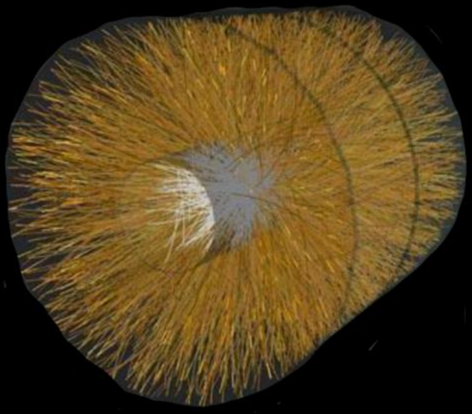
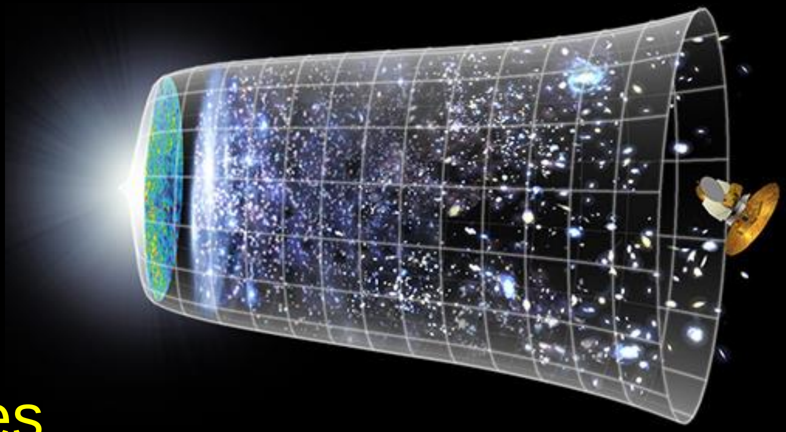
Far from equilibrium particle production and thermalization:

From cosmology to heavy-ion collisions


ALICE/CERN


WMAP Science Team

J. Berges

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ISOQUANT SFB1225

The Big Bang and the little bangs - Non-equilibrium phenomena in cosmology and in heavy-ion collisions, CERN TH, August 2016

Content

- Particle production and thermalization process from extreme conditions:

- Inflaton (p)reheating as a well understood quantum example
  $\rightarrow$ Bose condensation and fermion production far from equilibrium

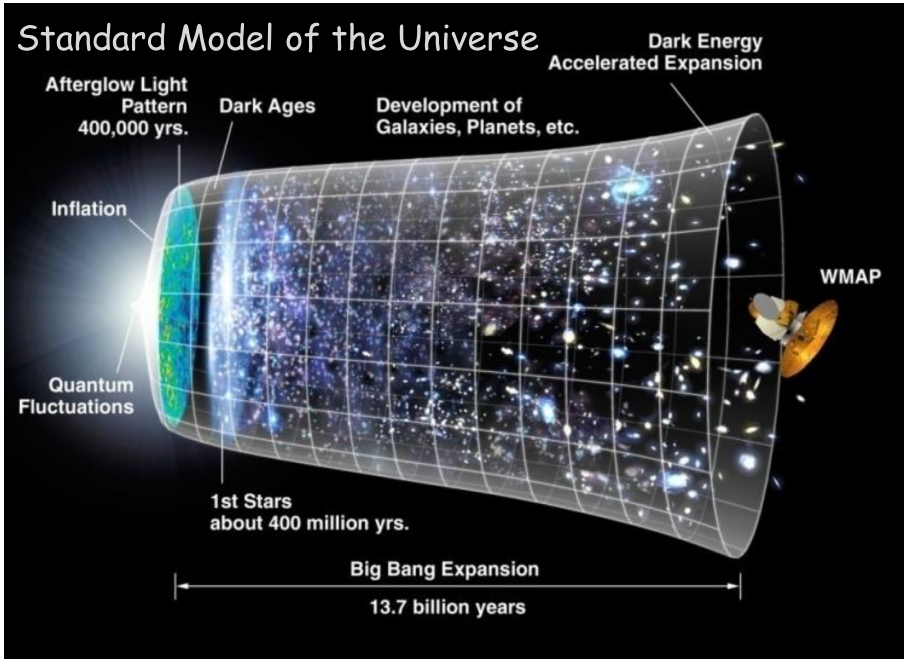
- Heavy-ion collisions in the high-energy limit
  $\rightarrow$ Gluon and quark production using real-time lattice simulations

- Universality far from equilibrium and the role of the nonperturbative infrared regime

- Anomalous fermion production in gauge theories

Space-time evolution of the Universe versus ultra-relativistic nuclear collisions in the laboratory

Schematic space-time evolution histories:



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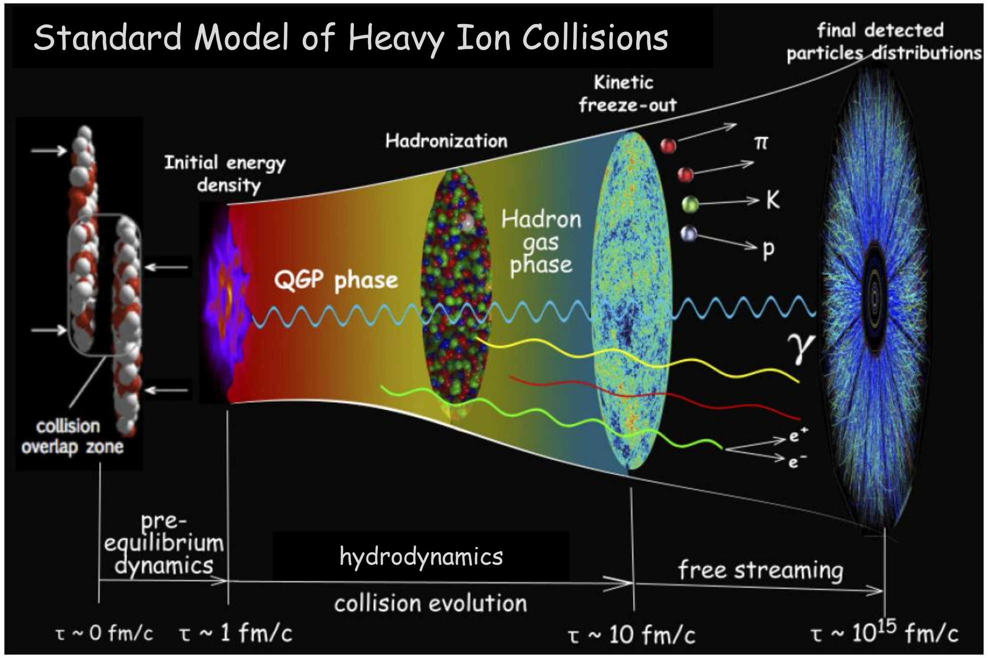


P. Sorensen & C. Shen

- initial-state quantum fluctuations → particle production and thermalization → hierarchy of decoupling processes → experimentally observed final state

Particle production and thermalization process from extreme conditions

Consider general class of models (including lattice gauge theories) with bosons coupled to Dirac fermions:

$$\mathcal{L} = \frac{1}{2}\partial\Phi^*\partial\Phi - V(\Phi) + \sum_k^{N_f}\left[i\bar{\Psi}_k\gamma^\mu\partial_\mu\Psi_k - \bar{\Psi}_k(\overset{m\,\to\,g\Phi(x)}{\overset{\downarrow}{M}}\underset{\frac{1}{2}(1-\gamma^5)}{\underset{\nearrow}{P_L}} + M^*\underset{\frac{1}{2}(1+\gamma^5)}{\underset{\nwarrow}{P_R}})\Psi_k\right]$$

e.g. $V(\Phi) = \frac{1}{2}m^2\Phi^2 + \frac{\lambda}{4!}\Phi^4$ for inflaton field

- starting from vacuum initial conditions for N_f fermion flavors
- with extreme initial conditions for bosonic fields:

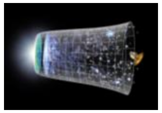
interaction strength *field² expectation value*

$$\frac{\lambda \cdot \langle\Phi^2\rangle}{Q^2} \sim 1$$

characteristic energy/momentum²

→ Extreme conditions can enhance the loss of details about microscopic properties (coupling strengths, initial conditions, …)

Examples of extreme initial conditions



Preheating after inflation

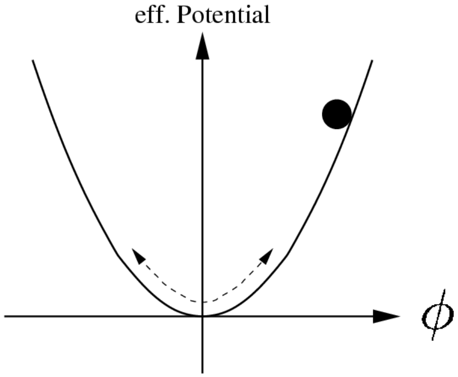
Parametric resonance preheating:
Kofman, Linde, Starobinsky…

Pure state density matrix ρ_0

Large inflaton **field amplitude** ($\lambda \lesssim 10^{-12}$):

$$\langle\Phi\rangle(t) = \mathrm{Tr}(\rho_0\Phi) \sim 0.3\, M_P \gg m/\sqrt{\lambda}$$

with vacuum fluctuations



Heavy-ion collisions in the high-energy limit

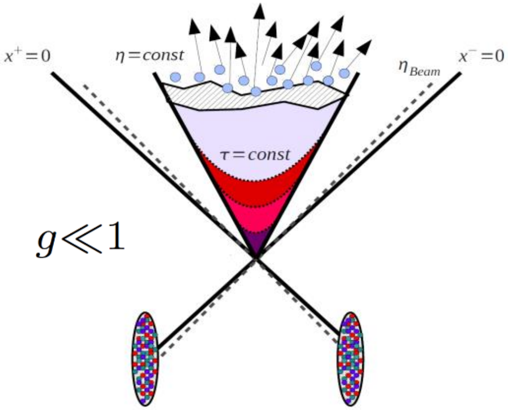
CGC: Lappi, McLerran, Dusling,
Gelis, Venugopalan, Epelbaum…

Large initial gauge **fields**:

$$\langle A \rangle \sim Q_s/g$$

with fluctuations:

$$\langle \delta A^2 \rangle \sim Q_s^2$$



Preheating and turbulent thermalization: A well understood *quantum* example

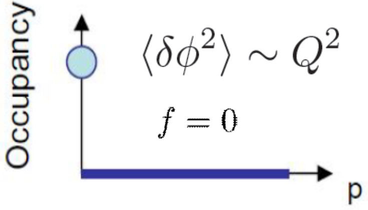
Schematic evolution at early times:

see also talk by Kyohei Mukaida

Large initial field → growth of particle number → over-occupation

$\langle\phi\rangle \sim Q/\sqrt{\lambda}$

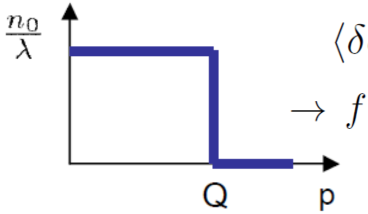
$\langle\delta\phi^2\rangle \sim Q^2$



$\boxed{f(Q) \sim e^{\gamma_Q \Delta t}}$

$\Delta t \sim Q^{-1}\log\lambda^{-1}$

instability



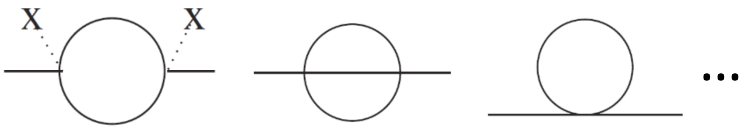
$\langle\delta\phi^2\rangle \sim Q^2/\lambda$

$\to f(Q) \sim 1/\lambda \gg 1$

Quantum field theory:

N-component field with $\lambda/(4!N)\ (\Phi_a\Phi_a)^2$

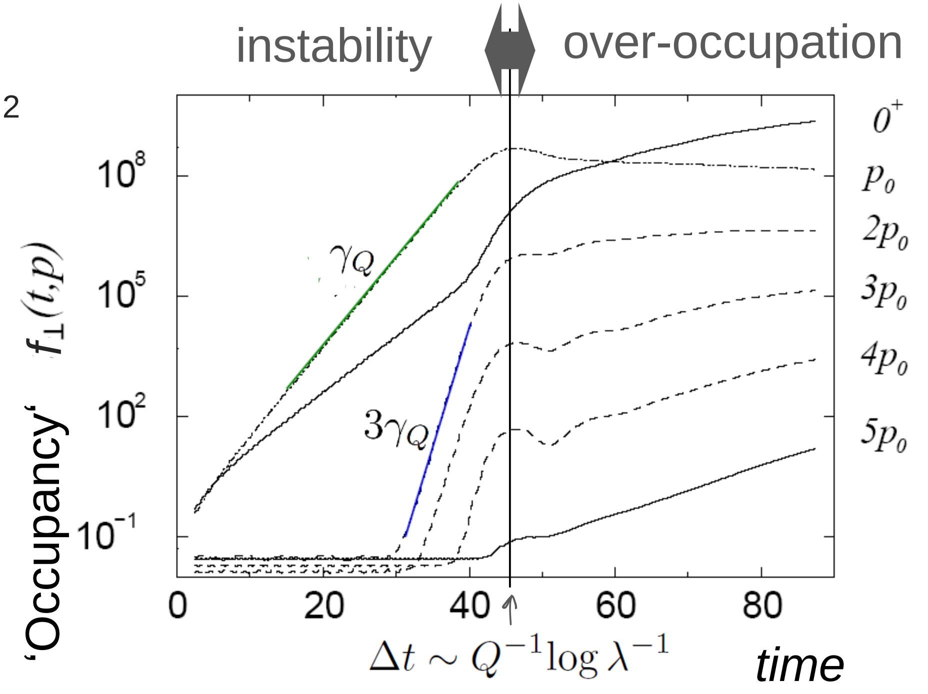
Dynamical power counting (2PI):



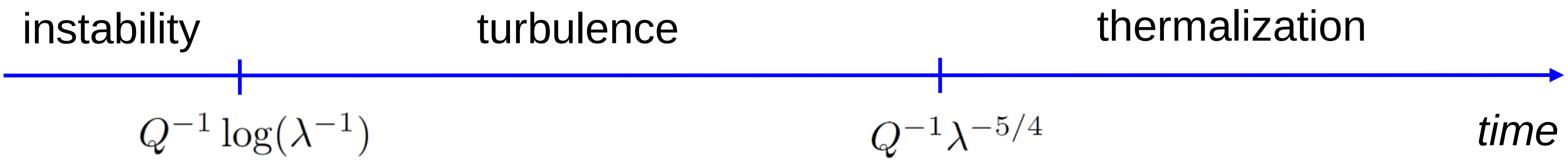
$f \sim 1/\lambda^{1/2}$ $f \sim 1/\lambda^{3/2}$ $f \sim 1/\lambda$

time

Berges, Serreau, PRL 91 (2003) 111601



Beyond instability regime: Non-perturbative bosonic sector

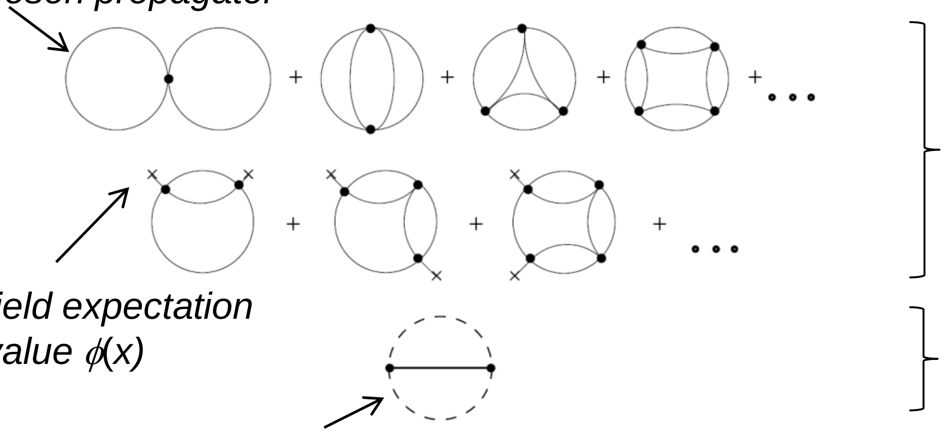


1) over-occupation at the end of instability regime preempts perturbative power counting

2) development of non-perturbative infrared/Bose condensation during turbulent regime

→ 1/*N* expansion to NLO (2PI) with quantum effective action:

Berges NPA 699 (2002) 847; Aarts et al. PRD 66 (2002) 045008



boson propagator

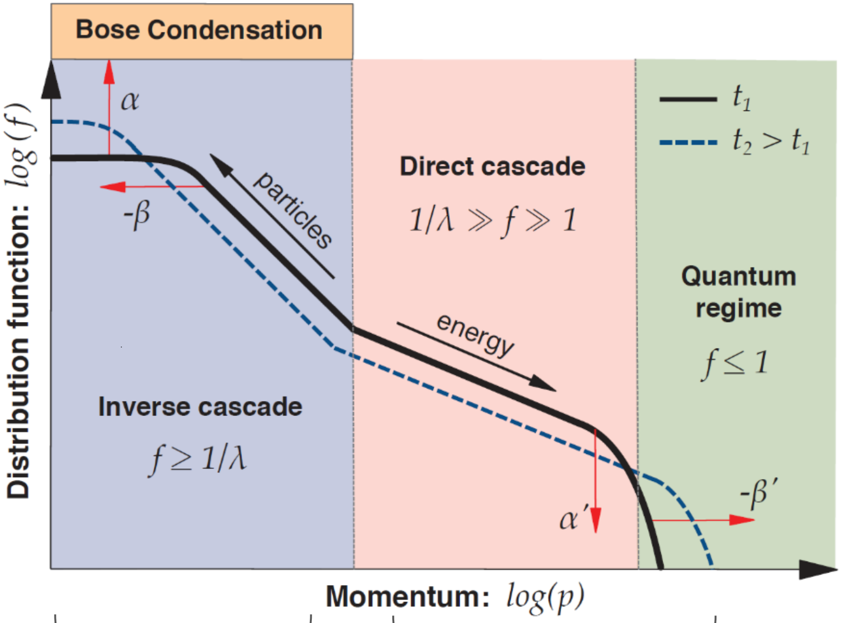
field expectation value $\phi(x)$

fermion propagator

infinite (geometric) series at NLO in the number of inflaton field components

$O(g^2)$

Turbulent regime: Self-similar scaling behavior



$$\alpha = \frac{3}{2}\,, \quad \beta = \frac{1}{2}$$

Piñeiro Orioli, Boguslavski, Berges, PRD 92 (2015) 025041

$$\alpha' = -\frac{4}{5}\,, \quad \beta' = -\frac{1}{5}$$

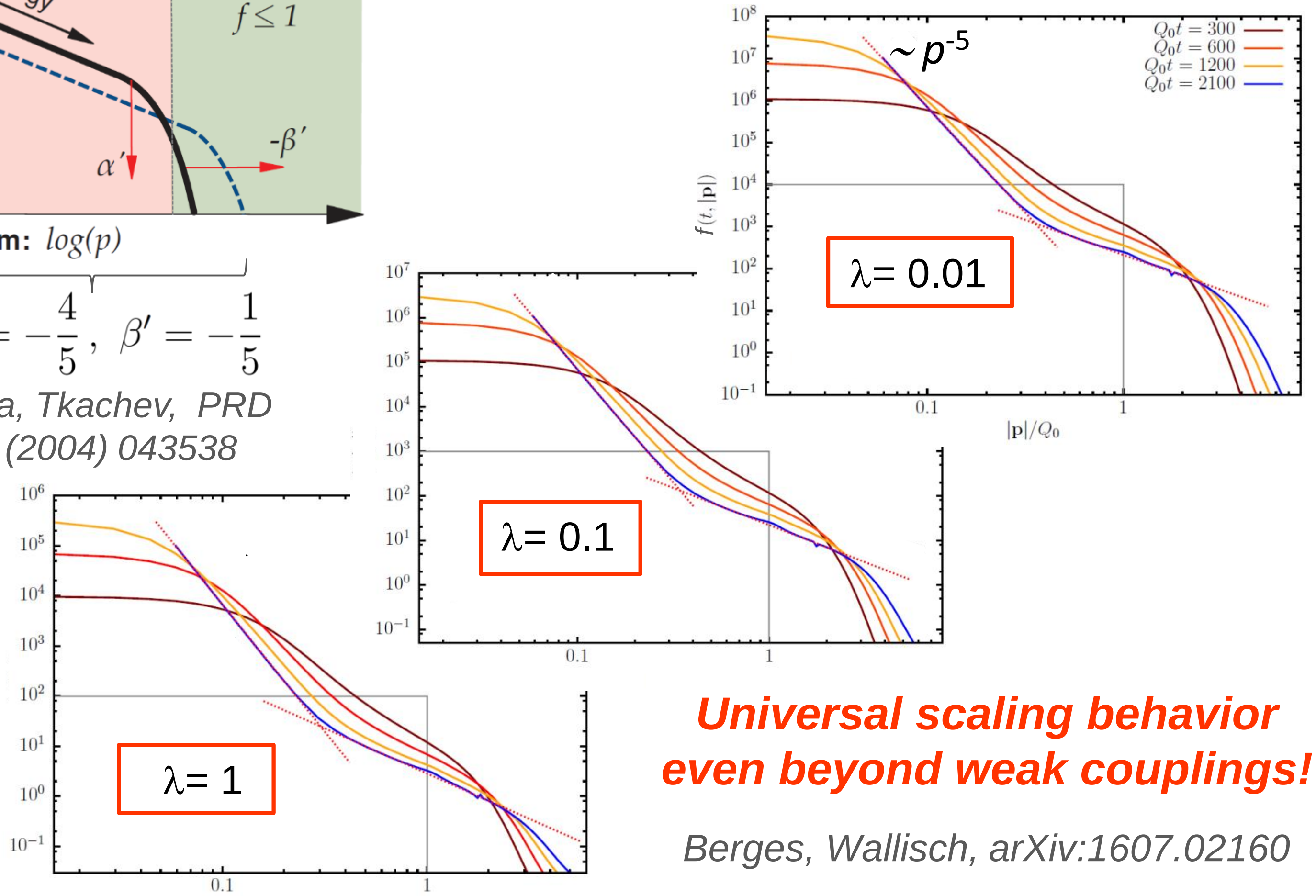
Micha, Tkachev, PRD 70 (2004) 043538

universal scaling exponents

$$f(t,\mathbf{p}) = t^{\alpha}\, f_S(t^{\beta}\mathbf{p})$$

nonthermal fixed point distribution

Scalar QFT at NLO 1/N with N = 4:



Universal scaling behavior even beyond weak couplings!

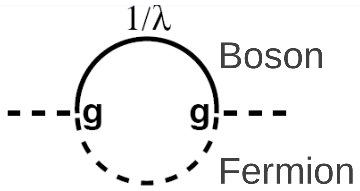
Berges, Wallisch, arXiv:1607.02160

Fermion production amplification from highly occupied bosons

Semi-classical: $iD^{-1}_{0,ij}(x,y) = \left[i\gamma^\mu \partial_\mu - m_\psi - \frac{g}{N_f}\phi(t) \right] \delta^{(4)}(x-y)\,\delta_{ij}$

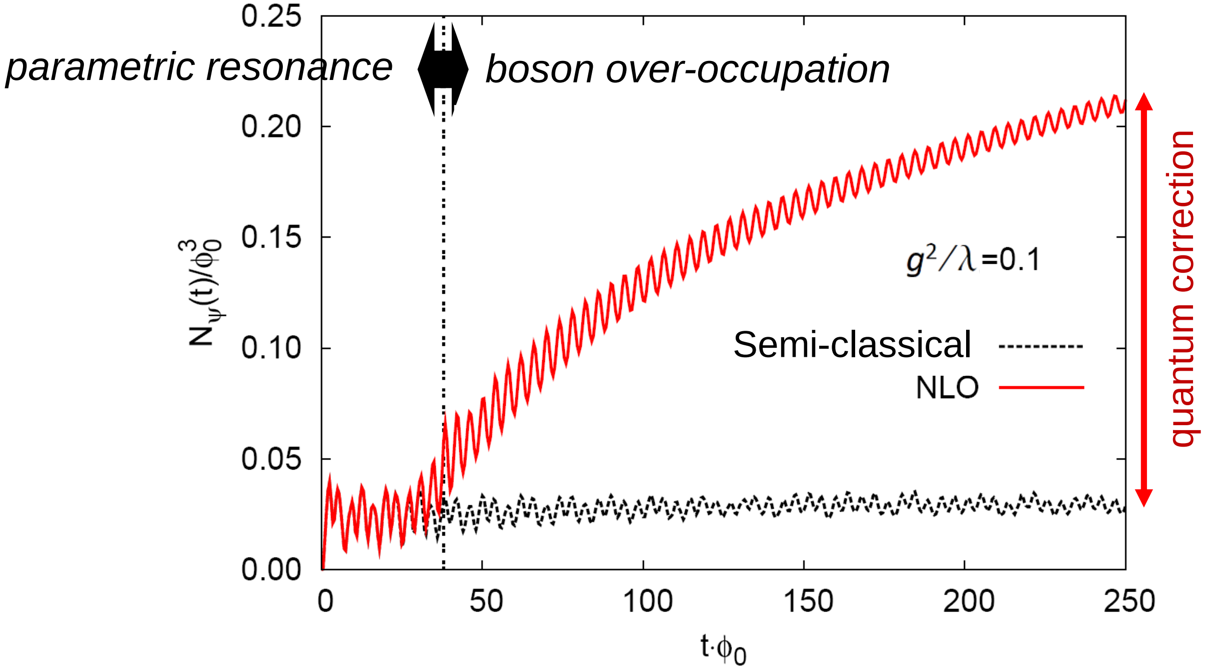
Baacke, Heitmann, Pätzold, PRD 58 (1998) 125013; Greene, Kofman, PLB 448 (1999) 6; Giudice, Peloso, Riotto, Tkachev, JHEP 9908 (1999) 014; Garcia-Bellido, Mollerach, Roulet, JHEP 0002 (2000) 034; …

NLO 1/*N* +



$\sim \frac{g^2}{\lambda}$

Berges, Gelfand, Pruschke, PRL 107 (2011) 061301



- strongly enhanced fermion production due to high boson cccupancies!

- backreaction on bosons controlled by small ~g^2

$\phi = \phi_0 \sqrt{6N_s/\lambda}$, $m_\psi = 0$

Impact of bosons on nonequilibrium fermion spectral function

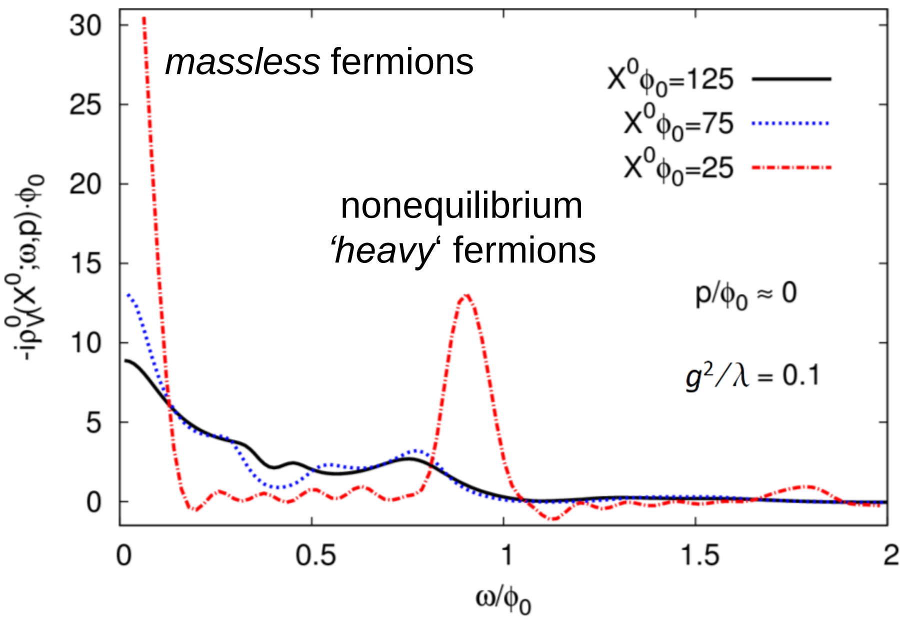
$$\rho(x,y) = i\,\langle\{\psi(x),\bar{\psi}(y)\}\rangle$$

$$\rho_V^\mu = \frac{1}{4}\mathrm{tr}\,(\gamma^\mu\rho)$$ vector components

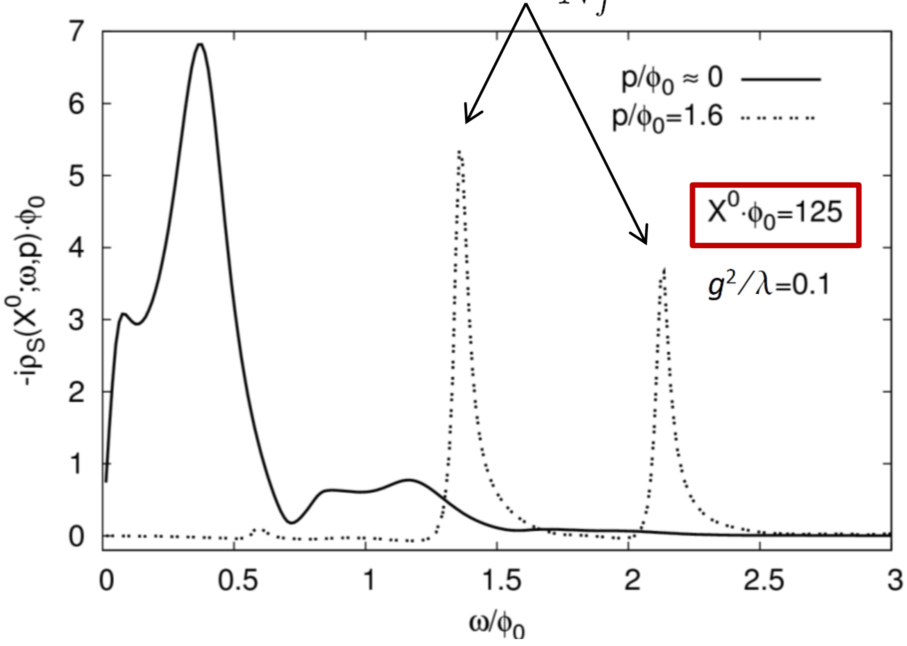
$$\rho_S = \frac{1}{4}\mathrm{tr}\,(\rho)$$ scalar component

quantum field anti-commutation relation: $-i\rho_V^0(t,t;\mathbf{p}) = 1$

Wigner transform: ($X^0 = (t + t')/2$)



$$M_\psi^{\mathrm{eff}}(t) \simeq \pm\frac{g}{N_f}|\phi(t)|$$



Real-time lattice simulations with fermions

$$\mathcal{L} = \frac{1}{2}\partial\Phi^*\partial\Phi - V(\Phi) \ + \sum_k^{N_f} \left[i\bar{\Psi}_k\gamma^\mu\partial_\mu\Psi_k - \bar{\Psi}_k(MP_L + M^*P_R)\Psi_k \right]$$

Highly-occupied bosons at weak coupling: employ classical-statistical simulations

Fermion quantum fluctuations:

$$\int \prod_k D\Psi_k^+ D\Psi_k e^{i\int \mathcal{L}(\Phi,\Psi^+,\Psi)} \quad \rightarrow \quad \boxed{\partial_x^2\Phi(x) + V'(\Phi(x)) + N_f J(x) = 0}$$

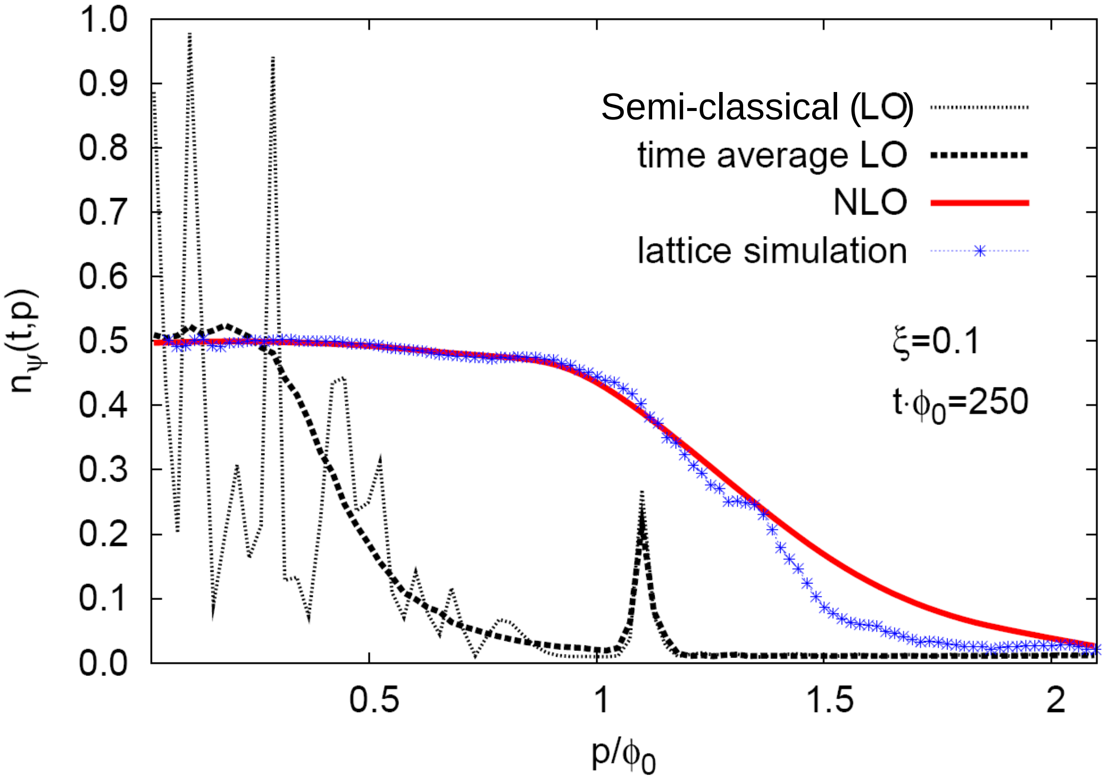
$$J(x) \ = \ J^{\mathrm{S}}(x) + J^{\mathrm{PS}}(x) \qquad \begin{aligned} J^{\mathrm{S}}(x) &= -g\left\langle \bar{\Psi}(x)\Psi(x) \right\rangle = g\mathrm{Tr}\, D(x,x)\,, \\ J^{\mathrm{PS}}(x) &= -g\left\langle \bar{\Psi}(x)\gamma^5\Psi(x) \right\rangle = g\mathrm{Tr}\, D(x,x)\gamma^5 \end{aligned}$$

For classical $\Phi(x)$ the exact equation for the fermion $D(x,y)$ reads:

$$\boxed{(i\gamma^\mu\partial_{x,\mu} - m \ + g\mathrm{Re}\,\Phi(x) - ig\mathrm{Im}\,\Phi(x)\gamma^5)D(x,y) = 0}$$

Aarts, Smit; Borsanyi, Hindmarsh; Berges, Gelfand, Pruschke; Saffin, Tranberg; Kasper, Hebenstreit; Mace, Mueller, Schlichting, Sharma, Tanji,…

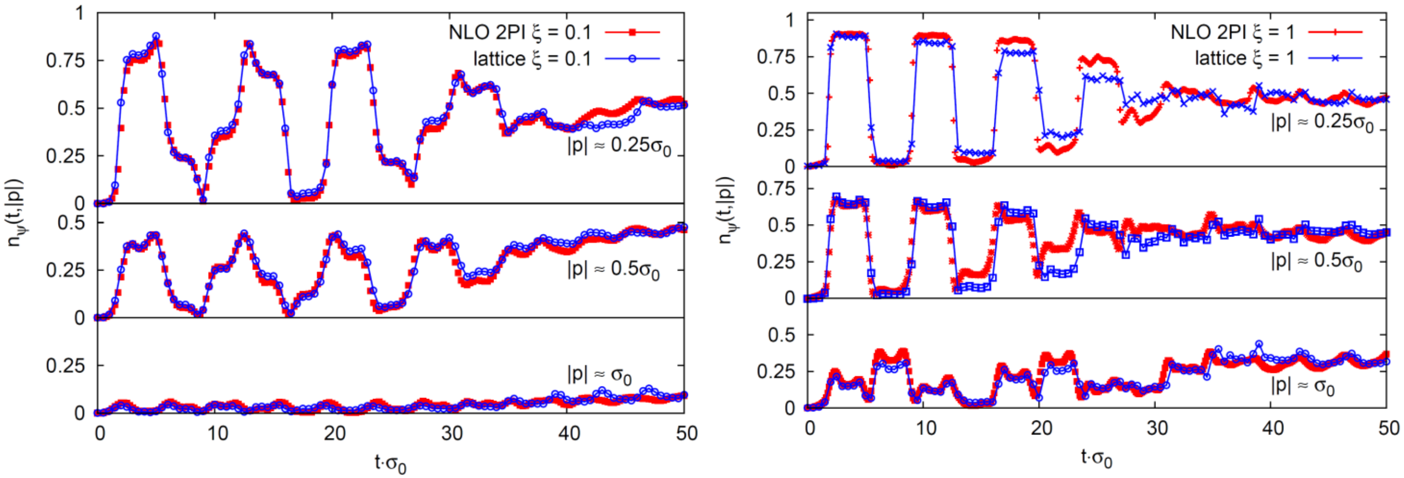
Comparing lattice simulations to NLO quantum results



- Wilson fermions on a 64^3 lattice

$$\xi = g^2/\lambda$$

Berges, Gelfand, Pruschke
PRL 107 (2011) 061301



good agreement even for ξ = 1

Berges, Gelfand, Sexty
PRD 89 (2014) 025001

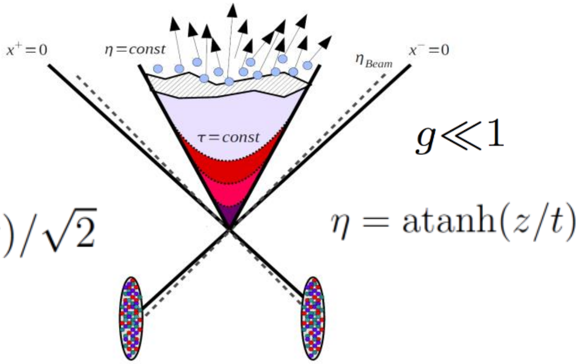
Heavy-ion collisions in the high-energy limit

Large initial gauge fields: $\langle A \rangle \sim Q_s/g$

with initial (vacuum) fluctuations: $\langle \delta A^2 \rangle \sim Q_s^2$

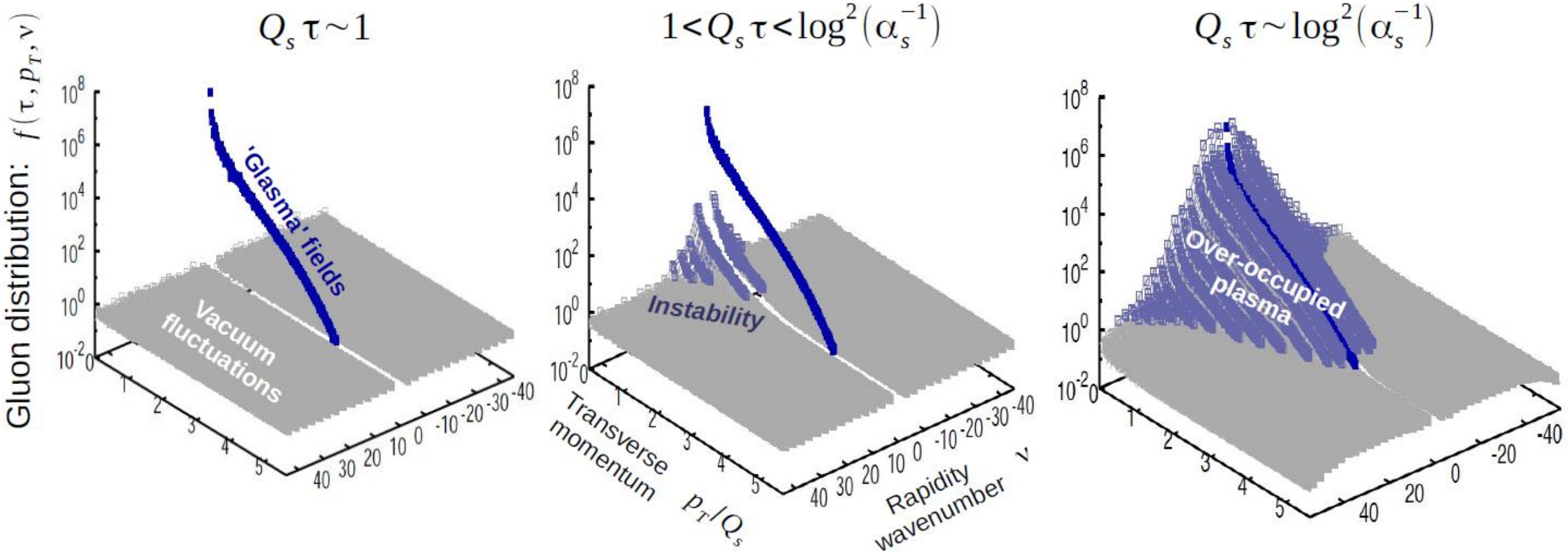
and longitudinal expansion: $\tau = \sqrt{t^2 - z^2}$ $x^{\pm} = (t \pm z)/\sqrt{2}$

→ ***plasma instabilities***



$g \ll 1$

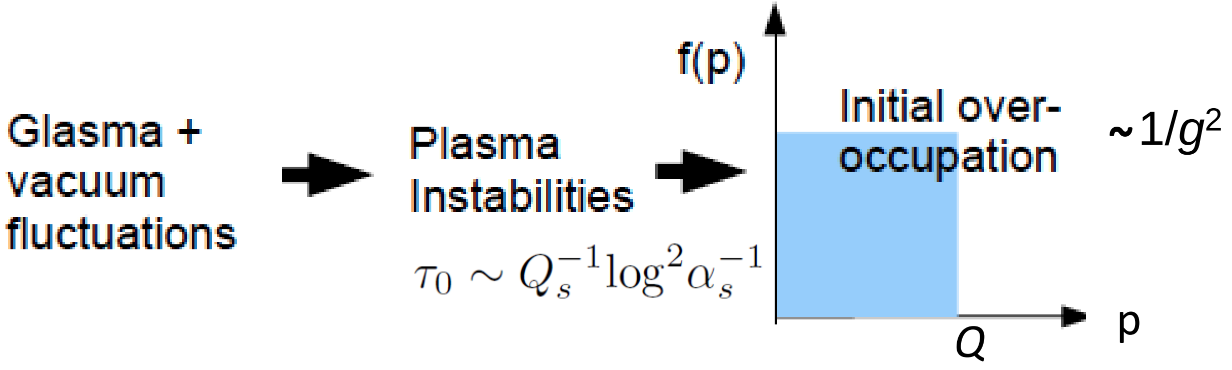
$\eta = \mathrm{atanh}(z/t)$



JB, Schenke, Schlichting, Venugopalan, NPA931 (2014) 348 for initial spectrum from *Epelbaum, Gelis, PRD88 (2013) 085015*. ***Plasma instabilities from wide range of initial conditions:***

Mrowczynski; Rebhan, Romatschke, Strickland; Arnold, Moore, Yaffe; Bödecker; Attems, … Romatschke, Venugopalan; J.B., Scheffler, Schlichting, Sexty; Fukushima, Gelis …

Overoccupied non-Abelian plasma



Glasma + vacuum fluctuations → Plasma Instabilities →

$\tau_0 \sim Q_s^{-1} \log^2 \alpha_s^{-1}$

To discuss attractor: Initial over-occupation described by family of distributions at τ_0

(read-out in ‘Coulomb gauge‘)

occupancy parameter

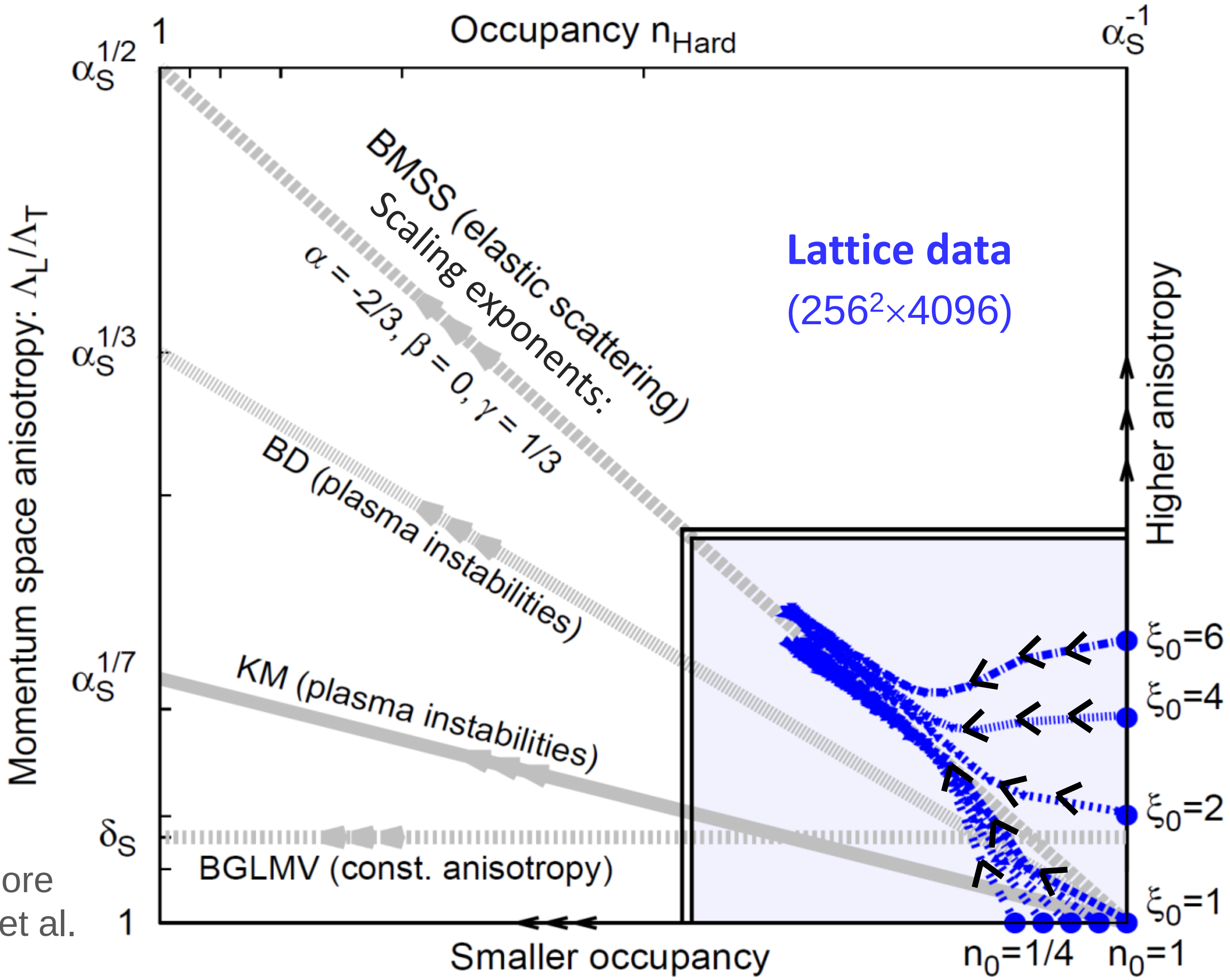
$$f(\mathrm{p_T}, \mathrm{p_z}, \tau_0) = \frac{n_0}{2g^2}\,\Theta\left(Q_s - \sqrt{\mathrm{p_T^2} + (\xi_0 \mathrm{p_z})^2}\right)$$

anisotropy parameter

(controls “prolateness” or “oblateness” of initial momentum distribution)

Nonthermal fixed point

Evolution in the `anisotropy-occupancy plane´



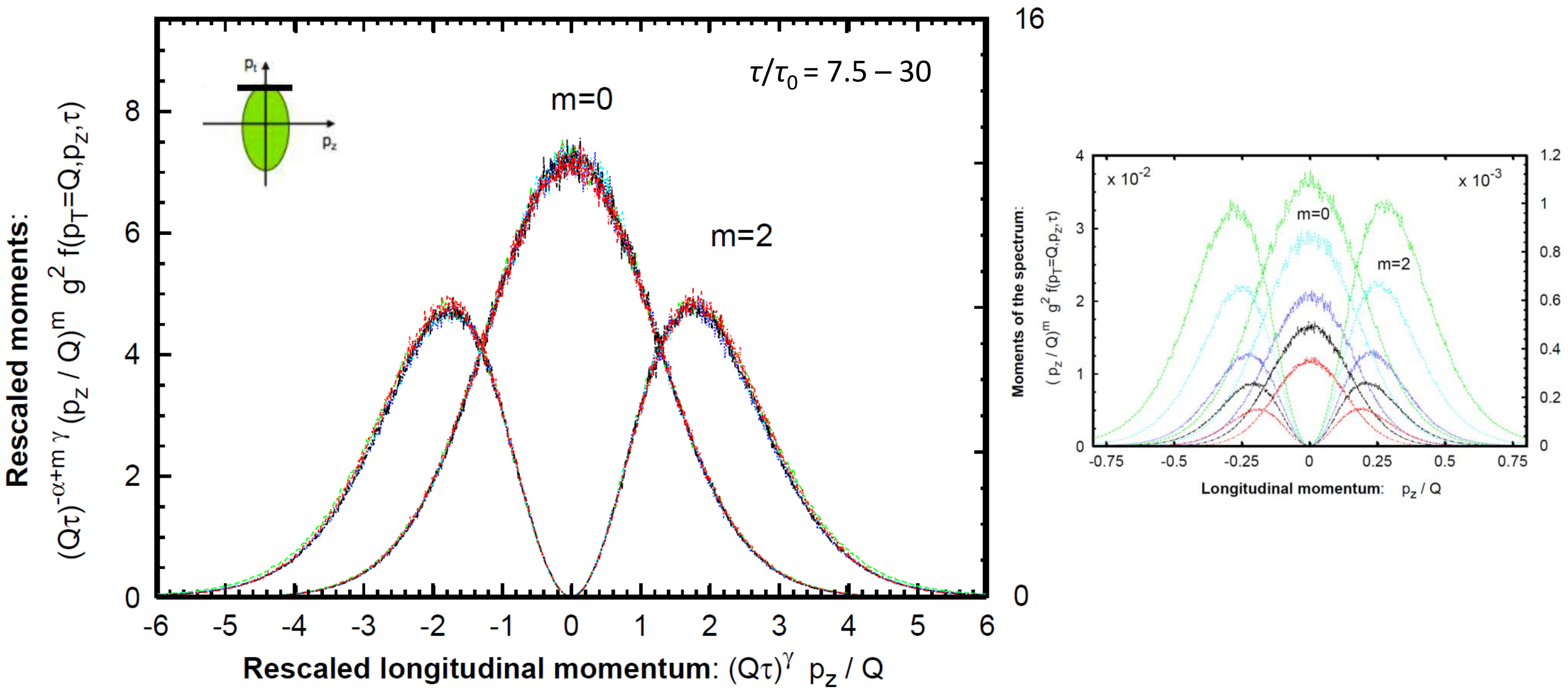
BD: Bodecker
KM: Kurkela, Moore
BGLMV: Blaizot et al.

J.B., Boguslavski, Schlichting, Venugopalan, PRD 89 (2014) 074011; ibid. 114007

Early stage of `bottom-up´ scaling emerges as a consequence of the attractor*

**Baier et al (BMSS), PLB 502 (2001) 51*

Self-similar evolution



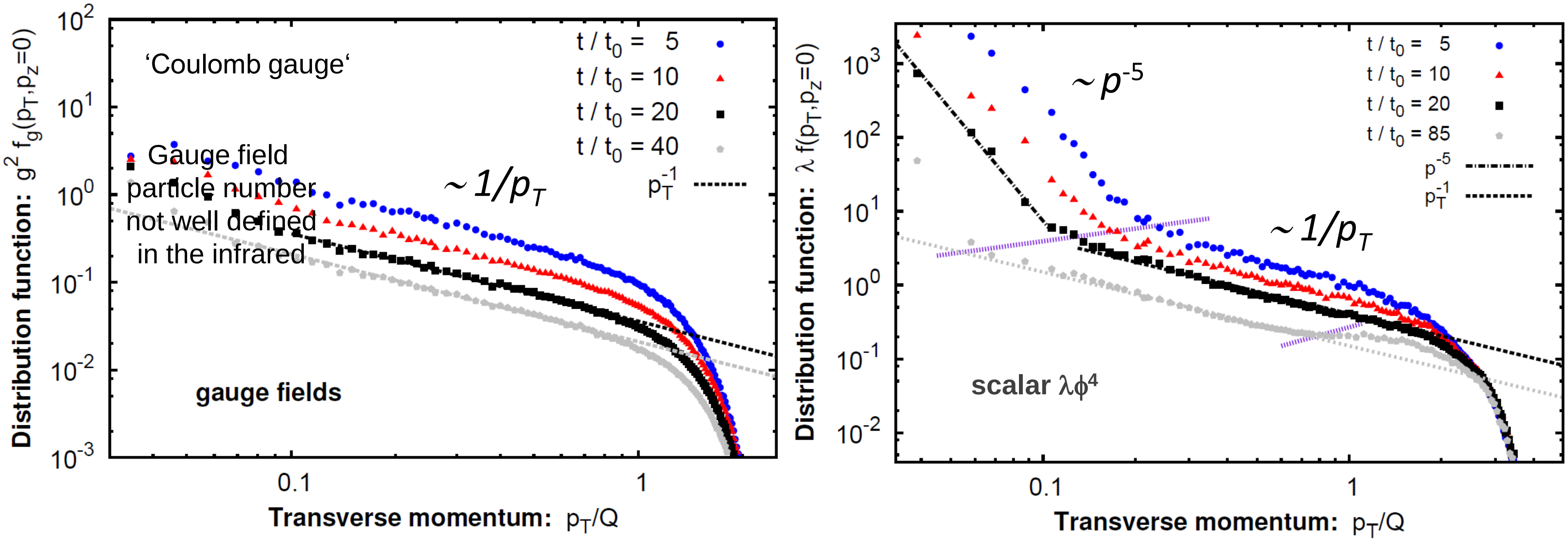
Scaling exponents: $\alpha = -2/3\ ,\qquad \beta = 0\ ,\qquad \gamma = 1/3$

and scaling distribution function f_S:

$$\boxed{f(\mathrm{p_T}, \mathrm{p_z}, \tau) = (Q\tau)^{\alpha} \underbrace{f_S\Big((Q\tau)^{\beta}\mathrm{p_T}, (Q\tau)^{\gamma}\mathrm{p_z}\Big)}_{\textit{nonthermal fixed-point distribution}}}$$

Comparing gauge and scalar field theories

with longitudinal expansion



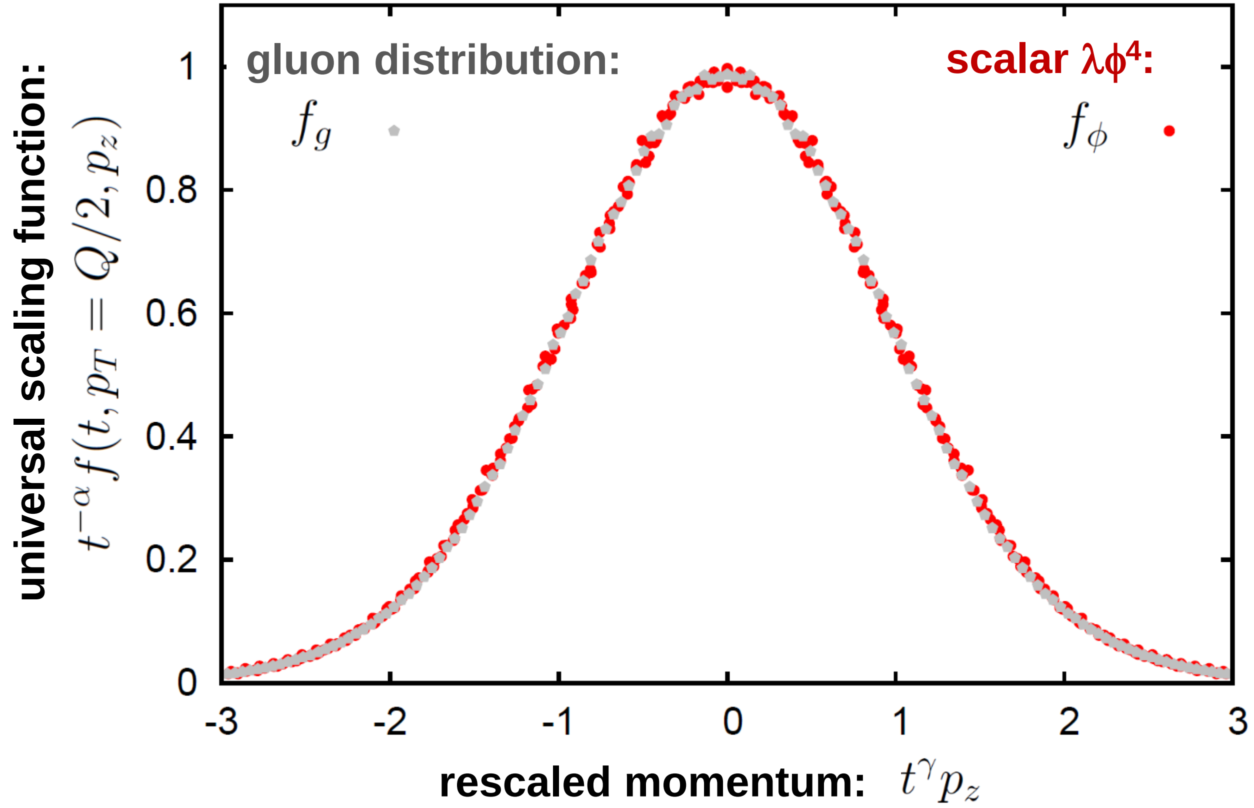
- For gauge & scalar fields: range of ***thermal-like transverse spectrum ~1/p_T*** even as longitudinal distribution is being \`squeezed´
- Strongly enhanced infrared regime for scalars: ***inverse particle cascade leading to Bose condensation***, $\sim 1/p^5$ as in ***isotropic superfluid turbulence***
- At latest available times for scalars a flat distribution for $p_T \gtrsim Q$ emerges

J.B., Boguslavski, Schlichting, Venugopalan, Phys. Rev. Lett. 114 (2015) 6, 061601

Universality far from equilibrium

- Same *universal exponents and scaling function* in common $1/p_T$ range

$$\alpha = -2/3 \quad , \quad \beta = 0 \quad , \quad \gamma = 1/3$$



→ Remarkably large universality class far from equilibrium!

Some puzzles and challenges

- The scaling solution seems well understood for the gauge theory (BMSS) – but the corresponding kinetic theory arguments fail for the scalar theory

Consider the Boltzmann equation

$$[\partial_\tau - \frac{p_Z}{\tau}\partial_{p_Z}]f(p_T, p_Z, \tau) = C[f](p_T, p_Z, \tau)$$

with a self-similar evolution

$$f(p_T, p_Z, \tau) = (Q\tau)^{\alpha} f_S((Q\tau)^{\beta} p_T, (Q\tau)^{\gamma} p_Z)$$

→ ***Non-thermal fixed point solution*** $(f \gg 1)$

$$[\alpha + \beta p_T \partial_{p_T} + (\gamma - 1) p_Z \partial_{p_Z}] f_S(p_T, p_Z) = Q^{-1} C[f_S](p_T, p_Z)$$

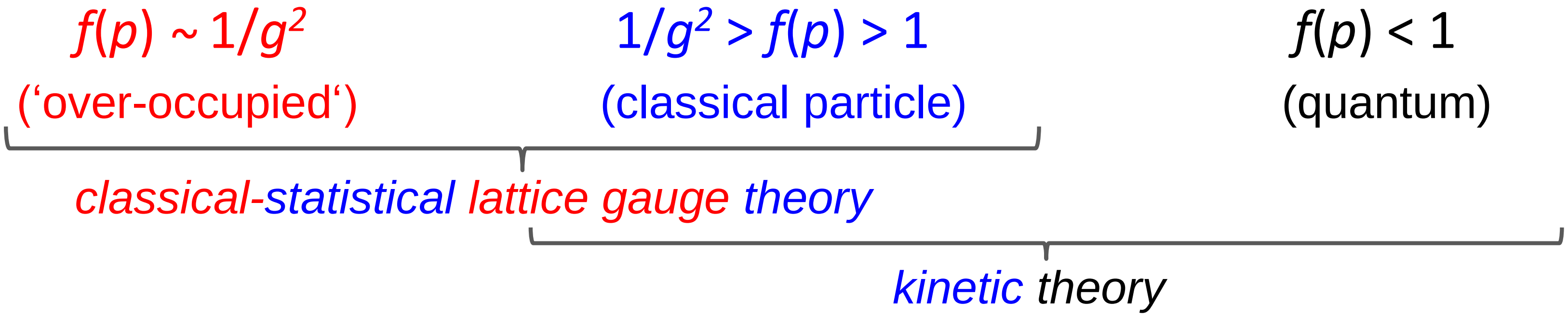
→ ***Scaling exponents determined by scaling relations for***

- Small angle elastic scattering $(2\alpha - 2\beta + \gamma = -1)$ ← for gauge theory
- Energy conservation $(\alpha - 3\beta - \gamma = -1)$
- Particle number conservation $(\alpha - 2\beta - \gamma = -1)$

→ $\alpha = -2/3, \beta = 0, \gamma = 1/3$ ***in excellent agreement with lattice data!***

However: No dominance for small angle scattering in scalar theory expected!
More general principle underlying nonthermal fixed point?

- In the weak-coupling limit, kinetic theory is expected to have an overlapping range of validity with classical-statistical simulations



$f(p) \sim 1/g^2$
('over-occupied')

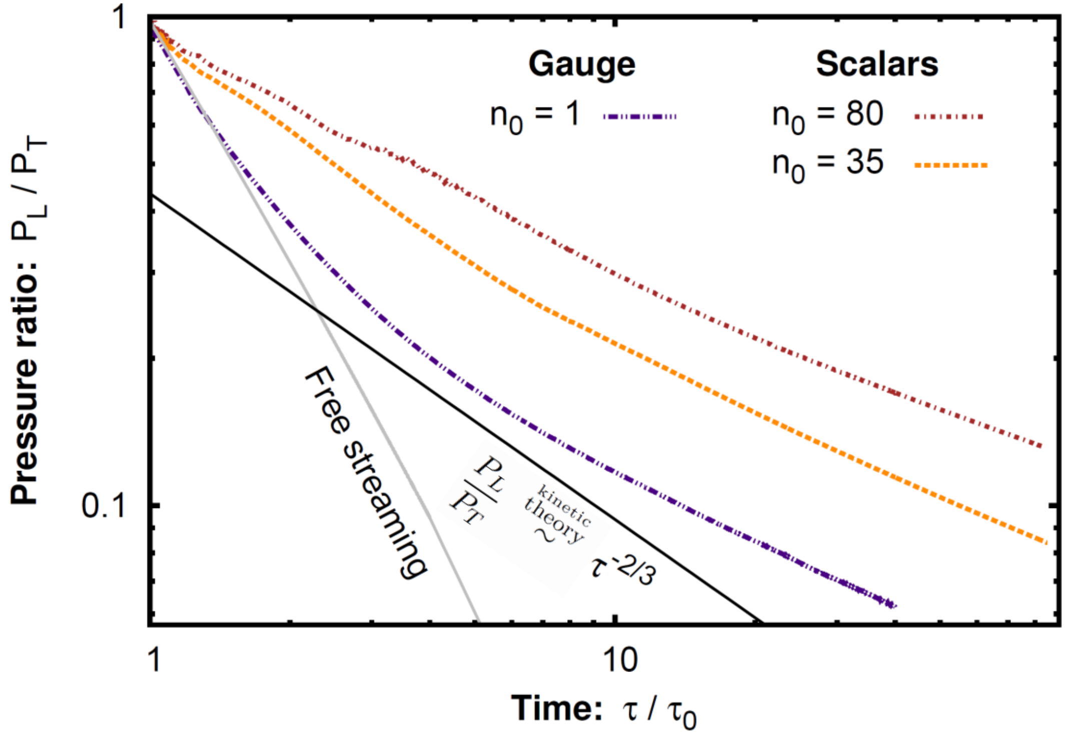
$1/g^2 > f(p) > 1$
(classical particle)

$f(p) < 1$
(quantum)

classical-statistical lattice gauge theory

kinetic theory

However, kinetic theory seems not to reproduce important quantities such as P_L / P_T characterizing isotropization of the longitudinally expanding plasma:

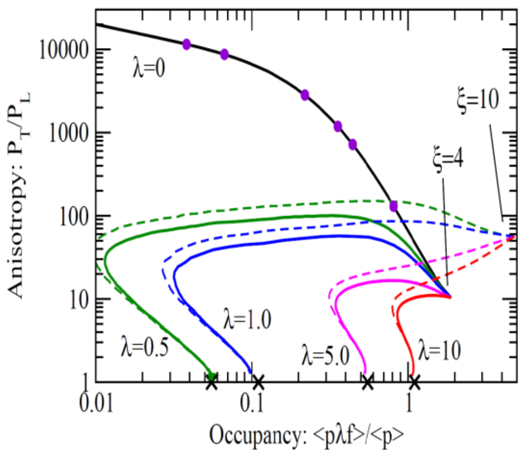


- Scaling behavior of P_L / P_T very similar for scalar and gauge field simulations!
- In scalar theory behavior of P_L / P_T known to arise from infrared contributions (Bose condensation)

→ Nonperturbative despite (weak) coupling parameter

- Perturbative estimates extrapolated beyond the weak-coupling regime suggest absence of transient universal scaling behavior

Gauge: Kurkela, Zhu, PRL 115 (2015) 182301:
Scalar: Epelbaum, Gelis, Jeon, Moore, Wu,
JHEP 09 (2015) 117



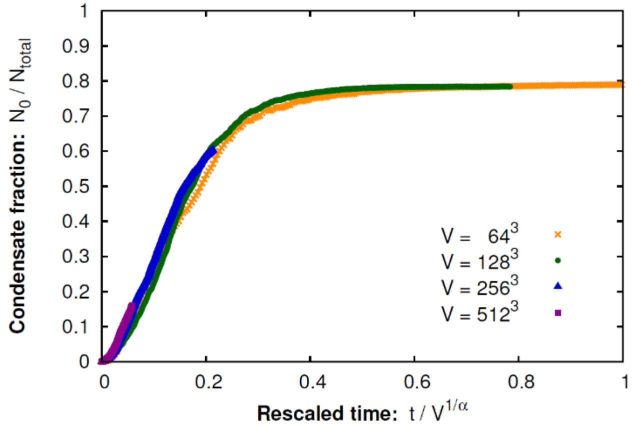
However, no such indications (yet no expansion) from nonperturbative estimates in scalar quantum field theory *JB, Wallisch, arXiv:1607.02160*

(and holographic superfluids?) *Ewerz et al., JHEP 18 (2015) 1505 (cf. also Adams, Chesler, Liu, Science 341 (2013) 368)*

- By now, detailed understanding of the infrared dynamics (Bose condensation) in scalar QFT (NLO 1/*N*)

condensation time:

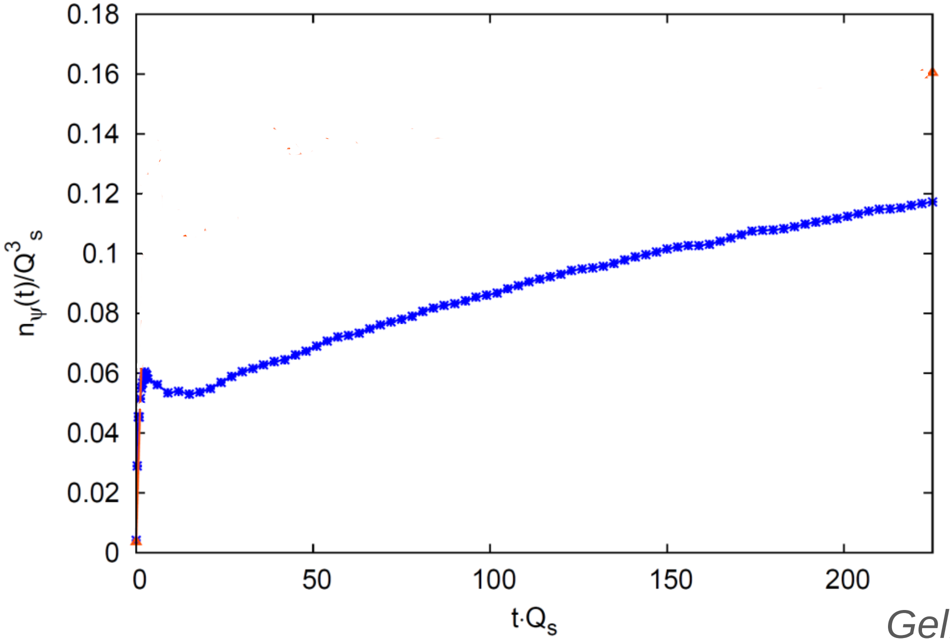
$$t_f \simeq t_0 \left(\frac{|\psi_0|^2(t_f)}{f(t_0,0)} \right)^{1/\alpha} V^{1/\alpha}$$



Piñeiro Orioli, Boguslavski, JB, PRD 92 (2015) 025041

However, no such understanding in gauge theories – despite indications for significant infrared contributions to gauge invariant quantities (P_L / P_T)

Quark production from over-occupied gauge fields

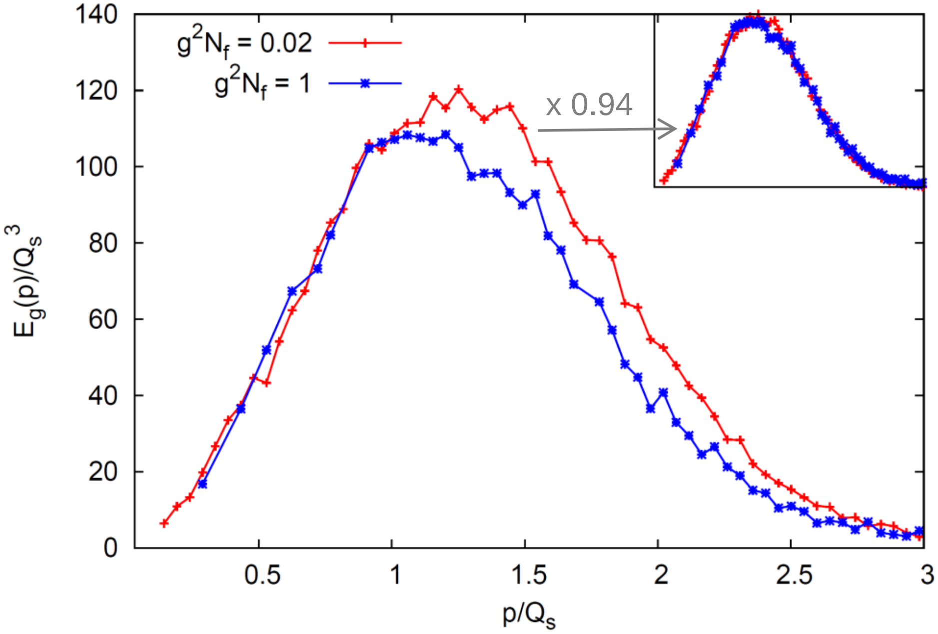


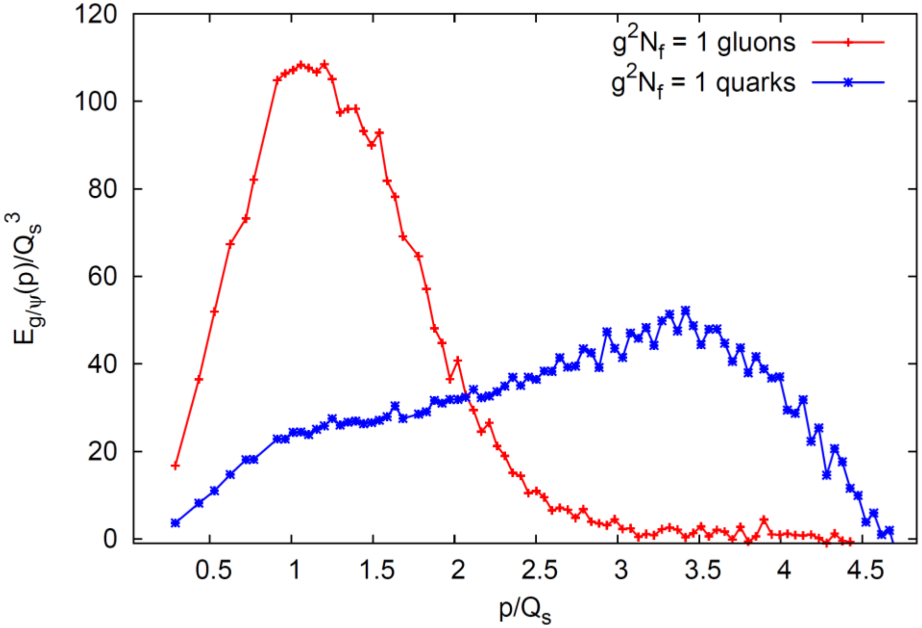
Lattice SU(2), 3+1 D, no expansion

- Significant quark production from highly occupied gauge fields
- Negligible back-reaction on the gauge fields for small enough g^2N_f

→ may consider large N_f with $g^2N_f \sim 1$

Gelfand, Hebenstreit, Berges, PRD 93 (2016) 085001





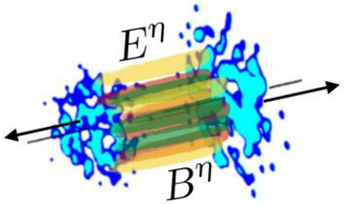
Anomalous fermion production with real-time lattice simulations

→ **see also talk by Soeren Schlichting**

Axial vector current: $j_5^\mu = \overline{\psi}\gamma^\mu\gamma_5\psi$

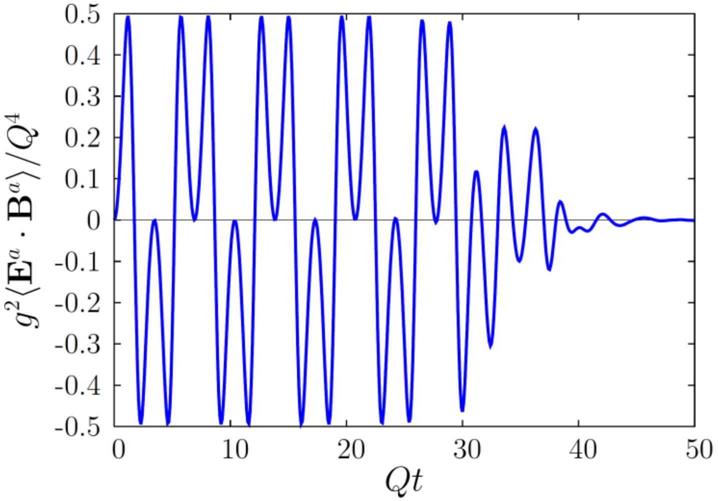
Non-conservation: $\partial_\mu j_5^\mu = 2m\overline{\psi}i\gamma_5\psi + \underbrace{\frac{g^2}{4\pi^2}\mathbf{E}^a\cdot\mathbf{B}^a}_{\text{anomalous}}$

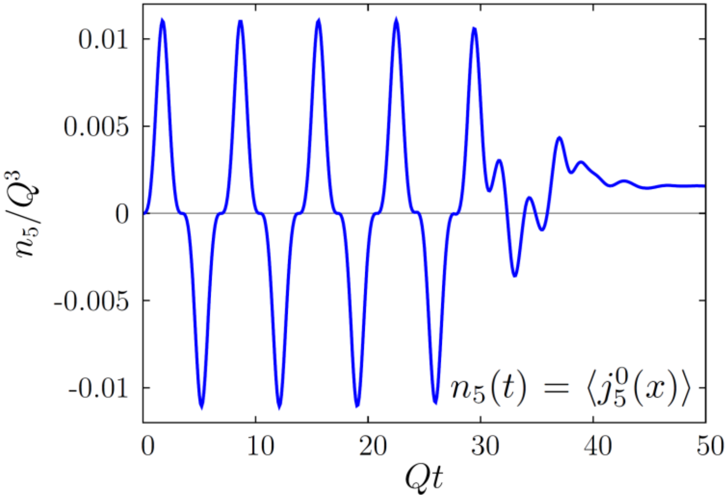
Adler; Bell, Jackiw



Precision test of real-time lattice implementation with Wilson fermions (SU(2)):

$$\langle E_x^1(0)\rangle = \langle E_y^2(0)\rangle = \langle E_z^3(0)\rangle \sim \frac{Q^2}{g}, \quad \langle B_x^1(0)\rangle = \langle B_y^2(0)\rangle = \langle B_z^3(0)\rangle = 0$$





- Very good agreement of lattice and early-time analytic results:

$$n_5(t) = \frac{Q^3}{3^{3/4}4\pi^2}\,\mathrm{cn}^3\left(\sqrt{\frac{2}{\sqrt{3}}}Qt - K(1/2), \frac{1}{2}\right)$$

- Anomaly induced axial charge can persist even beyond decoherence time

Tanji, Mueller, Berges, PRD 93 (2016) 074507

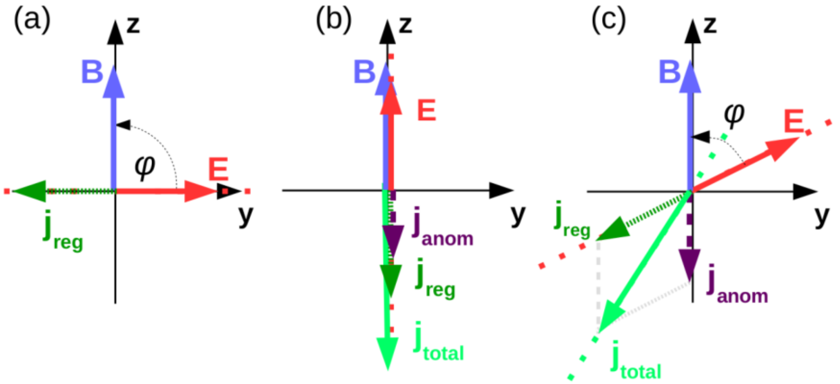
see also Saffin, Tranberg, JHEP 02 (2012) 102

Application: ‘Chiral magnetic effect’ in QED

→ **see also talk by Florian Hebenstreit**

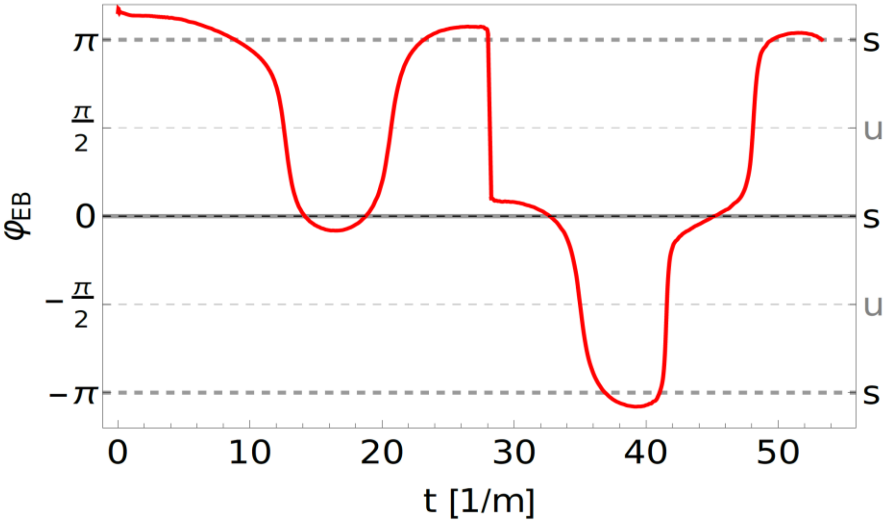
Mueller, Hebenstreit, JB, PRL 117 (2016) 061601

(cf. Kharzeev et al. Nature Physics (2016), Mueller, Schlichting, Sharma arXiv:1606.00342)

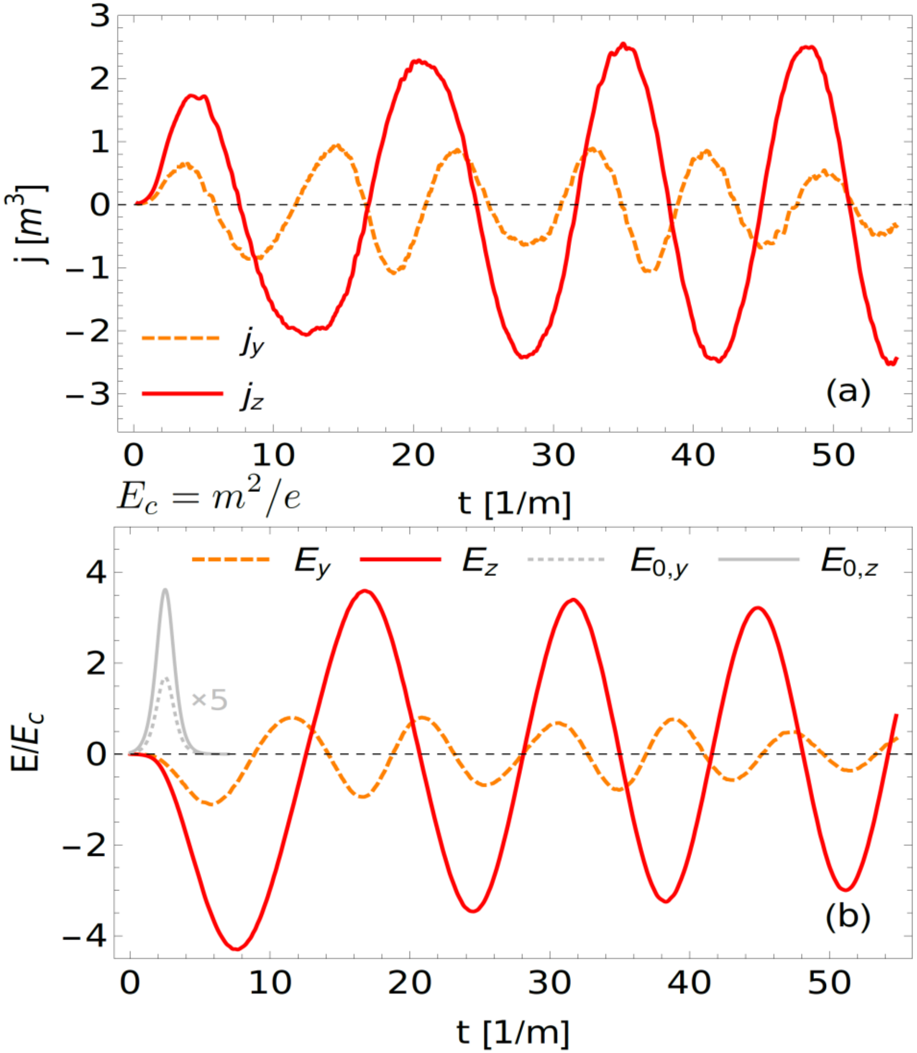


$$\vec{j}_{anom} = \sigma_5(n_5(t))\vec{B}$$

Dynamical tracking behavior of $\varphi = \measuredangle(\mathbf{E}, \mathbf{B})$
→ collinear configs with max. anom. current



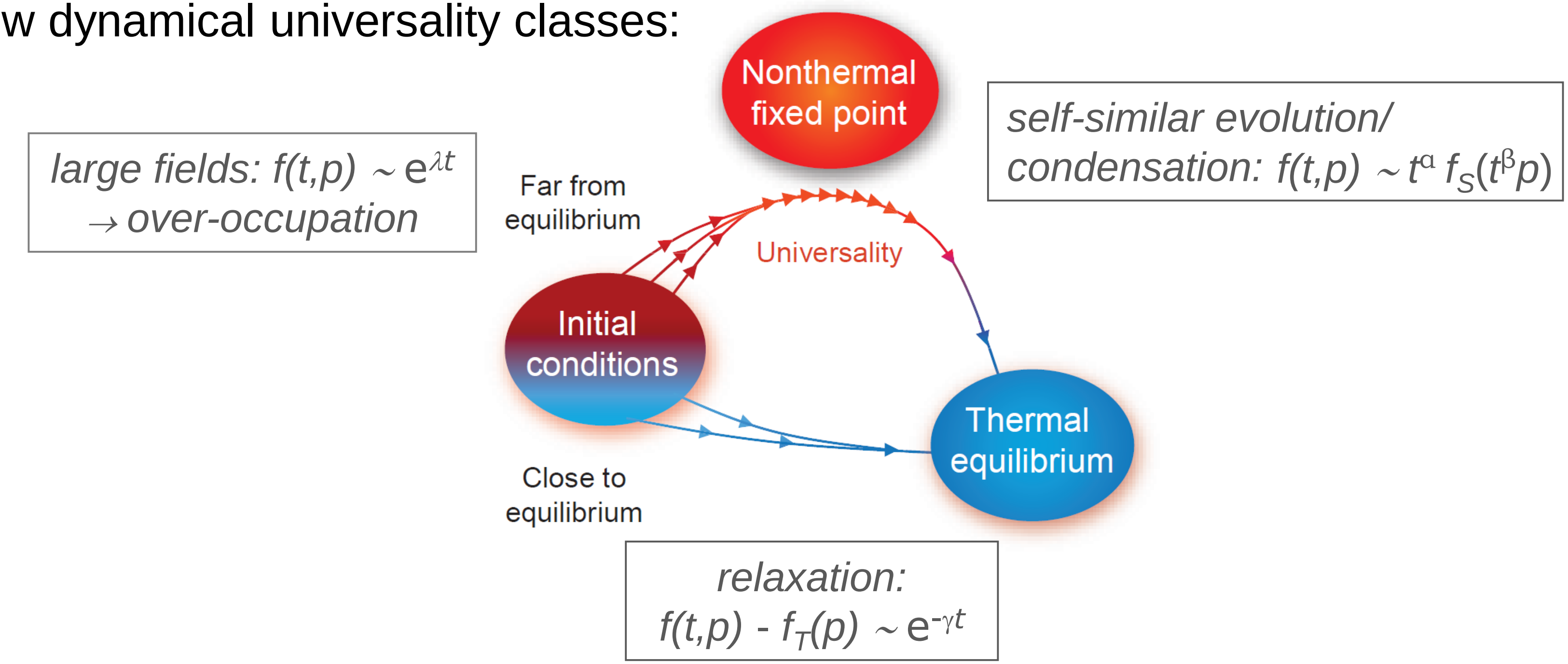
Induced currents and plasma oscillations:



Conclusions

Extreme conditions → loss of details about microscopic model & initial conditions:

- New dynamical universality classes:

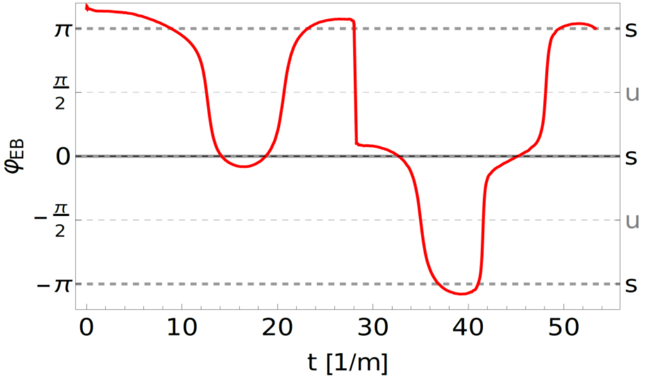


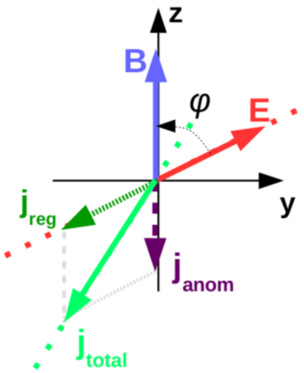
large fields: $f(t,p) \sim e^{\lambda t}$
$\rightarrow$ *over-occupation*

self-similar evolution/
condensation: $f(t,p) \sim t^{\alpha} f_S(t^{\beta}p)$

relaxation:
$f(t,p) - f_T(p) \sim e^{-\gamma t}$

- Anomalous fermion production with tracking behavior ($E > E_c = m^2/e$):





Macroscopic fields, condensates and fluctuations

- In a quantum theory the field amplitude corresponds to the expectation value of a (here relativistic, real) Heisenberg field operator $\Phi(t,\mathbf{x})$:

$$\phi(t) = \langle \Phi(t,\mathbf{x}) \rangle \equiv \mathrm{Tr}\left\{ \varrho_0 \, \Phi(t,\mathbf{x}) \right\}$$

time-dependent expectation value

density operator at some `initial´ time t = 0

- Fluctuations derive from correlation functions, e.g. spatially homogeneous:

`quantum-half´

volume: $(2\pi)^3 \delta(\mathbf{0}) \to V$

occupation number distribution

$$\frac{f(t,\mathbf{p}) + 1/2}{\omega(t,\mathbf{p})} + (2\pi)^3 \delta(\mathbf{p}) \phi_0^2(t)$$

dispersion

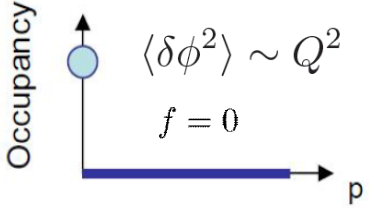
condensate^2

$$\equiv \frac{1}{2} \int \mathrm{d}^3 x \, e^{-i\mathbf{p}\mathbf{x}} \, \langle \Phi(t,\mathbf{x})\Phi(t,0) + \Phi(t,0)\Phi(t,\mathbf{x}) \rangle$$

Preheating: Insensitivity to initial condition details

Example: `*Inflaton*´ $\lambda\phi^4$ theory ($\lambda \ll 1$), $\phi = \phi_0 + \delta\phi$

1. Large initial field:



$\langle\phi\rangle \sim Q/\sqrt{\lambda}$

$\langle\delta\phi^2\rangle \sim Q^2$

$f = 0$

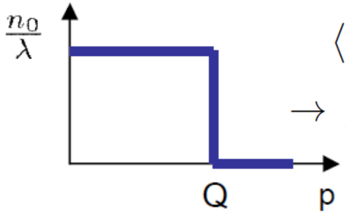
$$\boxed{f(Q) \sim e^{\gamma_Q \Delta t}}$$

$\Delta t \sim Q^{-1} \log \lambda^{-1}$

$\longrightarrow$

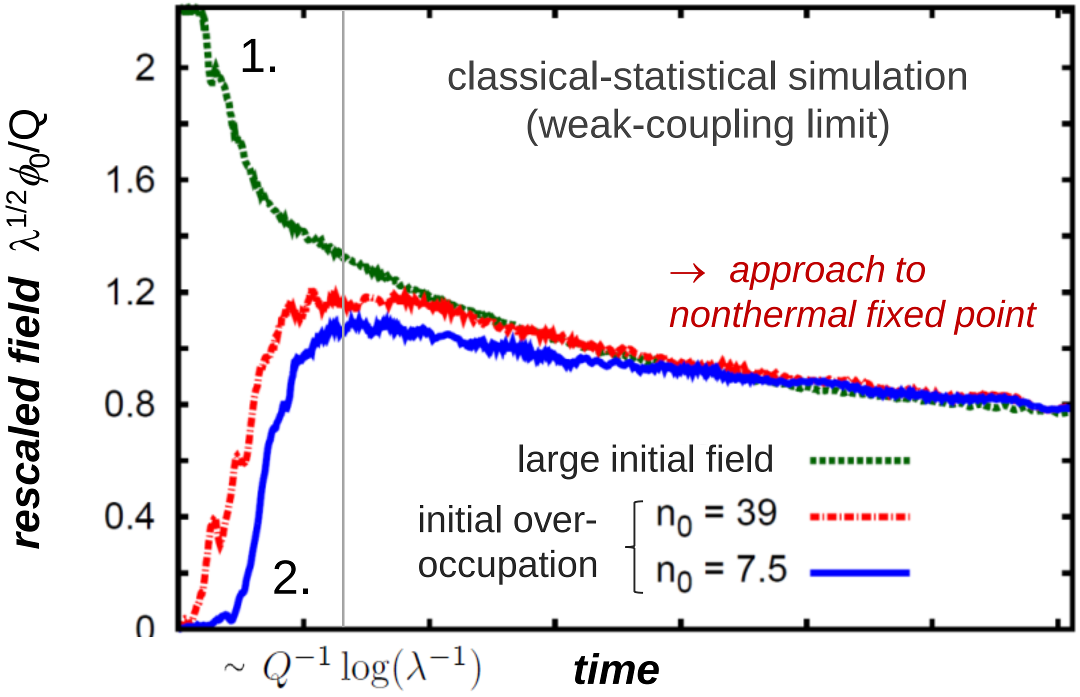
instability

2. High occupancy:



$\langle\delta\phi^2\rangle \sim Q^2/\lambda$

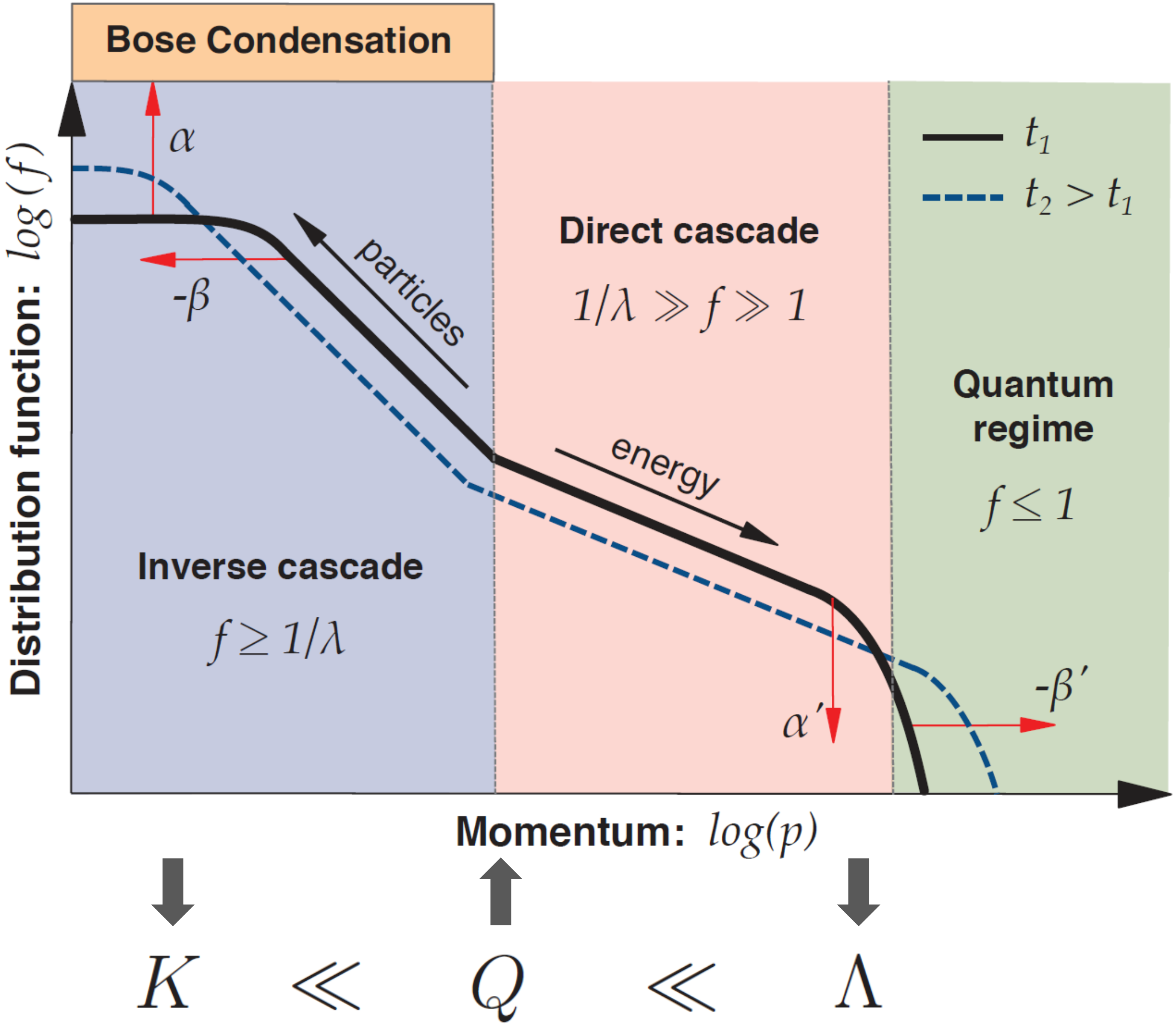
$\rightarrow f(Q) \sim 1/\lambda \gg 1$



$\rightarrow$ approach to nonthermal fixed point

Berges, Boguslavski, Schlichting, Venugopalan, JHEP 1405 (2014) 054

Schematic behavior near nonthermal fixed point: dual cascade



Particle versus energy transport

$$K \ll Q \ll \Lambda$$

number conservation:

$$\dot{n}_Q = \dot{n}_K + \dot{n}_\Lambda$$

energy conservation:

$$Q\dot{n}_Q = K\dot{n}_K + \Lambda\dot{n}_\Lambda$$

$$\dot{n}_K = \frac{\Lambda - Q}{\Lambda - K}\dot{n}_Q \simeq \dot{n}_Q$$

$$\dot{n}_\Lambda = \frac{Q - K}{\Lambda - K}\dot{n}_Q \simeq \frac{Q}{\Lambda}\dot{n}_Q$$

$$\Rightarrow \quad \Lambda\dot{n}_\Lambda \simeq Q\dot{n}_Q$$

Particles are transported towards lower scales, energy towards higher scales

Self-similarity

$$f(t,\mathbf{p}) = s^{\alpha/\beta} f(s^{-1/\beta} t, s\mathbf{p})$$

$$s^{-1/\beta} t = 1$$

$$\Rightarrow$$

$$\boxed{f(t,\mathbf{p}) = t^{\alpha} f_S(t^{\beta}\mathbf{p})}$$

Time-independent scaling function: $f_S(t^{\beta}\mathbf{p}) \equiv f(1, t^{\beta}\mathbf{p})$

Scaling exponents α and β determine rate and direction of transport:

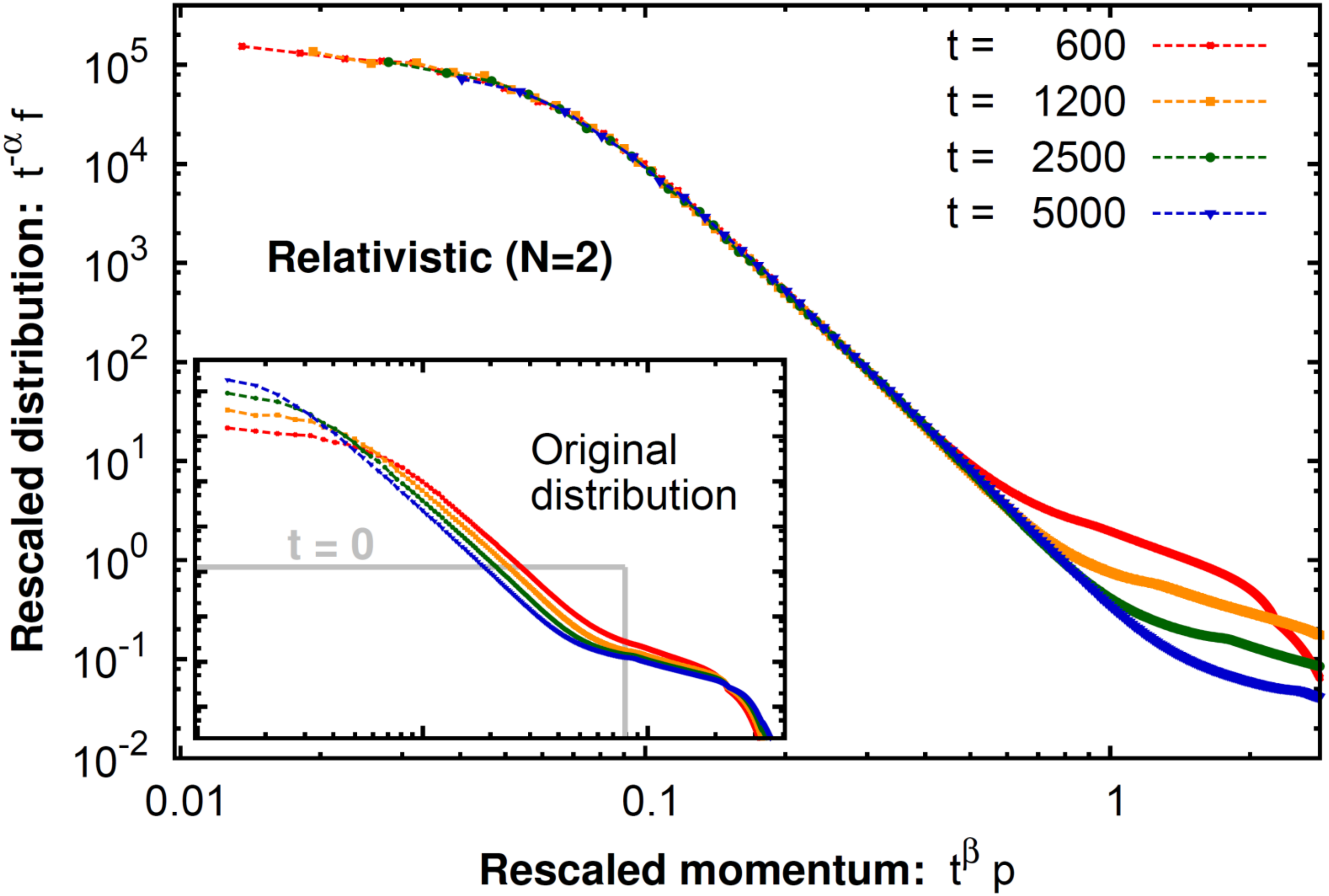
$$K(t_1) = K_1 \quad \Rightarrow \quad K(t) = K_1 (t/t_1)^{-\beta}$$

$$f(t, K(t)) \sim t^{\alpha}$$

e.g. $\alpha > 0$, $\beta > 0$: particle transport towards lower momentum scales

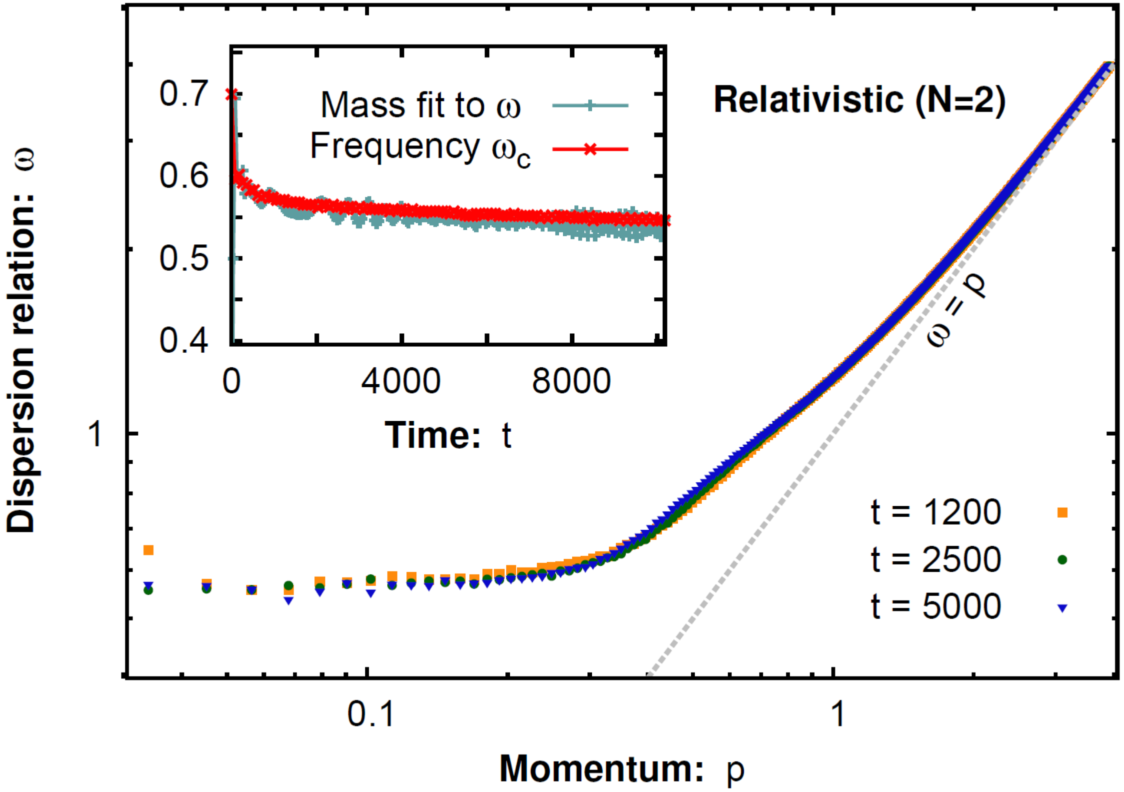
Self-similar dynamics: infrared scaling

$$f(t,\mathbf{p}) = t^{\alpha}\, f_S(t^{\beta}\mathbf{p}) \;, \quad \alpha = 1.51 \pm 0.13\,, \qquad \beta = \;0.51 \pm 0.04$$



Piñeiro Orioli, Boguslavski, Berges, PRD 92 (2015) 025041

Mass scale separating non-relativistic infrared regime

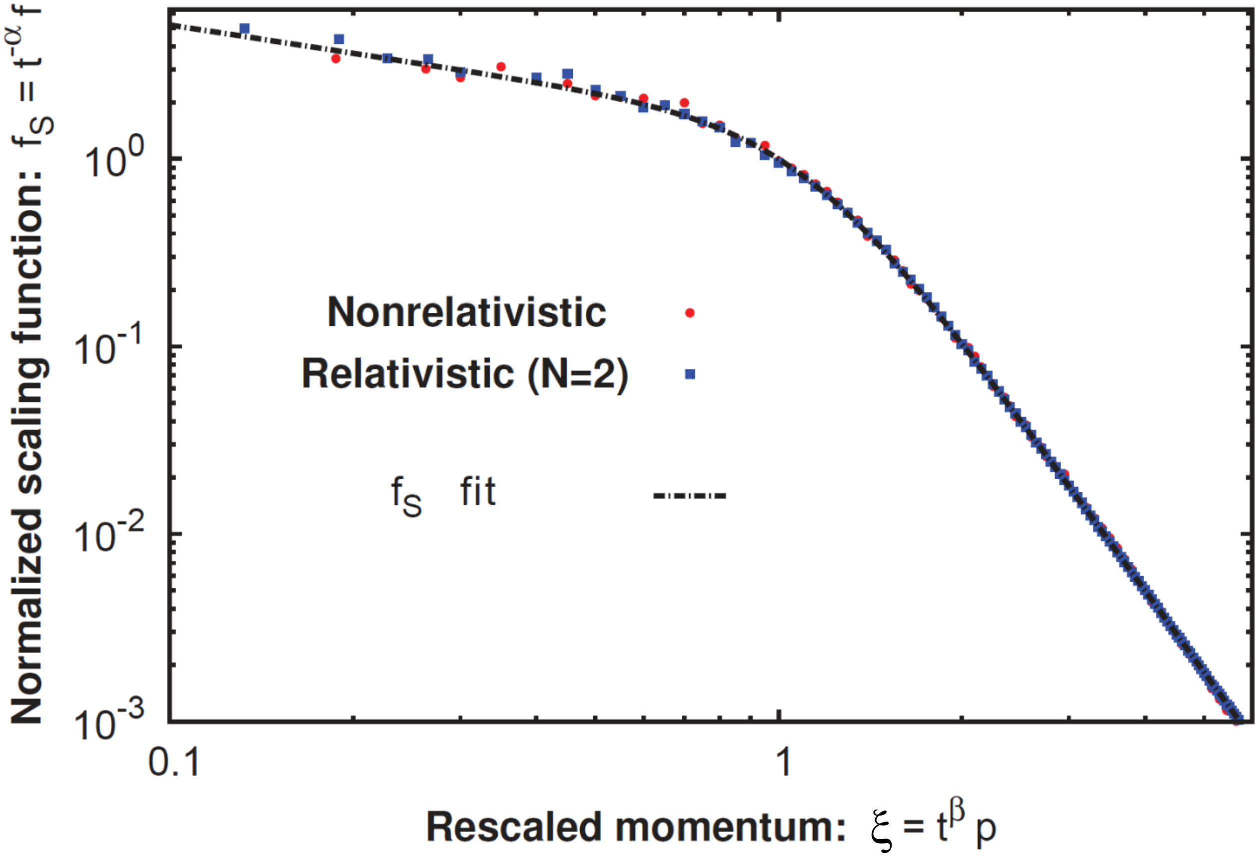


- ***non-relativistic infrared dynamics*** expected because of the generation of a ***mass gap*** (condensate + medium)

→ relativistic & non-relativstic field theories have same infrared scaling

Piñeiro Orioli, Boguslavski, Berges, PRD 92 (2015) 025041

Universal scaling form of the distribution function



Piñeiro Orioli, Boguslavski, Berges, PRD 92 (2015) 025041

$$f_S(\xi) \simeq \frac{A(\kappa_> - \kappa_<)}{(\kappa_> - 2)(\xi/B)^{\kappa_<} + (2 - \kappa_<)(\xi/B)^{\kappa_>}} \quad , \; \kappa_< \simeq 0.5 \;\; , \;\; \kappa_> \simeq 4.5$$

$$f_S(\xi = B) = A \quad , \quad df_S(\xi = B)/d\xi = -2A/B$$

Estimating scaling properties

$$\frac{\partial f(t,\mathbf{p})}{\partial t} = C\,[f]\,(t,\mathbf{p})$$ *`collision integral´*

$$C[f](t,\mathbf{p}) \underset{\substack{\uparrow \\ f(t,\mathbf{p}) = s^{\alpha/\beta} f(s^{-1/\beta}t,\, s\mathbf{p})}}{=} s^{-\mu}\, C[f](s^{-1/\beta}t,\, s\mathbf{p}) \underset{\substack{\uparrow \\ s^{-1/\beta}t \,=\, 1}}{=} t^{-\beta\mu}\, C[f_S](1, t^{\beta}\mathbf{p})$$

Use $$\frac{\partial}{\partial t}\left[t^{\alpha}\, f_S(t^{\beta}\mathbf{p})\right] = t^{\alpha-1}\left[\alpha + \beta\,\mathbf{q}\cdot\nabla_{\mathbf{q}}\right] f_S(\mathbf{q})\big|_{\mathbf{q}=t^{\beta}\mathbf{p}}$$

Time-independent *fixed point equation:*

$$\boxed{\left[\alpha + \beta\,\mathbf{p}\cdot\nabla_{\mathbf{p}}\right] f_S(\mathbf{p}) = C[f_S](1,\mathbf{p})}$$

+ scaling relation: $$\boxed{\alpha - 1 = -\beta\mu}$$

Micha, Tkachev, PRD 70 (2004) 043538

Conservation laws

$$n = \int \frac{d^d p}{(2\pi)^d}\, f(t,\mathbf{p}) = t^{\alpha-\beta d} \int \frac{d^d q}{(2\pi)^d}\, f_S(\mathbf{q})$$

$$\Rightarrow \quad \boxed{\textit{particle conservation:} \quad \alpha = \beta d}$$

$$\epsilon = \int \frac{d^d p}{(2\pi)^d}\, \omega(\mathbf{p}) f(t,\mathbf{p}) = t^{\alpha-\beta(d+z)} \int \frac{d^d q}{(2\pi)^d}\, \omega(\mathbf{q}) f_S(\mathbf{q})$$

$$\Rightarrow \quad \boxed{\textit{energy conservation:} \quad \alpha = \beta(d+z)}$$

$$\omega(\mathbf{p}) = s^{-z} \omega(s\mathbf{p})$$

Perturbative estimate (Gross-Pitaevskii)

$$C^{2\leftrightarrow 2}[f](t,\mathbf{p}) = \int \mathrm{d}\Omega^{2\leftrightarrow 2}(\mathbf{p},\mathbf{l},\mathbf{q},\mathbf{r})$$

$$\times \left[(f_{\mathbf{p}}+1)(f_{\mathbf{l}}+1)f_{\mathbf{q}}f_{\mathbf{r}} - f_{\mathbf{p}}f_{\mathbf{l}}(f_{\mathbf{q}}+1)(f_{\mathbf{r}}+1)\right]$$

$$\overset{1\,\ll\, f_{\mathbf{p}}}{\simeq} \int \mathrm{d}\Omega^{2\leftrightarrow 2}(\mathbf{p},\mathbf{l},\mathbf{q},\mathbf{r})\ \left[(f_{\mathbf{p}}+f_{\mathbf{l}})f_{\mathbf{q}}f_{\mathbf{r}} - f_{\mathbf{p}}f_{\mathbf{l}}(f_{\mathbf{q}}+f_{\mathbf{r}})\right]$$

Using:

$$\int \mathrm{d}\Omega^{2\leftrightarrow 2}(\mathbf{p},\mathbf{l},\mathbf{q},\mathbf{r}) \sim g^2 \int \frac{d^d l}{(2\pi)^d}\frac{d^d q}{(2\pi)^d}\frac{d^d r}{(2\pi)^d}$$

$$\times (2\pi)^{d+1}\,\delta^{(d)}(\mathbf{p}+\mathbf{l}-\mathbf{q}-\mathbf{r})\,\delta(\omega_{\mathbf{p}}+\omega_{\mathbf{l}}-\omega_{\mathbf{q}}-\omega_{\mathbf{r}})$$

gives scaling relation: $\alpha - 1 = -\beta\left[3d - (d+2) - 3\alpha/\beta\right]$

$$\overset{\alpha\,=\,\beta d}{\Rightarrow}\quad \textit{nonrel. particle transport:}\ \ \alpha = -\frac{d}{2},\ \ \beta = -\frac{1}{2}$$

Negative perturbative exponents do not account for inverse particle cascade!

Beyond perturbation theory: large-*N* expansion to NLO

$$\begin{aligned}\int \mathrm{d}\Omega^{2\leftrightarrow 2}(\mathbf{p},\mathbf{l},\mathbf{q},\mathbf{r}) \longrightarrow \int d\Omega^{\mathrm{NLO}}[f](t,\mathbf{p},\mathbf{l},\mathbf{q},\mathbf{r}) &\sim \int \frac{d^d l}{(2\pi)^d}\frac{d^d q}{(2\pi)^d}\frac{d^d r}{(2\pi)^d} \\ &\times (2\pi)^{d+1}\,\delta^{(d)}(\mathbf{p}+\mathbf{l}-\mathbf{q}-\mathbf{r})\,\delta(\omega_{\mathbf{p}}+\omega_{\mathbf{l}}-\omega_{\mathbf{q}}-\omega_{\mathbf{r}}) \\ &\times g^2_{\mathrm{eff}}[f](t,\mathbf{p},\mathbf{q})\,,\end{aligned}$$

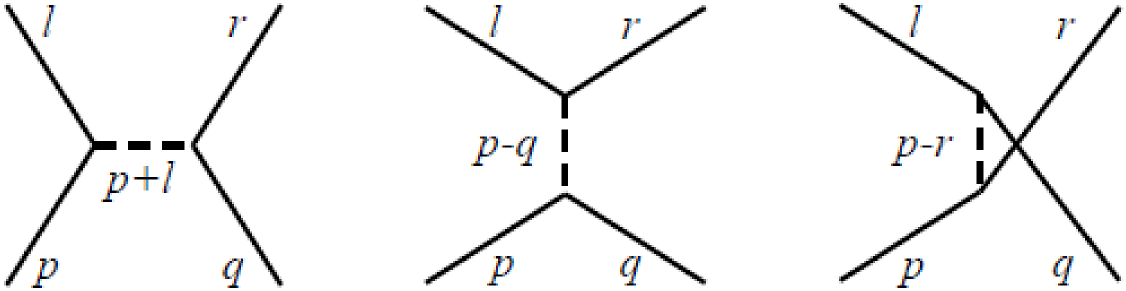


FIG. 11. Illustration of different scattering channels. The vertex correction at NLO may be viewed as an effective interaction, which involves the exchange of an intermediate particle.

based on Berges Nucl. Phys. A 699 (2002) 847; Aarts et al. Phys. Rev. D 66 (2002) 045008

Vertex correction (NLO 1/*N*)

$$g^2_{\text{eff}}(t,\mathbf{p},\mathbf{q}) \equiv \frac{g^2}{|1+\Pi^R_{\text{nr}}(t,\omega_{\mathbf{p}}-\omega_{\mathbf{q}},\mathbf{p}-\mathbf{q})|^2}$$

one-loop retarded self-energy:

$$\Pi^R_{\text{nr}}(t,\omega,\mathbf{p}) \;=\; g\int \frac{\mathrm{d}^d q}{(2\pi)^d}\, f(t,\mathbf{p}-\mathbf{q})$$

$$\times\left[\frac{1}{\omega_{\mathbf{q}}-\omega_{\mathbf{p}-\mathbf{q}}-\omega-i\epsilon}+\frac{1}{\omega_{\mathbf{q}}-\omega_{\mathbf{p}-\mathbf{q}}+\omega+i\epsilon}\right]$$

scaling behavior:

$$\Pi^R_{\text{nr}}(t,\omega_{\mathbf{p}},\mathbf{p}) = s^{\alpha/\beta-d+2}\,\Pi^R_{\text{nr}}(s^{-1/\beta}t,\omega_{s\mathbf{p}},s\mathbf{p})$$

$\alpha/\beta \geq d$, $\Pi^R_{\text{nr}}(t,\omega_{\mathbf{p}},\mathbf{p}) \gg 1$ in the infrared:

$$\Rightarrow \qquad g^2_{\text{eff}}(t,\mathbf{p},\mathbf{q},\mathbf{r}) = s^{-2(\alpha/\beta-d+2)}\,g^2_{\text{eff}}(s^{-1/\beta}t,s\mathbf{p},s\mathbf{q},s\mathbf{r})$$

Scaling solution at NLO 1/*N*

$$C_{\mathrm{nr}}^{\mathrm{NLO}}[f](t,\mathbf{p}) = s^{-(2-\alpha/\beta)}\, C_{\mathrm{nr}}^{\mathrm{NLO}}[f](s^{-1/\beta}t, s\mathbf{p})$$

$$= t^{\alpha-2\beta}\, C_{\mathrm{nr}}^{\mathrm{NLO}}[f_S](1, t^{\beta}\mathbf{p})$$

gives scaling relation: $\alpha - 1 = \alpha - 2\beta$

$$\Rightarrow \boxed{\textit{nonrel. transport: } \beta = \frac{1}{2} \;\; \textit{of} \begin{cases} \textit{particles:} & \alpha = d/2 \\ \textit{energy:} & \alpha = (d+2)/2 \end{cases}}$$

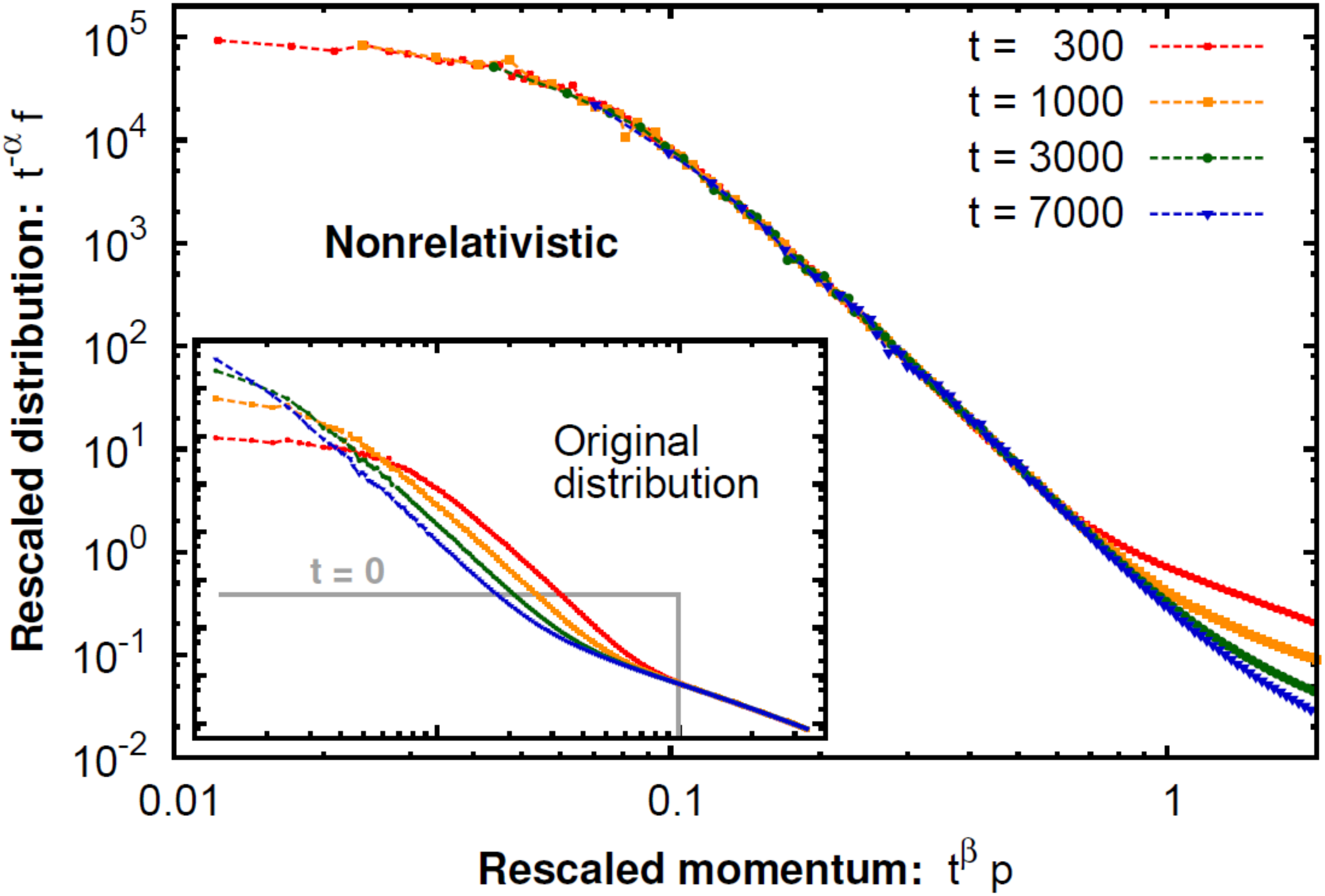
Positive nonperturbative exponents can describe inverse cascade!

NLO result in good agreement with full numerical simulation

Piñeiro Orioli, Boguslavski, Berges, PRD 92 (2015) 025041

Self-similar dynamics from classical-statistical simulations

$$f(t,\mathbf{p}) = t^{\alpha}\, f_S(t^{\beta}\mathbf{p}) \;, \qquad \alpha = 1.66 \pm 0.12\,, \qquad \beta = \; 0.55 \pm 0.03$$



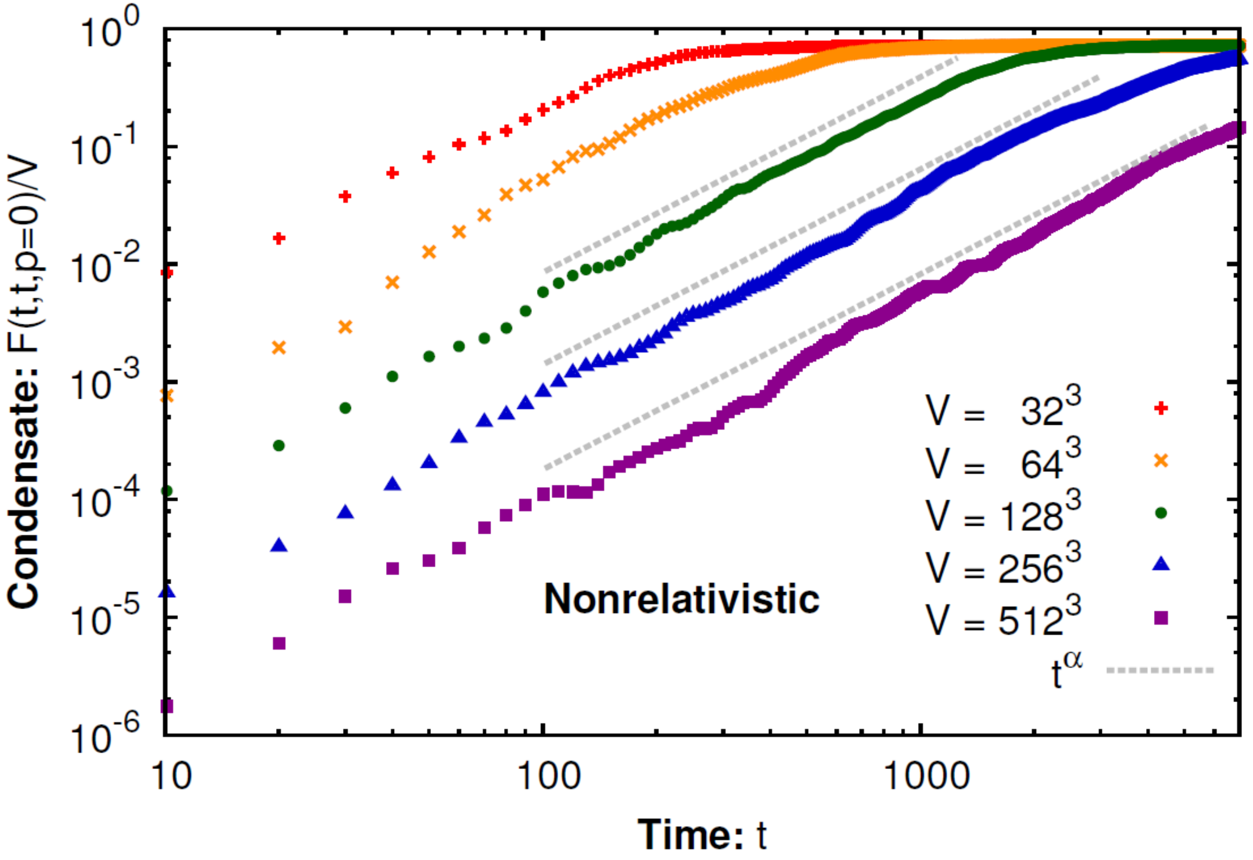
Piñeiro Orioli, Boguslavski, Berges, PRD 92 (2015) 025041

Condensation far from equilibrium

$$F(t,t',\mathbf{x}-\mathbf{x}') = \frac{1}{2}\langle \psi(t,\mathbf{x})\psi^*(t',\mathbf{x}') + \psi(t',\mathbf{x}')\psi^*(t,\mathbf{x})\rangle$$

$$f(t,\mathbf{p}) + (2\pi)^3\delta^{(3)}(\mathbf{p})|\psi_0|^2(t) \equiv \int d^3x\, e^{-i\mathbf{px}}\, F(t,t,\mathbf{x})$$

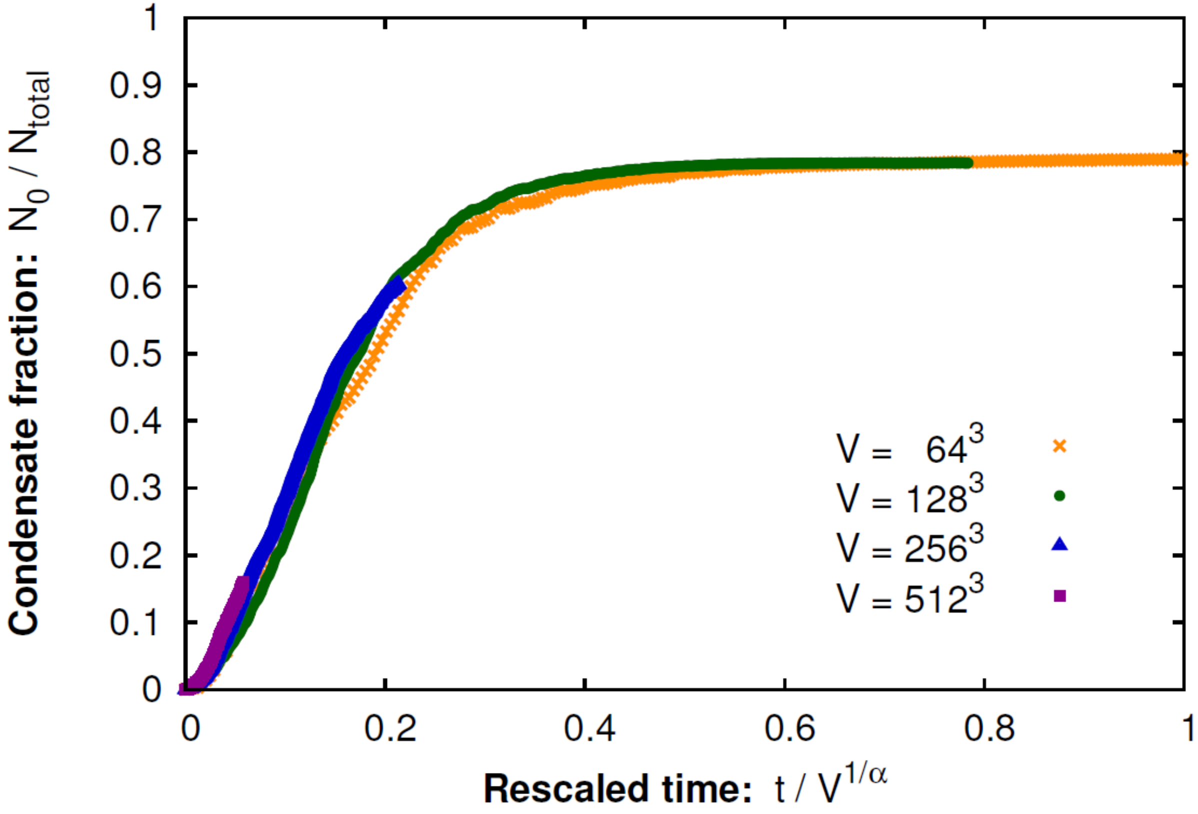
volume: $(2\pi)^3\delta(\mathbf{0}) \to V$



Condensation time

$$\frac{N_0(t)}{N_{\rm total}} = \frac{|\psi_0|^2(t)}{\int d^3p/(2\pi)^3 f(t,\mathbf{p}) + |\psi_0|^2(t)} \quad , \quad V^{-1}F(t,t,\mathbf{p}=0) \sim t^{\alpha}$$

$$\Rightarrow \qquad \boxed{t_f \simeq t_0 \left(\frac{|\psi_0|^2(t_f)}{f(t_0,0)} \right)^{1/\alpha} V^{1/\alpha}}$$



Analytic estimates agree well with simulations!

Nonthermal fixed point in non-Abelian gauge theory

Interpret scaling condition with **energy/number conserving*** Fokker-Planck-type dynamics for the collisional broadening of longitudinal momentum distribution:

$$C^{(\mathrm{elast})}[p_T, p_z; f] = \hat{q}\ \partial^2_{p_z} f(p_T, p_z, t)$$

with **momentum diffusion parameter**: $\hat{q} \sim \alpha_S^2 \int \frac{d^2 p_T}{(2\pi)^2} \int \frac{dp_z}{2\pi}\ f^2(p_T, p_z, t)$

□ 1) $\mu = 3\alpha - 2\beta + \gamma \xrightarrow{\alpha - 1 = \mu(\alpha, \beta, \gamma)} 2\alpha - 2\beta + \gamma + 1 = 0$

2) number conservation $\longrightarrow \alpha - 2\beta - \gamma + 1 = 0$

3) energy conservation $\longrightarrow \alpha - 3\beta - \gamma + 1 = 0$

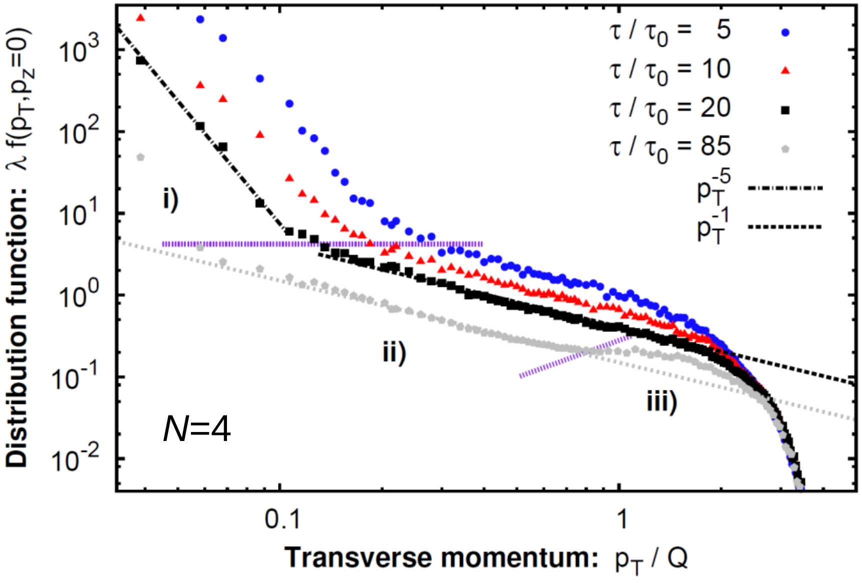
remarkable agreement with lattice data!

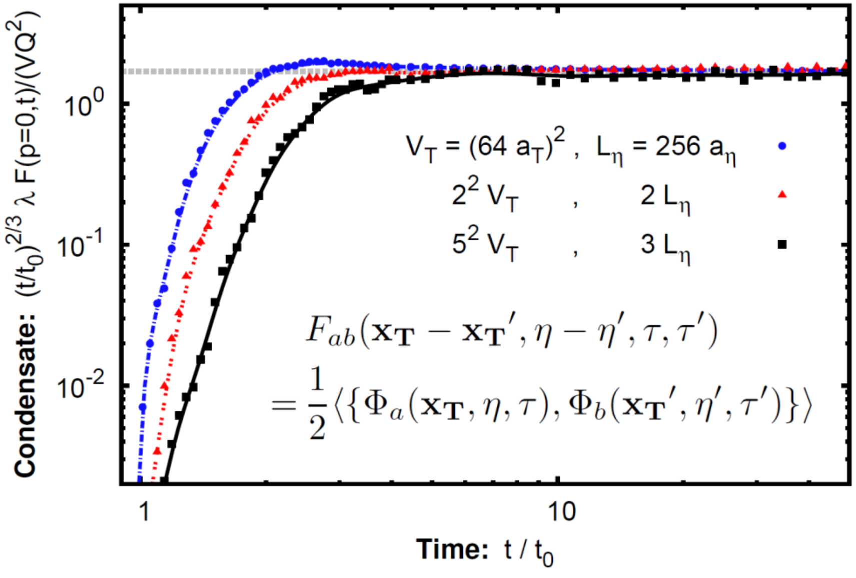
$$\boxed{\alpha = -2/3\ , \qquad \beta = 0\ , \qquad \gamma = 1/3}$$

*cf. early stages of Baier, Mueller, Schiff, Son, PLB 502 (2001) 51 (‘BMSS‘)

A closer look at scalar *N*-component field theory

with longitudinal expansion





$$F_{ab}(\mathbf{x_T}-\mathbf{x_T}',\eta-\eta',\tau,\tau') = \frac{1}{2}\langle\{\Phi_a(\mathbf{x_T},\eta,\tau),\Phi_b(\mathbf{x_T}',\eta',\tau')\}\rangle$$

Range	α	β	γ	$f_S(p_T,p_z)$
i)	1	$2/3$	$2/3$	$(\|\mathbf{p}\|^{1/2}+\|\mathbf{p}\|^5)^{-1}$
ii)	$-2/3$	0	$1/3$	$\exp(-p_z^2/(2\sigma_z^2))/p_T$
iii)	$-1/2$	0	$1/2$	$\mathrm{sech}(p_z/\sigma_z)$

- ***Isotropic scaling*** properties and ***condensation*** in the infrared regime