z = 0$ for various values of k_m.

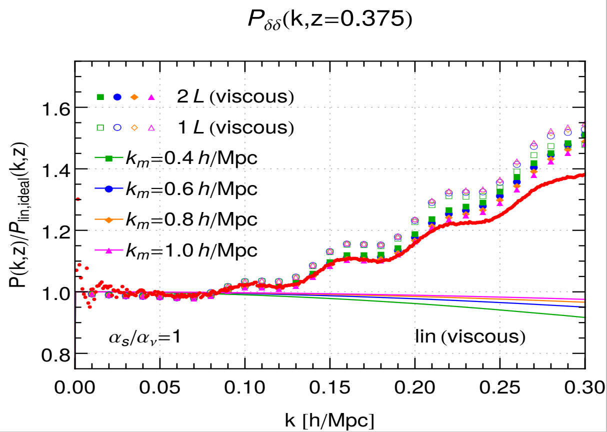


Figure: One-loop spectrum in the effective theory at $z = 0.375$ for various values of k_m.

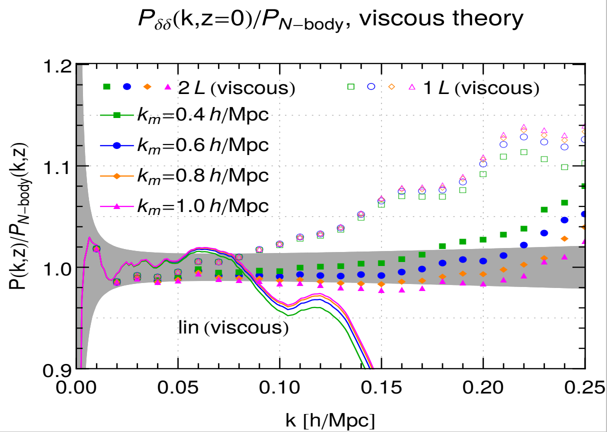


Figure: Comparison of results for the power spectrum obtained within the viscous theory normalized to the N-body result at $z = 0$. The grey band corresponds to an estimate for the error of the N-body result.

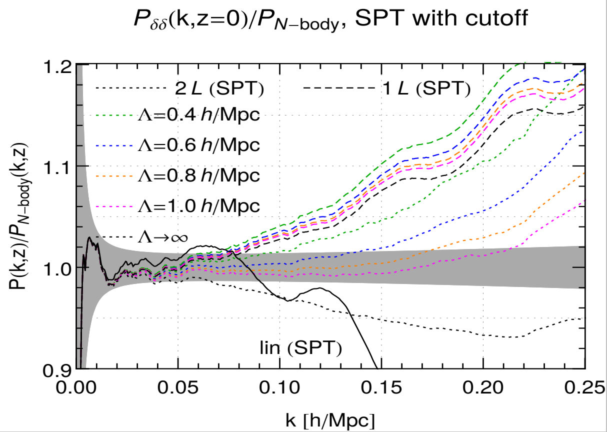


Figure: One- and two-loop results in standard perturbation theory (shown as dashed and dotted lines, respectively), computed with various values of an ad-hoc cutoff Λ (coloured lines), as well as in the limit $\Lambda \to \infty$ (black lines).

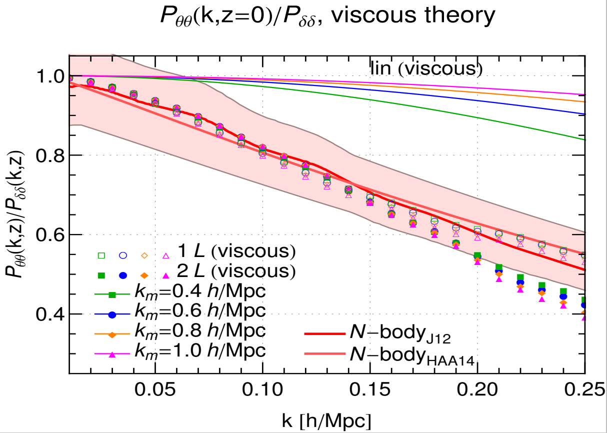


Figure: The velocity-velocity spectrum obtained within the viscous theory, compared to results from N-body simulations at $z = 0$. The pink band corresponds to a 10% error for the N-body results.

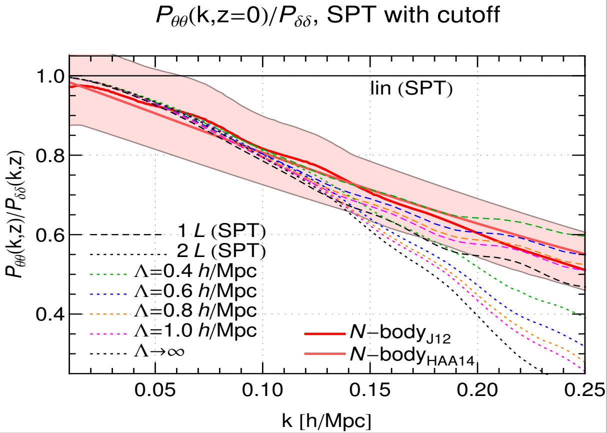


Figure: One- and two-loop results in standard perturbation theory (shown as dashed and dotted lines, respectively), computed with various values of an ad-hoc cutoff Λ (coloured lines), as well as in the limit $\Lambda \to \infty$ (black lines).

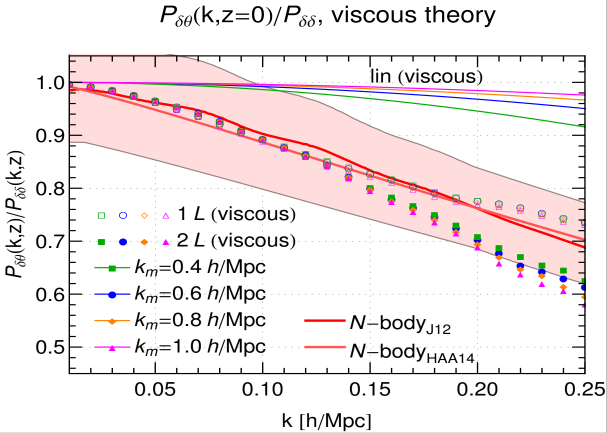


Figure: The density-velocity spectrum obtained within the viscous theory, compared to results from N-body simulations at $z = 0$. The pink band corresponds to a 10% error for the N-body results.

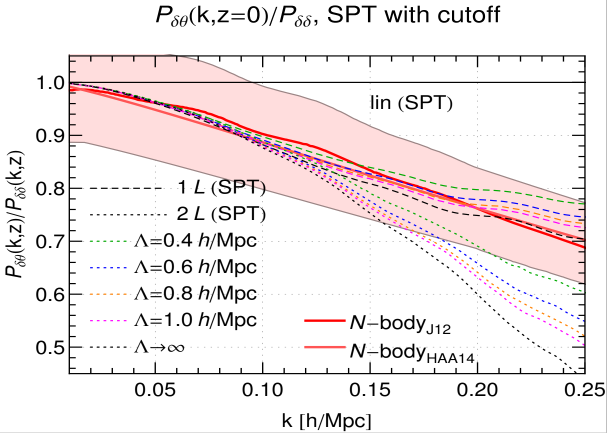


Figure: One- and two-loop results in standard perturbation theory (shown as dashed and dotted lines, respectively), computed with various values of an ad-hoc cutoff Λ (coloured lines), as well as in the limit $\Lambda \to \infty$ (black lines).

- We guessed
  $\nu(\tau)\mathcal{H} = \frac{3}{4}\alpha_\nu(\tau)\frac{\mathcal{H}^2}{k_m^2}, \quad c_s^2(\tau) = \alpha_s(\tau)\frac{\mathcal{H}^2}{k_m^2}$, with $\alpha_\nu, \ \alpha_s \lesssim 1$.
- At one loop we found
  $\nu\mathcal{H} = \frac{3}{4}\,\beta_\nu\, e^{2\eta}\,\frac{\mathcal{H}^2}{k_m^2}\,, \quad c_s^2 = \beta_s\, e^{2\eta}\,\frac{\mathcal{H}^2}{k_m^2}$, with $\beta_s + \beta_\nu = \frac{27}{10}\,k_m^2\sigma_{dk}^2(0)$.
- We generalize the framework by considering
  $\nu\mathcal{H} = \frac{3}{4}\lambda_\nu(k)\, e^{\kappa(k)\eta}\mathcal{H}^2, \quad c_s^2 = \lambda_s(k)\, e^{\kappa(k)\eta}\mathcal{H}^2$.
- The scale dependence of the parameters can be determined through renormalization-group methods. This requires a functional representation of the problem.

Functional representation

S. Floerchinger, M. Garny, N. T., U. Wiedemann
arXiv:1607.03453[astro-ph.CO]

- We follow and extend Matarrese, Pietroni 2007.
- We are interested in solving

$$\partial_\eta \phi_a(\mathbf{k}) = -\Omega_{ab}(\mathbf{k},\eta)\phi_b(\mathbf{k}) + \int d^3p\, d^3q\, \delta^{(3)}(\mathbf{k}-\mathbf{p}-\mathbf{q})\gamma_{abc}(\mathbf{p},\mathbf{q},\eta)\,\phi_b(\mathbf{p})\,\phi_c(\mathbf{q})$$

with stochastic initial conditions determined by the primordial power spectrum $P^0_{ab}(k)$.

- This can be achieved by computing the generating functional

$$Z[J,\, K;\, P^0] = \int \mathcal{D}\phi\mathcal{D}\chi \exp\left\{ -\frac{1}{2}\chi_a(0)P^0_{ab}\chi_b(0) + i\int d\eta\,[\chi_a(\delta_{ab}\partial_\eta + \Omega_{ab})\phi_b - \gamma_{abc}\chi_a\phi_b\phi_c + J_a\phi_a + K_b\chi_b] \right\}$$

- One can now define the generating functional of connected Green's functions

$$W[J,K;P^0] = -i \log Z[J,K;P^0].$$

- The full power spectrum P_{ab} and the propagator G_{ab} can be obtained through second functional derivatives of W,

$$\begin{aligned}
\left.\frac{\delta^2 W}{\delta J_a(-\mathbf{k},\eta)\,\delta J_b(\mathbf{k}',\eta')}\right|_{J,\,K=0} &= i\delta(\mathbf{k}-\mathbf{k}')\,P_{ab}(\mathbf{k},\eta,\eta')\,,\\
\left.\frac{\delta^2 W}{\delta J_a(-\mathbf{k},\eta)\,\delta K_b(\mathbf{k}',\eta')}\right|_{J,\,K=0} &= -\delta(\mathbf{k}-\mathbf{k}')G^R_{ab}(\mathbf{k},\eta,\eta')\,,\\
\left.\frac{\delta^2 W}{\delta K_a(-\mathbf{k},\eta)\,\delta K_b(\mathbf{k}',\eta')}\right|_{J,\,K=0} &= 0\,.
\end{aligned}$$

- The effective action is the Legendre transform

$$\Gamma[\phi, \chi; P^0] = \int d\eta d^3\mathbf{k} \, \{J_a\phi_a + K_b\chi_b\} - W[J, K; P^0],$$

where $\phi_a(\mathbf{k}, \eta) = \delta W / \delta J_a(\mathbf{k}, \eta), \quad \chi_b(\mathbf{k}, \eta) = \delta W / \delta K_b(\mathbf{k}, \eta).$

- The inverse retarded propagator satisfies

$$\int d\eta' \; D^R_{ab}(\mathbf{k}, \eta, \eta') G^R_{bc}(\mathbf{k}, \eta', \eta'') = \delta_{ac} \, \delta(\eta - \eta'') \, .$$

- It can be computed from the effective action

$$\left. \frac{\delta^2 \Gamma}{\delta\phi_a(-\mathbf{k}, \eta) \, \delta\phi_b(\mathbf{k}', \eta')} \right|_{J,\, K=0} = 0 \, ,$$

$$\left. \frac{\delta^2 \Gamma}{\delta\chi_a(-\mathbf{k}, \eta) \, \delta\phi_b(\mathbf{k}', \eta')} \right|_{J,\, K=0} = -\delta(\mathbf{k} - \mathbf{k}') D^R_{ab}(\mathbf{k}, \eta, \eta') \, ,$$

$$\left. \frac{\delta^2 \Gamma}{\delta\chi_a(-\mathbf{k}, \eta) \, \delta\chi_b(\mathbf{k}', \eta')} \right|_{J,\, K=0} = -i\delta(\mathbf{k} - \mathbf{k}') H_{ab}(\mathbf{k}, \eta, \eta')$$

- “Renormalized” field equations

$$\frac{\delta}{\delta\phi_a(\mathbf{x},\eta)}\Gamma[\phi,\chi] = J_a(\mathbf{x},\eta),$$
$$\frac{\delta}{\delta\chi_a(\mathbf{x},\eta)}\Gamma[\phi,\chi] = K_a(\mathbf{x},\eta),$$

- For vanishing source fields $J = K = 0$, we have $\chi = 0$ and the first equation is trivially satisfied.

Renormalization-group improvement

- Modify the initial power spectrum so that it includes only modes with wavevectors $|\mathbf{q}|$ larger than the coarse-graining scale k:

$$P_k^0(\mathbf{q}) = P^0(\mathbf{q})\,\Theta(|\mathbf{q}| - k).$$

- The coarse-grained effective action satisfies an exact RG equation

$$\partial_k \Gamma_k[\phi,\chi] = \frac{1}{2}\,\mathrm{Tr}\left\{\left(\Gamma_k^{(2)}[\phi,\chi] - i\left(P_k^0 - P^0\right)\right)^{-1}\partial_k P_k^0\right\}.$$

- Use an ansatz of the form

$$\Gamma_k[\phi,\chi] = \int d\eta \Big[\int d^3q\, \chi_a(-\mathbf{q},\eta) \left(\delta_{ab}\partial_\eta + \hat{\Omega}_{ab}(\mathbf{q},\eta)\right) \phi_b(\mathbf{q},\eta)$$
$$- \int d^3k\, d^3p\, d^3q\, \delta^{(3)}(\mathbf{k}-\mathbf{p}-\mathbf{q}) \gamma_{abc}(\mathbf{k},\mathbf{p},\mathbf{q}) \chi_a(-\mathbf{k},\eta)\phi_b(\mathbf{p},\eta)$$
$$- \frac{i}{2} \int d^3q\, \chi_a(\mathbf{q},\eta) H_{ab,k}(\mathbf{q},\eta,\eta')\chi_b(\mathbf{q},\eta') + \ldots \Big],$$

where

$$\hat{\Omega}(\mathbf{q},\eta) = \begin{pmatrix} 0 & -1 \\ -\frac{3}{2}\Omega_m + \lambda_s(k)\, e^{\kappa(k)\eta}\, \mathbf{q}^2 & 1 + \frac{\mathcal{H}'}{\mathcal{H}} + \lambda_\nu(k)\, e^{\kappa(k)\eta}\, \mathbf{q}^2 \end{pmatrix}.$$

- Derive differential equations for the k-dependence of $\lambda_\nu(k)$, $\lambda_s(k)$, $\kappa(k)$.

Comments

- A prescription is needed in order to project the general form of the inverse retarded propagator

$$D^R_{ab}(\mathbf{q},\eta,\eta') = \delta_{ab}\delta'(\eta-\eta') + \Omega_{ab}(\mathbf{q},\eta)\delta(\eta-\eta') + \Sigma^R_{ab}(\mathbf{q},\eta,\eta')$$

to the form

$$D^R_{ab}(\mathbf{q},\eta,\Delta\eta) = \left(\delta_{ab}\partial_\eta - \hat{\Omega}_{ab}(\mathbf{q},\eta)\right)\,\delta(\eta-\eta')\,.$$

- The projection is performed through a Laplace transform.
- At the first order of an iterative solution of the exact RG equation, one finds

$$\lambda_s(k) = \frac{31}{70}\sigma^2_{dk}, \qquad \lambda_\nu(k) = \frac{78}{35}\sigma^2_{dk}, \qquad \kappa(k) = 2.$$

This validates our intuitive matching through the propagator.

- At the next level, the k-dependence of $\lambda_\nu(k)$, $\lambda_s(k)$, $\kappa(k)$ can be derived.

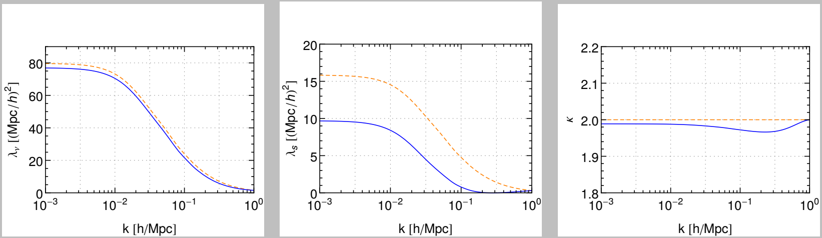


Figure: RG evolution of $\lambda_\nu(k)$, $\lambda_s(k)$ and $\kappa(k)$. We have initialized the flow at $k = \Lambda = 1\ h/$ Mpc with the one-loop values. The solid lines correspond to the solution of the full flow equations, while the dashed lines correspond to the solution of the one-loop approximation.

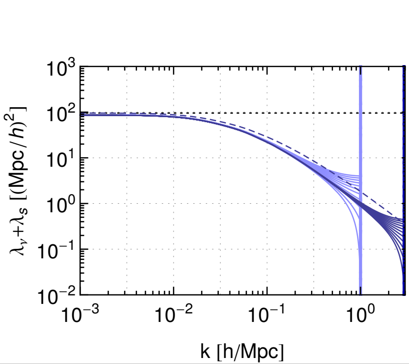


Figure: RG evolution of the sum $\lambda_\nu(k) + \lambda_s(k)$. The various lines correspond to the RG evolution obtained when imposing initial values at $\Lambda = 1\ h/\text{Mpc}$ (light blue) or $\Lambda = 3\ h/\text{Mpc}$ (dark blue), respectively. The dashed line shows the perturbative one-loop estimate for comparison.

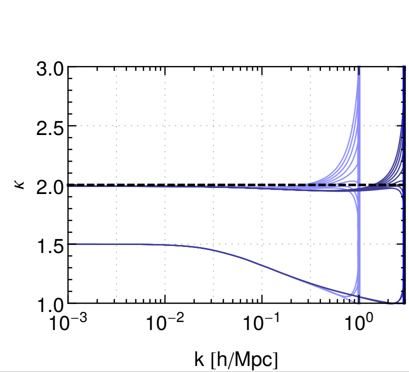


Figure: RG evolution of the power law index $\kappa(k)$ characterizing the time-dependence of the effective sound velocity and viscosity. The various blue lines show the RG evolution when initializing the RG flow at $\Lambda = 1\ h/\mathrm{Mpc}$ (light blue) or $\Lambda = 3\ h/\mathrm{Mpc}$ (dark blue), respectively.

Conclusions

- The nature of dark matter is still unknown. It is reasonable to consider possibilities beyond an ideal, pressureless fluid. The first-order formalism includes bulk and shear viscosities.
- In the presence of significant shear viscosity, the fluctuations of the cosmological fluid can backreact on the average energy density. The backreaction can accelerate the cosmological expansion. The growth of large-scale structure is also affected.
- Standard perturbation theory cannot describe reliably the short-distance cosmological perturbations.
- It is possible to "integrate out" the short-distance modes in order to obtain an effective description of the long-distance modes. One must allow for nonzero speed of sound and viscosity, whose form and time-dependence are determined uniquely.
- The nonlinear spectrum computed through the effective theory is in good agreement with results from N-body simulations.
- Perturbation theory seems to converge quickly for the effective theory if the UV cutoff is taken in the region $0.4 - 1\ h/$ Mpc.