Forced turbulence in non-Abelian plasmas

(In preparation)

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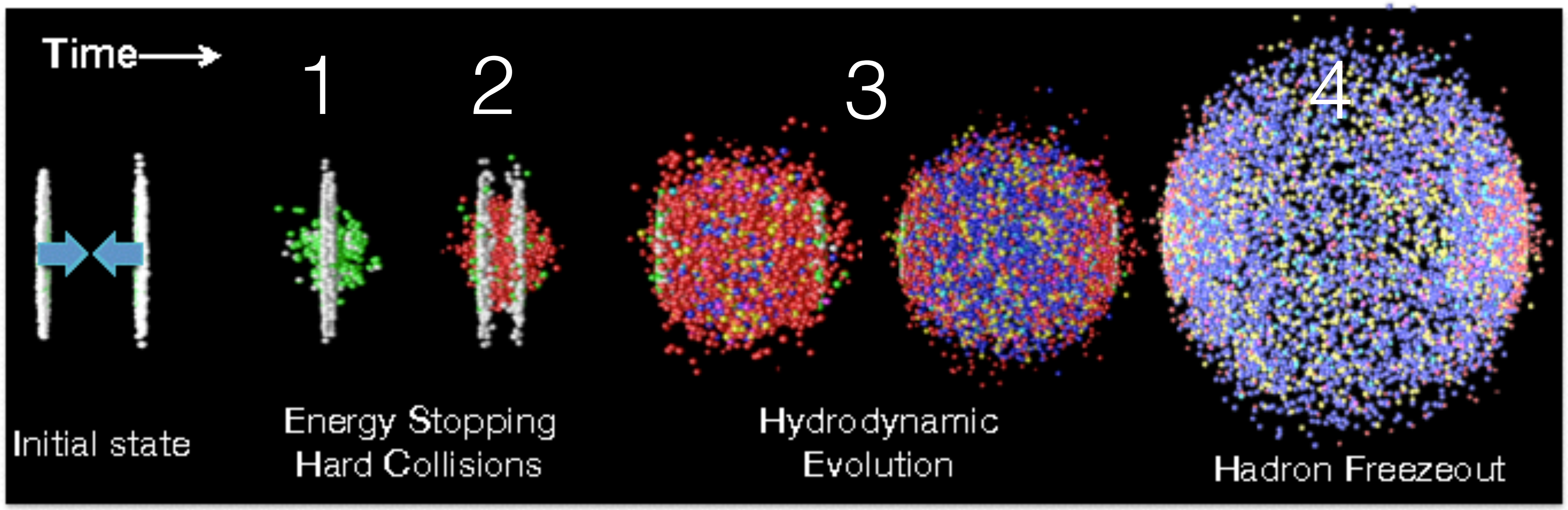
The Big Bang and the little bangs
August 26, 2016, CERN

Non-equilibrium QCD dynamics in Heavy Ion Collisions



1. **τ ~ 0 fm/c :** strong gauge fields (Glasma) $A \sim 1/g$

2. **τ ~ 0.1-2 fm/c (if weak coupling)** quasi-particles (gluons) $A \ll 1/g$

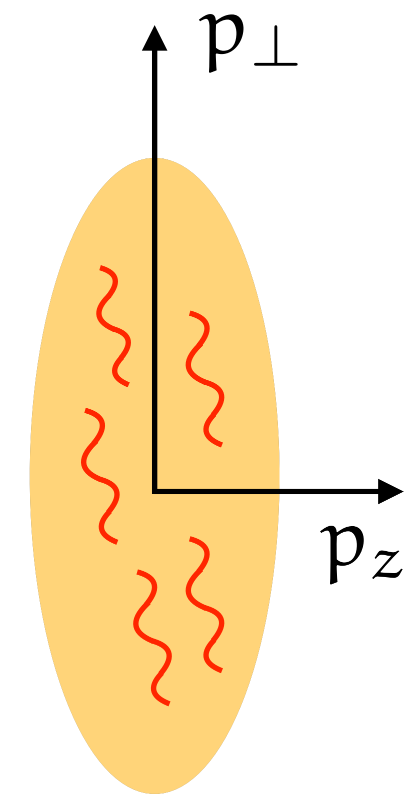
Hydrodynamization, isotropization, thermalization. Instabilities, Chaoticity, turbulence, etc.

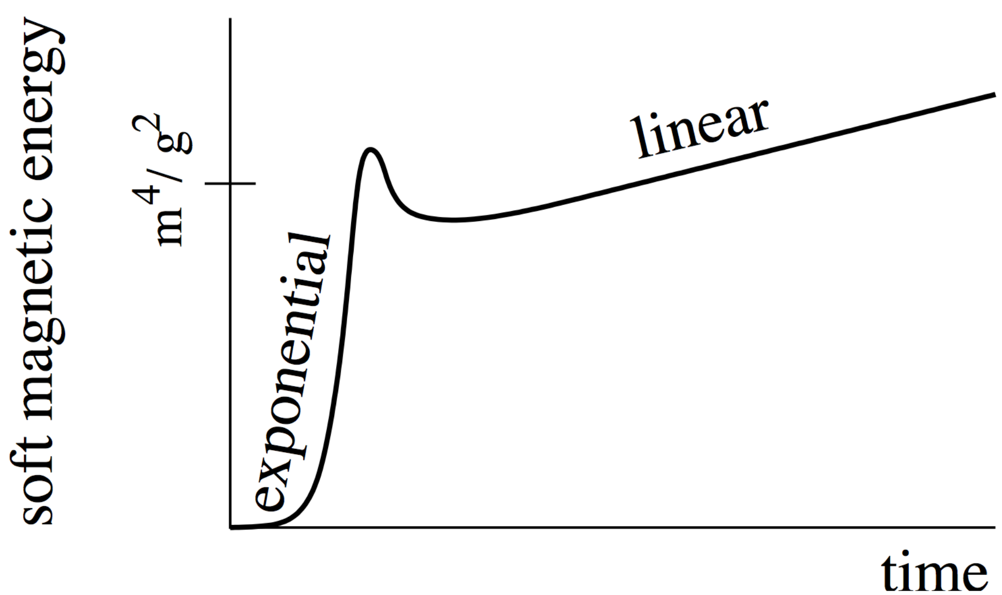
3. **τ ~ 2-10 fm/c** hydrodynamic evolution then feezout

Turbulence in early stages of Heavy Ion Collisions

- Immediately after the collision the system is far from equilibrium. Anisotropic particle distribution in momentum space.

- Chromo-Weibel Instabilities : Momentum anisotropy induces exponential growth of soft magnetic modes (early stage: as in abelian plasmas) which turns into a linear growth due to nonlinear interactions inherent to non-abelian plasmas



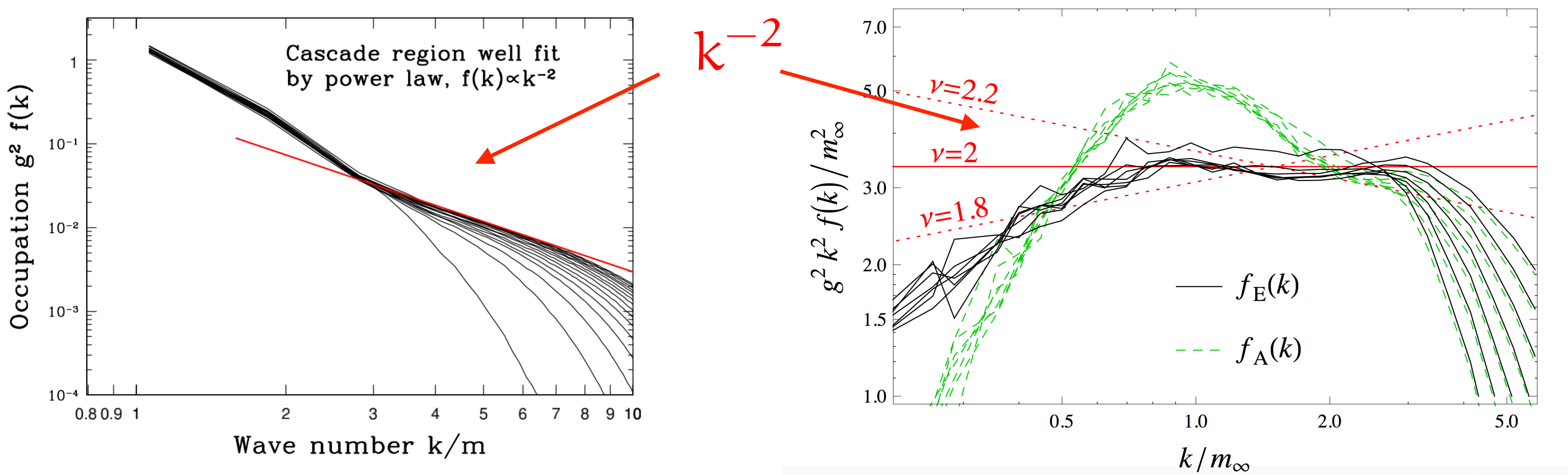


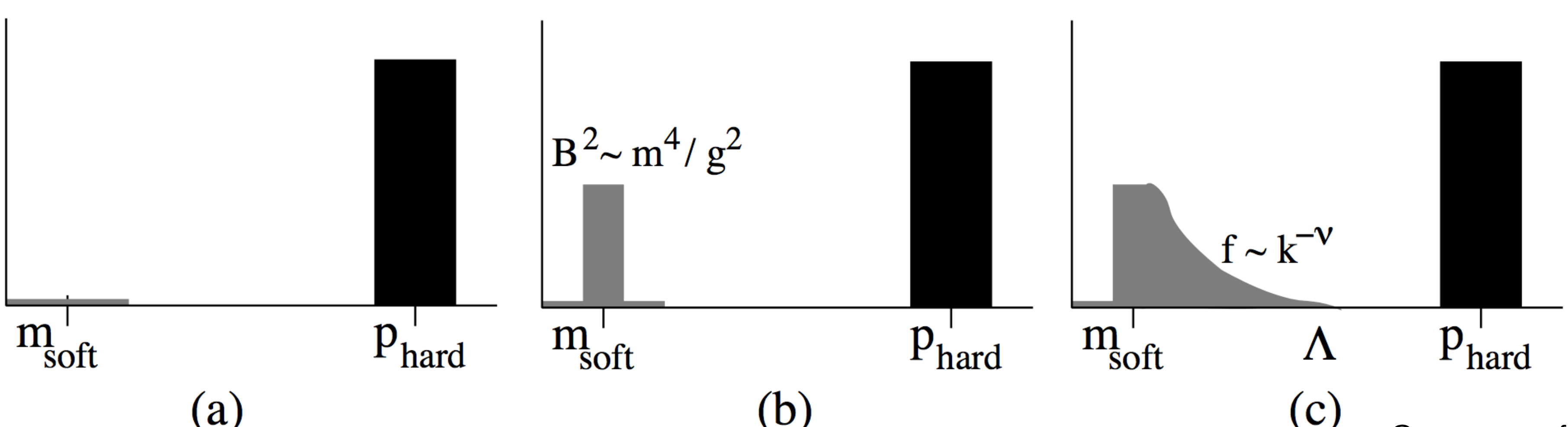
E. S. Weibel (1959)
S. Mrowczynski (1993)
P. Arnold, J. Lenaghan, G. Moore, L. Yaffe (2005)
A. Rebhan, P. Romatschke, and M. Strickland (2005)
D. Bödeker, K. Rummukainen (2005)
P. Arnold, G. D. Moore (2005)

Turbulence in early stages of Heavy Ion Collisions

- Hard-loop simulations (large scale separation between hard modes and soft excitations) : Nonlinear interactions develop a **turbulent cascade in the UV with exponent -2**

P.Arnold, G. D. Moore (2005)   A. Ipp, A. Rebhan, M. Strickland (2011)

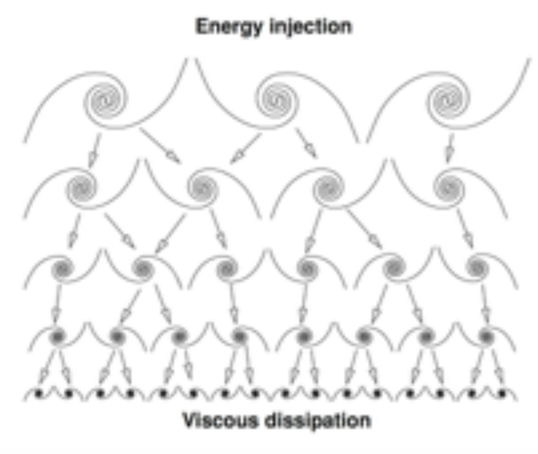


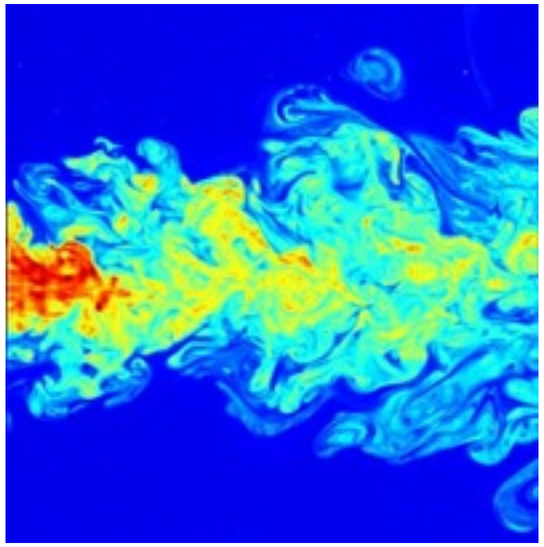


Courtesy of Arnold and Moore

Wave Turbulence (I)

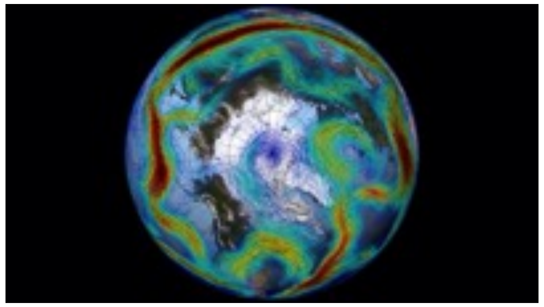
- **Out-of-equilibrium** statistics of random **non-linear waves**

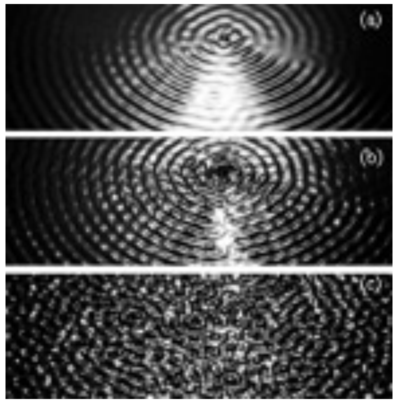


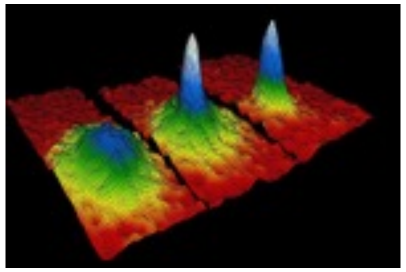


- Similarity with fluid turbulence: **inviscid transport of conserved quantities** from large to small scales through the so-called transparency window (or **inertial range**)

- Some examples:
  - Atmospheric Rossby waves
  - Water surface gravity and capillary waves
  - Waves in plasmas
  - Nonlinear Schrödinger equation (NL Optics, BEC)







Wave Turbulence (II)

- **Waves are excited by external processes. Driven turbulence:** Open system with source and sink → **away from thermodynamical equilibrium**

- Steady states characterized by **constant fluxes P and Q** rather than temperature and thermodynamical potentials

- **Kolmogorov-Obukhov (KO41) theory relies on Locality of interactions: Only eddies (waves) with comparable sizes (wavelengths) interact**. Steady state power spectra in momentum space depend on fluxes and not on the pumping and dissipation scales

- Weak (Wave) Turbulence Theory: **Kinetic description in the limit of weak nonlinearity**

V. E. Zakharov, V. S. L'vov, G. Falkovich (Springer- Verlag, 1992)

Wave Turbulence: Classical Yang-Mills

- **Can one understand the power spectrum k^{-2} from first principles?**
- From Arnold and Moore (2005): parametric argument:
  diffusion + energy conservation yield the exponent - 2
- From Mueller, Shoshi, Wong (2006): diffusion conserves particle number and not energy

  Turbulence in QCD is nonlocal $\Rightarrow$ $n(k) \sim k^{-1}$

- **Some caveats (in this work):**
  - **Homogeneous** and **isotropic** system of gluons
  - **Forcing:** Energy injection with constant rate P at $k_f \gg m$:
    Dispersion relation $\omega(k) \equiv |k|$
  - Weak nonlinearities in the **classical limit** (high occupancy):

$$g^2 \ll 1 \qquad \text{and} \qquad 1 \ll n(k) \ll \frac{1}{g^2}$$

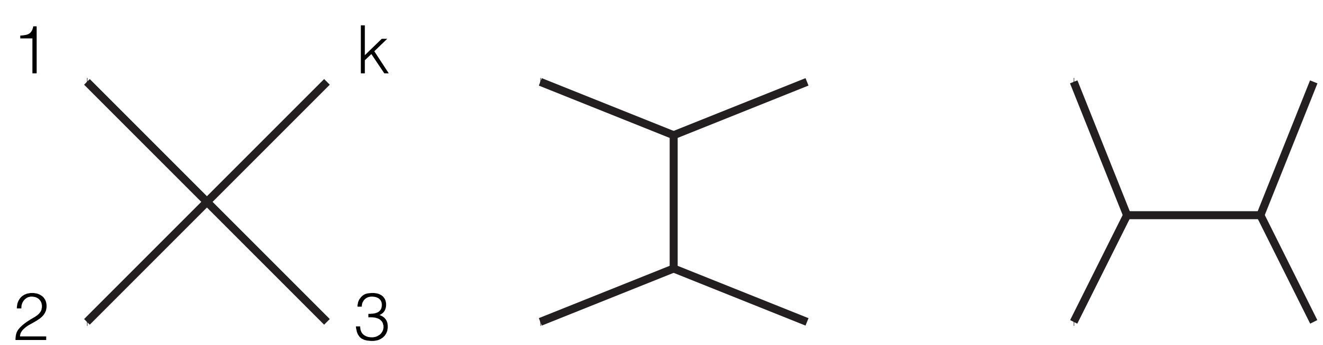
Elastic 2 to 2 process (4-waves interaction)

- Elastic gluon-gluon scattering

$$\frac{\partial}{\partial t} n_k = \frac{1}{2} \int_{k_1, k_2, k_3} \frac{1}{2\omega(k)} |\mathcal{M}_{12\to 3k}|^2 \, \delta(\sum_i k_i) \delta(\sum_i \omega_i) \; F[n]$$

$$F[n] \equiv [n_{k_1} n_{k_2} n_k + n_{k_1} n_{k_2} n_{k_3} - n_{k_1} n_{k_3} n_k - n_{k_2} n_{k_3} n_k] \sim n^3$$

$\mathcal{M}_{12\to 3k} \equiv$



- Two constant of motion: **particle number** and **energy** $\Rightarrow$ **Two fluxes**

$$Q(k) \equiv \dot{N} = \int^k d^3k' \, \dot{n}(k') \qquad\qquad P(k) \equiv \dot{E} = \int^k d^3k' \, |k'| \, \dot{n}(k')$$

Elastic 2 to 2 process (4-waves interactions)

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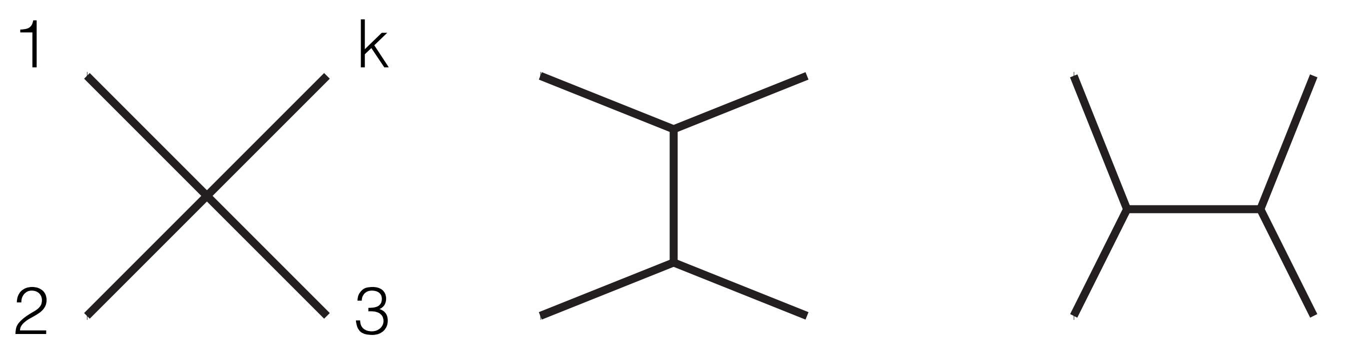
$\mathcal{M}_{12\to 3k} \equiv$



- **H-theorem** $\Rightarrow$ **Thermal fixed-point** (vanishing fluxes) $P = Q = 0$

$$n_k = \frac{T}{\omega(k) - \mu}$$

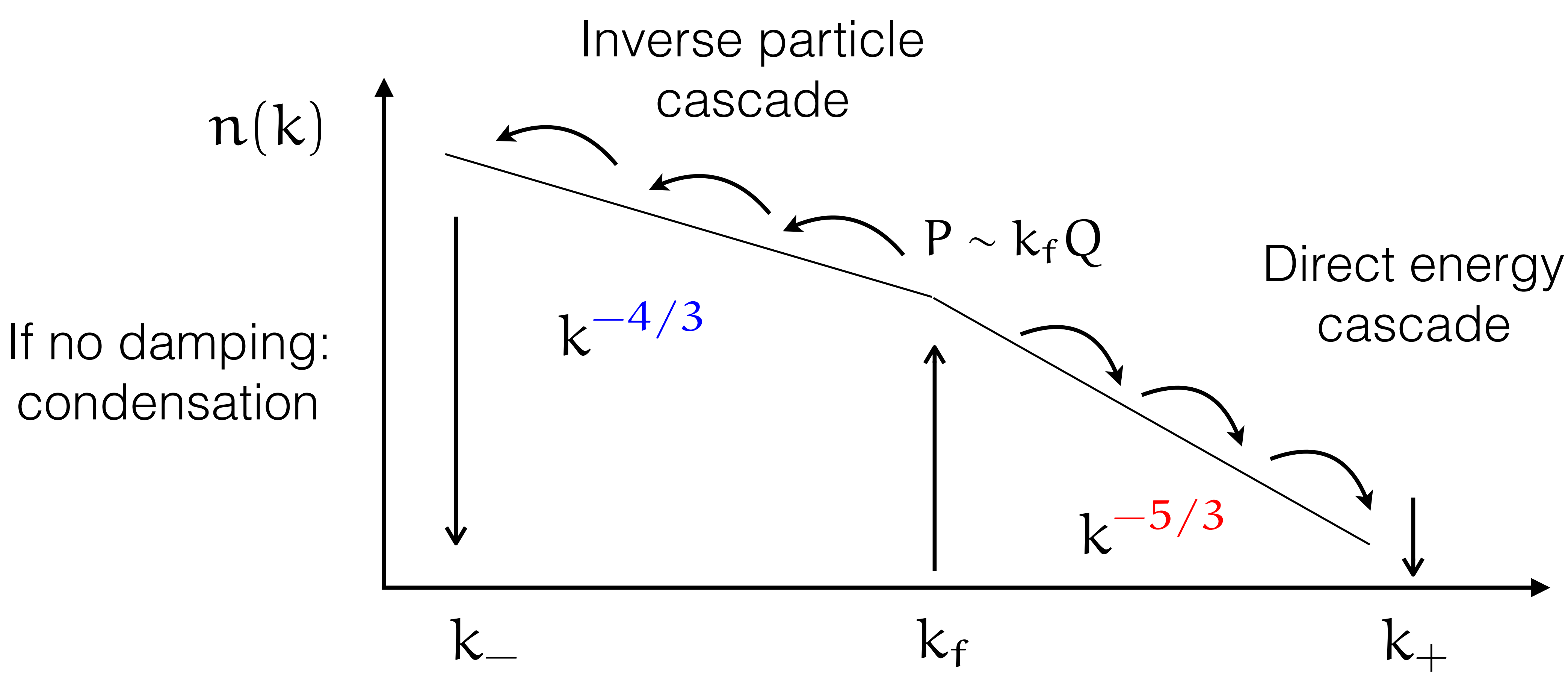
(Rayleigh-Jeans distribution)

Kolmogorov-Zakharov (KZ) Spectra

- From collision integral the flux scales as the cube of the the occupation number: nonlinear 4-wave interactions

$$P \sim Q \sim \dot{n} \sim n^3 \quad \Rightarrow n \sim P^{1/3} \sim Q^{1/3}$$

- Dim. analysis + scale invariance (locality of interactions) **⇒ KZ spectra**

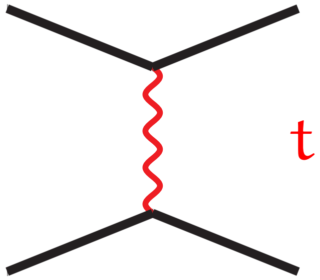


Are KZ spectra in QCD physically relevant in non-Abelian plasmas?

Elastic scattering in the small angle approximation

- Coulomb interaction is singular at small momentum transfer $k \gg q \geq m$

$$|\mathcal{M}_{k1\to 23}|^2 \sim \alpha^2 \frac{s^2}{t^2}$$



- **Dominant interaction: small angle scattering**
- Fokker-Planck equation — Diffusion and drag

$$\frac{\partial}{\partial t} n_k \equiv \frac{\hat{q}}{4k^2} \frac{\partial}{\partial k} k^2 \left[\frac{\partial}{\partial k} n_k + \frac{n_k^2}{T_*} \right]$$

L. D. Landau (1937) B. Svetitski (1988) Blaizot, Liao, McLerran (2012)

Diffusion coefficient	Screening mass	Effective temperature
$\hat{q} \equiv \sim \alpha^2 \int d^3k\, n_k^2$	$m^2 \sim \alpha \int \frac{d^3k}{\lvert k \rvert} n_k$	$T_* \sim \frac{\hat{q}}{\alpha m^2}$

Steady state solutions

$$\frac{\partial}{\partial t} n_k \equiv \frac{\hat{q}}{4k^2}\frac{\partial}{\partial k}k^2\left[\frac{\partial}{\partial k}n_k + \frac{n_k^2}{T_*}\right] + F - D$$

Forcing Damping

- Thermal fixed point: $\frac{T_*}{\omega - \mu}$
- Non-thermal fixed point (inverse particle cascade): $n(\omega) \sim \frac{A}{\omega} > \frac{T_*}{\omega} \quad (\omega = k)$

$$A \equiv \frac{1}{2}T_*\left(1 + \sqrt{1 + \frac{16Q}{\hat{q}T_*}}\right)$$

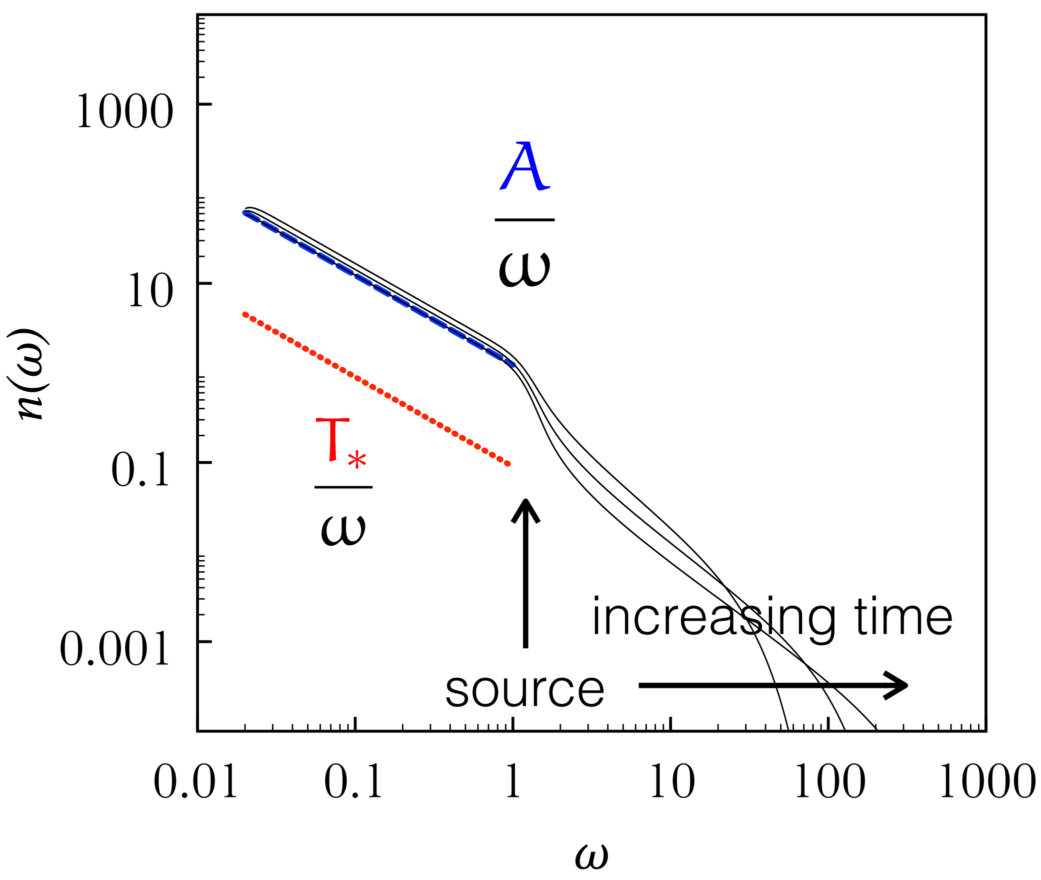
- **No KZ spectra.** Warm cascade behavior:

2-D Optical turbulence: S. Dyachenko, A.C. Newell, A. Pushkarev, V.E. Zakharov (1992)

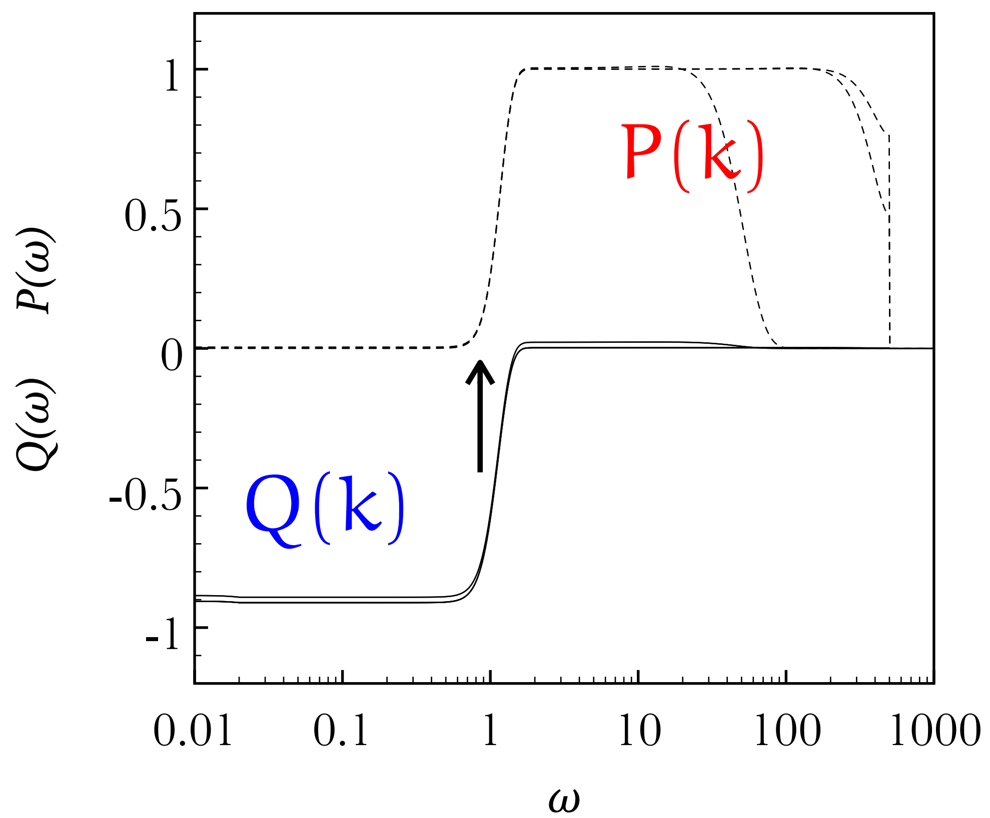
Boltzmann equation: D. Proment, S. Nazarenko, P. Asinari, and M. Onorato (2011)

- Parametrically $A \sim Q^{1/3}\omega_f^{-1/3}$: depends on the forcing ⇒ **nonlocality**

Numerical simulation of FK equation with forcing



- Particle and energy fluxes

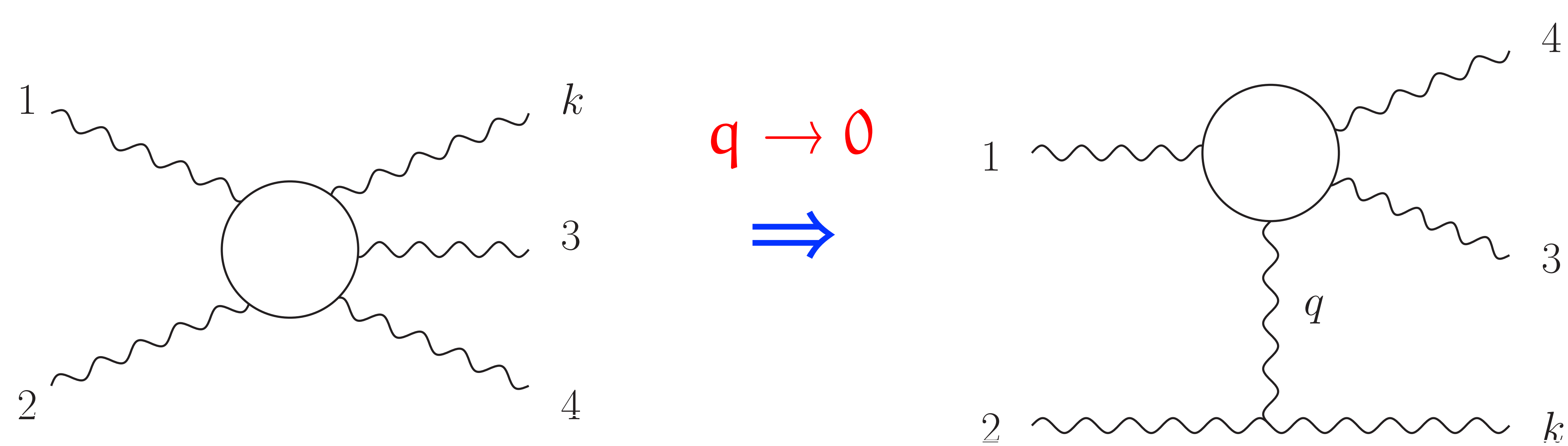


- The occupation number (left) and, the energy and particle number fluxes (right) at late times. Above the forcing scale the spectrum vanishes asymptotically

- Constant particle flux at $\omega=0$ $\Rightarrow$ Bose-Einstein condensation

Contribution from inelastic processes?

- Naively one would expect inelastic processes to be suppressed by powers of the coupling constant g
- In non-Abelian plasmas **inelastic processes are enhanced due to collinear divergences** and hence cannot be neglected compared to elastic processes
- **Small angle approximation:** 2 → 3 process reduces to an **effective 1 → 2**



R. Baier, Y. Dokshitzer, A. H. Mueller, D. Schiff, D. T. Son (2000)
P. Arnold, G. D. Moore, L. G. Yaffe (2002)

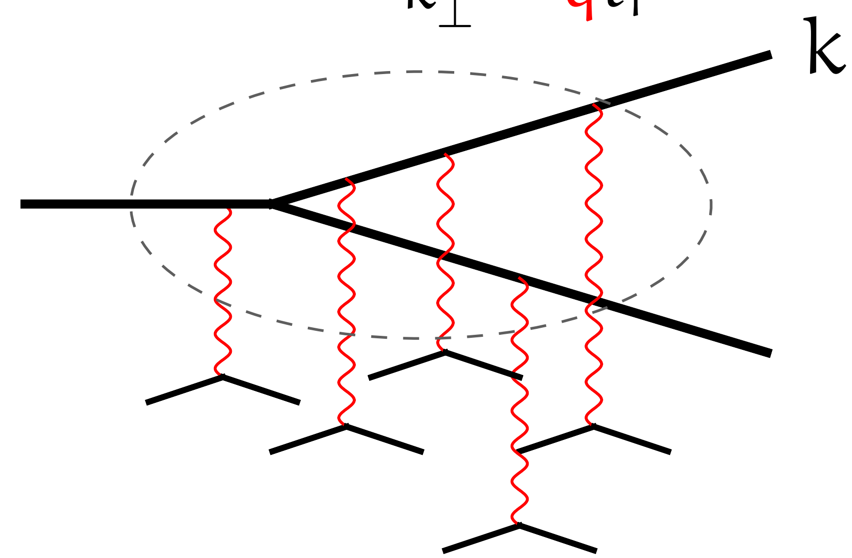
Effective 3 waves interaction (1 to 2 scattering)

- Landau-Pomeranchuk-Migdal (LPM) regime: many scatterings can cause a gluon to branch with the rate

formation time:

$$t_f(k) \sim \frac{k}{k_\perp^2} \sim \frac{k}{\hat{q} t_f}$$

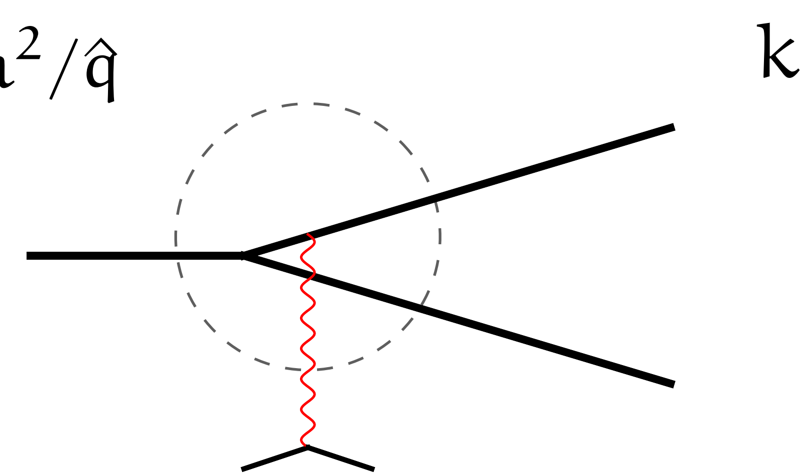
$$k\frac{d\Gamma}{dk} \sim \frac{\alpha}{t_f(k)} \sim \alpha\sqrt{\frac{\hat{q}}{k}}$$



R. Baier, Y. Dokshitzer, A. H. Mueller, S. Peigné, D. Schiff (1995) V. Zakharov (1996)

- Bethe-Heitler regime for $t_f(k) < \ell_{mfp} \sim m^2/\hat{q}$

$$k\frac{d\Gamma}{dk} \sim \frac{\alpha}{\ell_{mfp}}$$

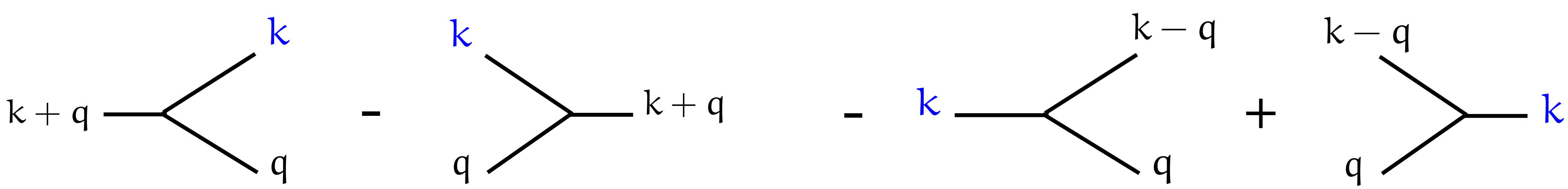


J. F. Gunion and G. Bertsch (1982)

Effective 3 waves interaction (1 to 2 scatterings)

- General form of the kinetic equation

$$\frac{\partial}{\partial t} n_k \equiv \frac{1}{k^3}\left[\int_0^\infty dq K(k+q,q)F(k+q,q) - \int_0^k dq K(k,q)F(k,q)\right]$$

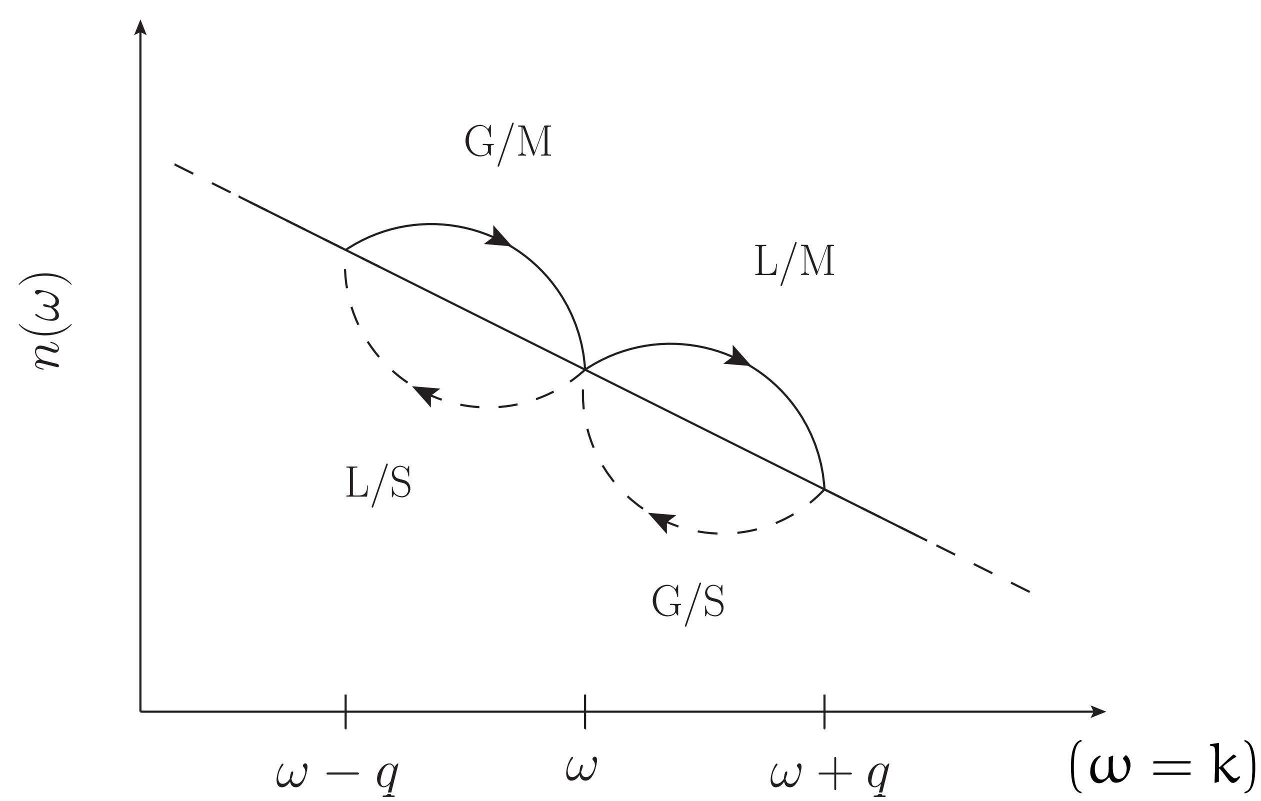


Gain/Splitting   Loss/Merging   Loss/Splitting   Gain/Merging

- **4 contributions** to the time evolution of the occupation number n(k)

- At thermal equilibrium F[n]=0 (detailed balance)

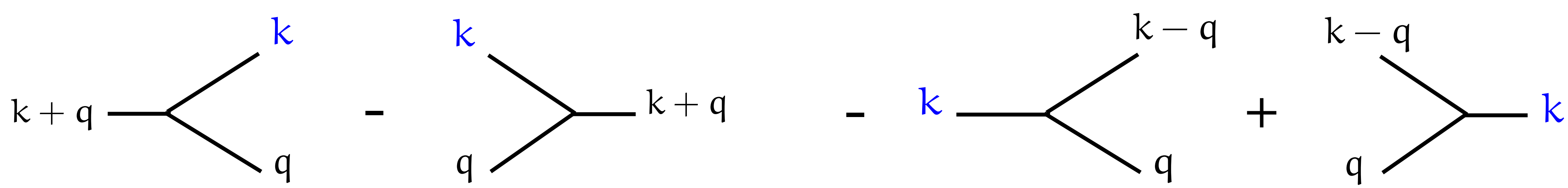
$$F(k,q) \equiv n_{k+q}n_k + (n_{k+q} - n_k)n_q \sim n^2$$



Effective 3 waves interaction (1 to 2 scatterings)

- General form of the kinetic equation

$$\frac{\partial}{\partial t} n_k \equiv \frac{1}{k^3}\left[\int_0^{\infty} dq K(k+q,q)F(k+q,q) - \int_0^{k} dq K(k,q)F(k,q)\right]$$



$$F(k,q) \equiv n_{k+q}n_k + (n_{k+q} - n_k)n_q \sim n^2$$

- Kernel in the LPM regime: $K(k,q) \equiv \alpha\sqrt{\hat{q}}\,\frac{(k+q)^{7/2}}{k^{1/2}q^{3/2}}$

R. Baier, Y. Dokshitzer, A. H. Mueller, D. Schiff, D. T. Son (2000)
P. Arnold, G. D. Moore, L. G. Yaffe (2002)

- Direct energy cascade (if interactions were local!)

$$n_k \sim \frac{P^{1/2}}{\hat{q}^{1/4}\,k^{7/4}}$$

P. Arnold, G. D. Moore (2005)

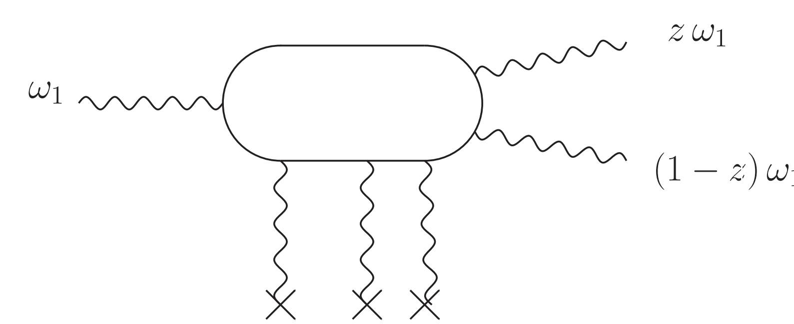
Non-locality of interactions in momentum space

- Assume a power spectrum $n \sim k^{-x}$ and require the energy flux to be independent of k

- We obtain the KZ exponent $x = 7/4$ (in the BH regime we find $x = 2$)

- The corresponding energy flux reads

$$P = \alpha\sqrt{\hat{q}} \int_0^1 dz \, \frac{(1-z)^x + z^x - 1}{z^{x+1/2}(1-z)^{x+3/2}} \ln\frac{1}{z}$$



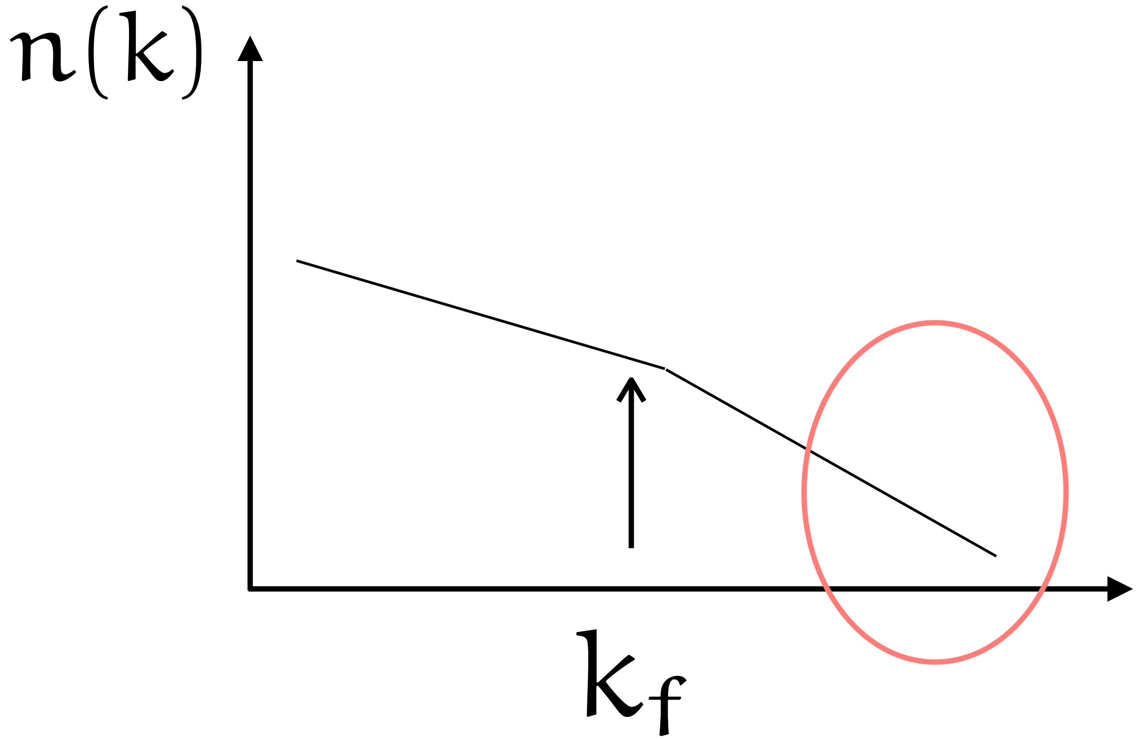
Locality: z ~ 1/2

- The above integral diverges on the KZ spectrum when z ⟶ 0 or z ⟶ 1

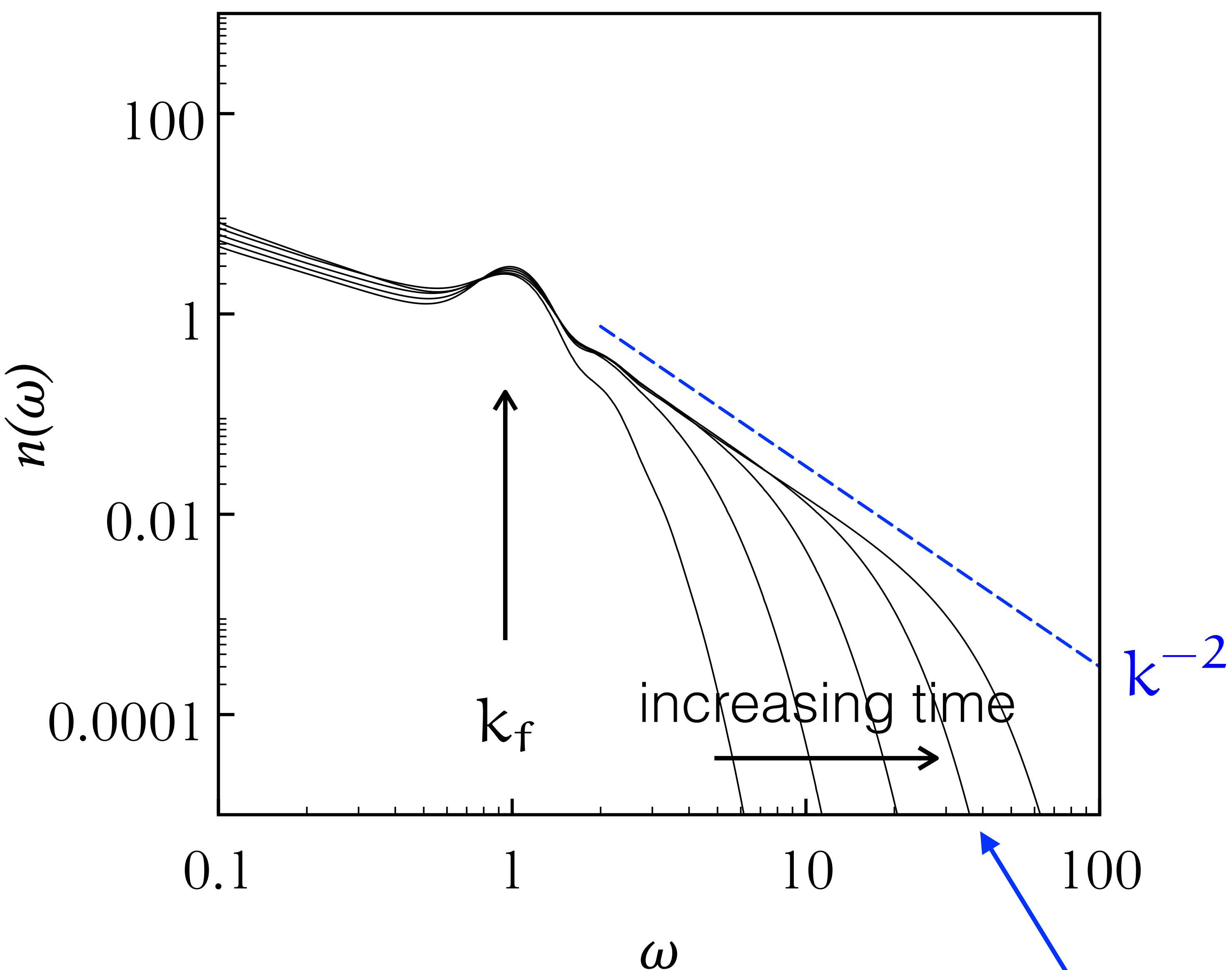
⇒ Effective 3 waves interaction is nonlocal in momentum space

and the KZ spectrum cannot be realized

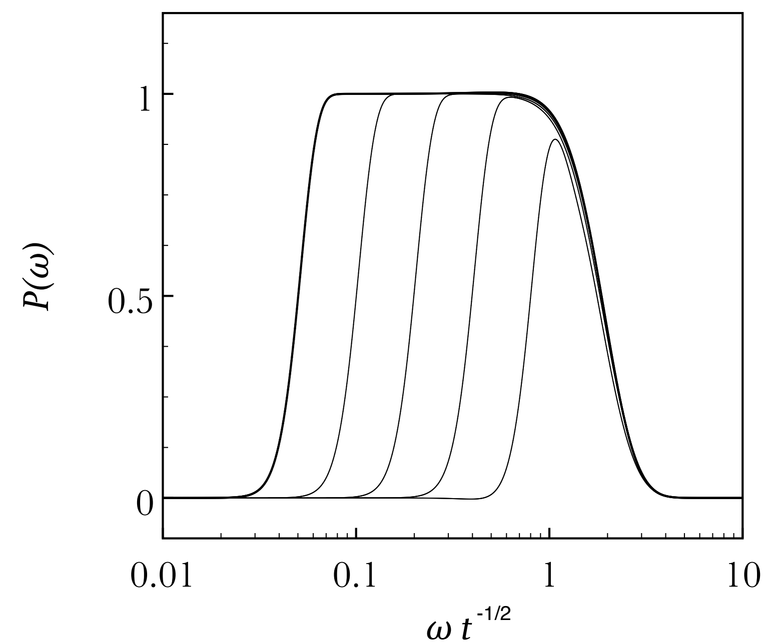
What is the physical steady state spectrum?



Numerical simulation with forcing (BH regime)



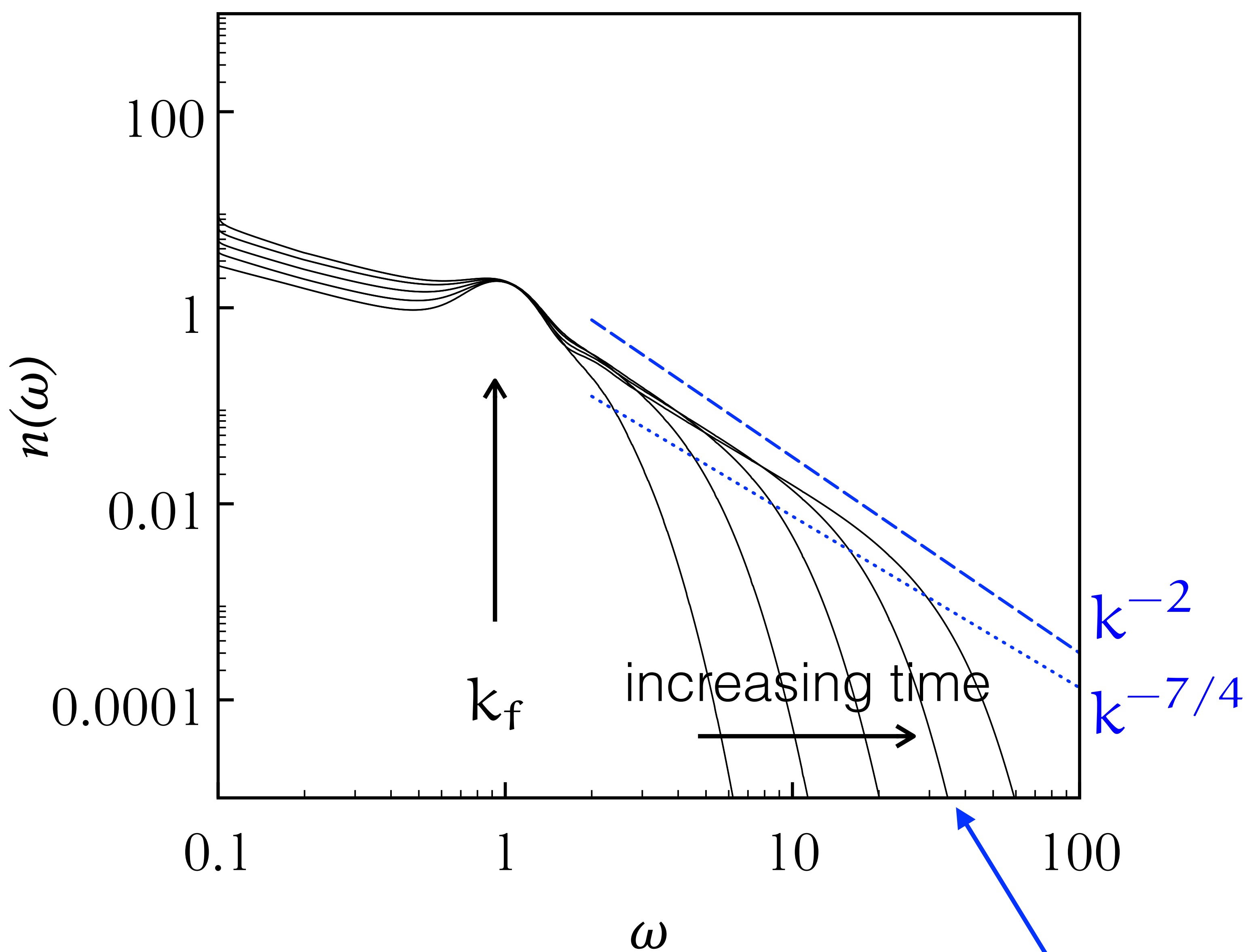
- Wave front moving towards the UV leaving in its wake the **steady power spectrum ~ k^{-2}**



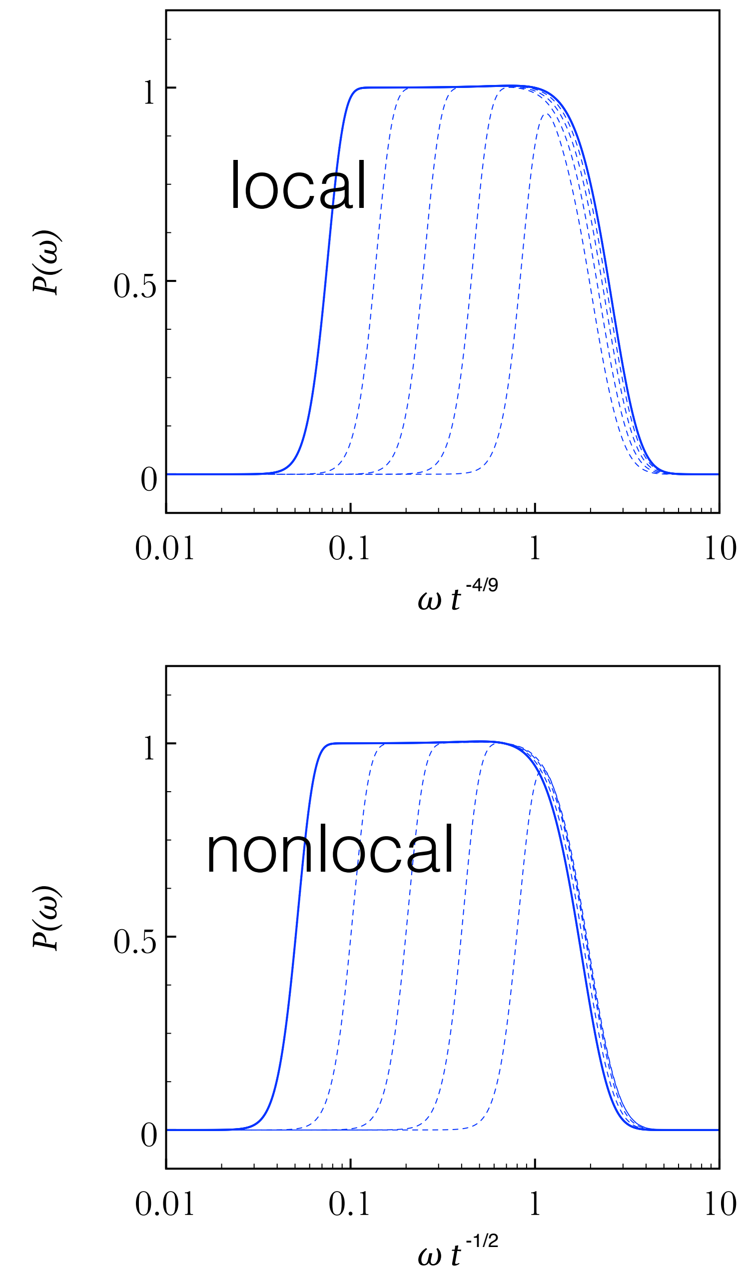
wave front evolution (diffusion-like): $\omega_*(t) \sim t^{1/2}$

Numerical simulation with forcing (LPM regime)

Quasi-Steady spectrum



Scaling of the wave front



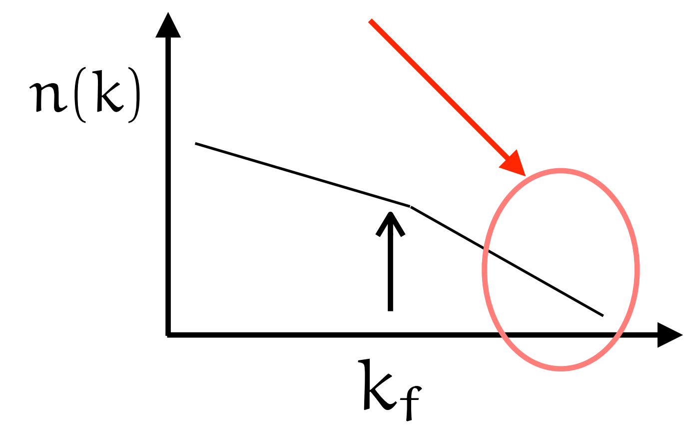
wave front evolution (diffusion-like): $\omega_*(t) \sim t^{1/2}$

Nonlocal energy cascade in the UV ($k \gg k_f$)

Approximation: strongly asymmetric splitting/merging

- In hard sector: $k \gg k_f$ we perform a gradient expansion around $k \gg q$

To the right of the source



- We obtain a **diffusion equation** in “4-D”

$$\frac{\partial}{\partial t} n(k) \simeq \frac{\hat{q}_{inel}}{4k^3} \frac{\partial}{\partial k} k^3 \frac{\partial}{\partial k} n(k)$$

- with the inelastic diffusion coefficient (in the LPM regime)

$$\hat{q}_{inel} = \alpha \sqrt{\hat{q}} \int_0^\infty dq \sqrt{q}\, n(q)$$

- Same equation in the BH regime!

Nonlocal energy cascade in the UV ($k \gg k_f$)

Approximation: strongly asymmetric splitting/merging

$$\frac{\partial}{\partial t} n(k) \simeq \frac{\hat{q}_{inel}}{4k^3} \frac{\partial}{\partial k} k^3 \frac{\partial}{\partial k} n(k)$$

- Recall that 3-D diffusion conserves number of particles: $N \sim \int dk\, k^2\, n(k)$

Its fixed point (inverse particle cascade): $n(k) \sim \frac{1}{k}$

- 4-D diffusion conserves energy: $E \sim \int dk\, k^3\, n(k)$

Its **fixed point (direct energy cascade):**

$$n(k) \sim \frac{1}{k^2}$$

Nonlocal energy cascade in the UV ($k \gg k_f$)

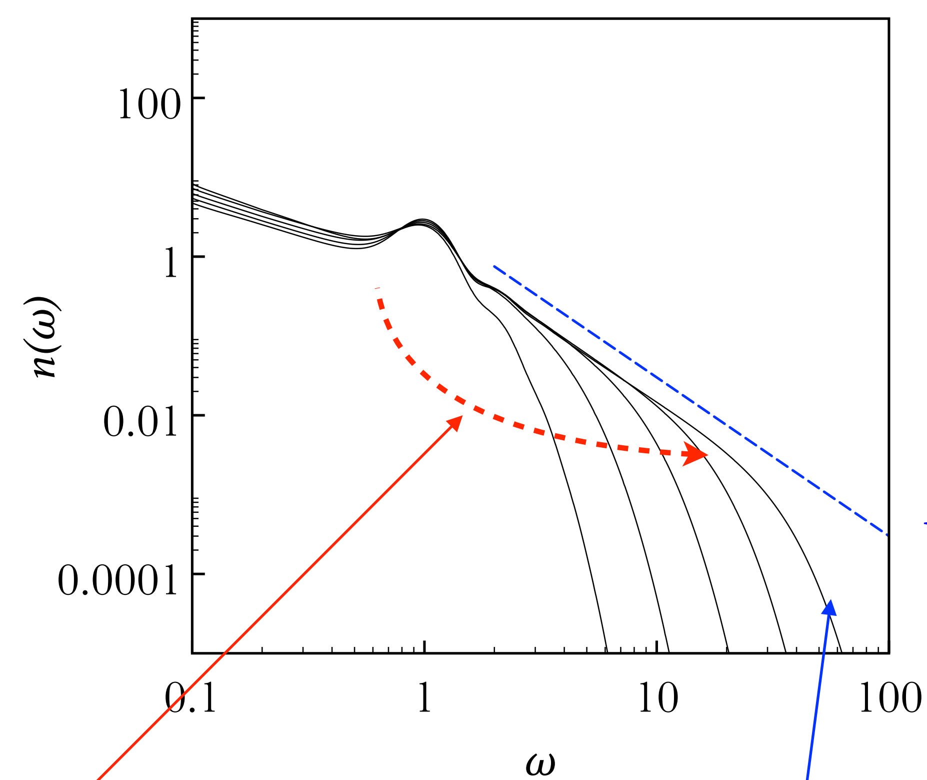
Approximation: strongly asymmetric splitting/merging

- wave front moves towards the UV leaving in its wake the nonlocal steady state spectrum: hard gluons in the inertial range interact dominantly with gluons at the forcing scale (energy gain)

Parametric estimate: $\hat{q} \sim k_f^3 n^2$

- The steady state spectrum depends on the forcing scale

$$n(k) \sim \frac{P}{\hat{q} k^2} \sim \frac{P^{1/3} k_f^{1/3}}{k^2}$$



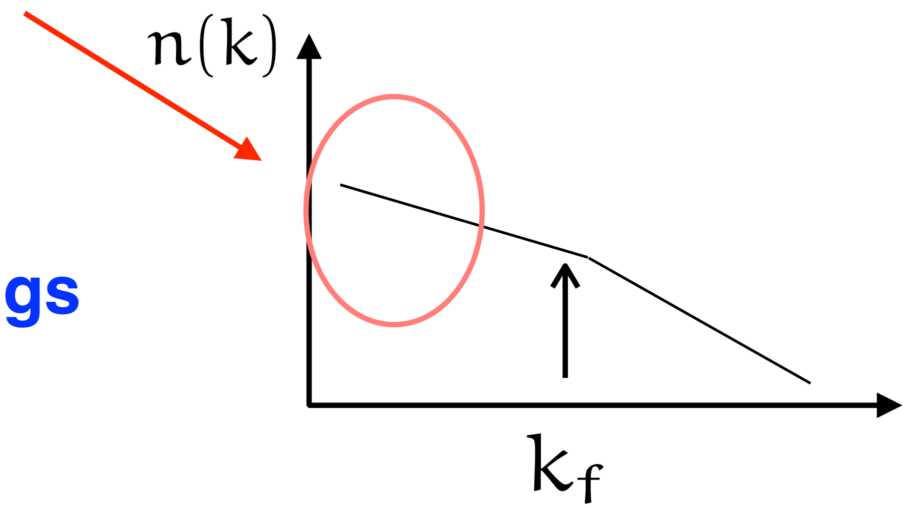
Nonlocal interactions of hard modes with the source

$\omega_*(t) \sim t^{1/2}$

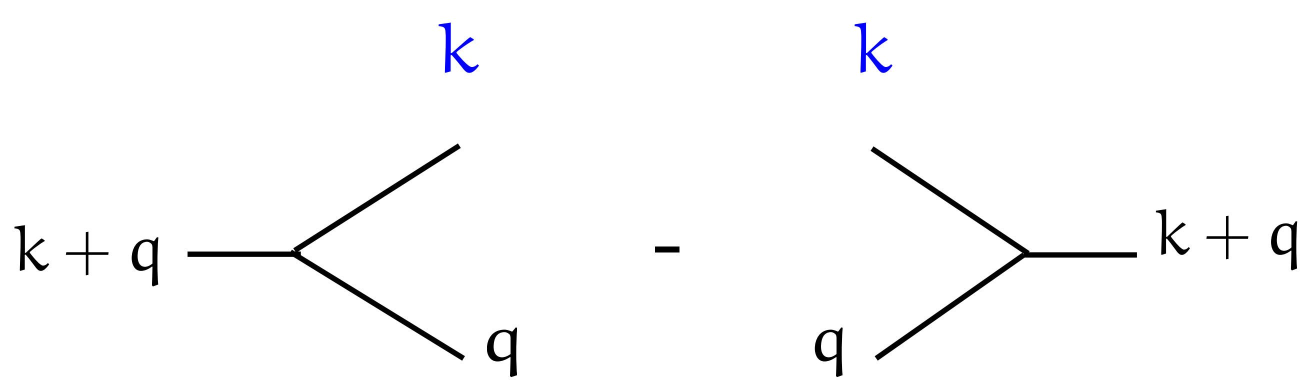
Thermalization of the soft sector ($k \ll k_f$)

- In soft sector: $k \ll k_f$

to the left of the source



- **Nonlocality** ⇒ The collision integral is dominated by **strongly asymmetric branchings**

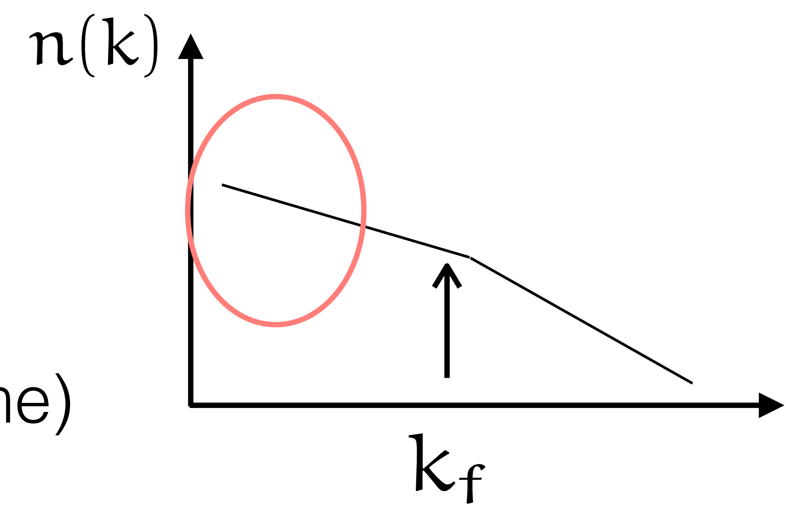


radiation absorption

$k \ll q$ and $n(k) \gg n(q) \sim n(k_f)$

Thermalization of the soft sector (k ≪ k_f)

- Relaxation to equilibrium: $n_{eq}(k) \equiv \frac{T_*}{k}$



$$\frac{\partial}{\partial t} n(k) \simeq \frac{1}{\tau_{rel}(k)} \left[n_{eq}(k) - n(k) \right]$$

- Relaxation time decreases with k (in the BH regime)

$$\tau_{rel}(k) \equiv \frac{k^2}{\hat{q}} \sim \frac{1}{\alpha^2 \, n_f^2 \, k_f} \left(\frac{k}{k_f} \right)^2$$

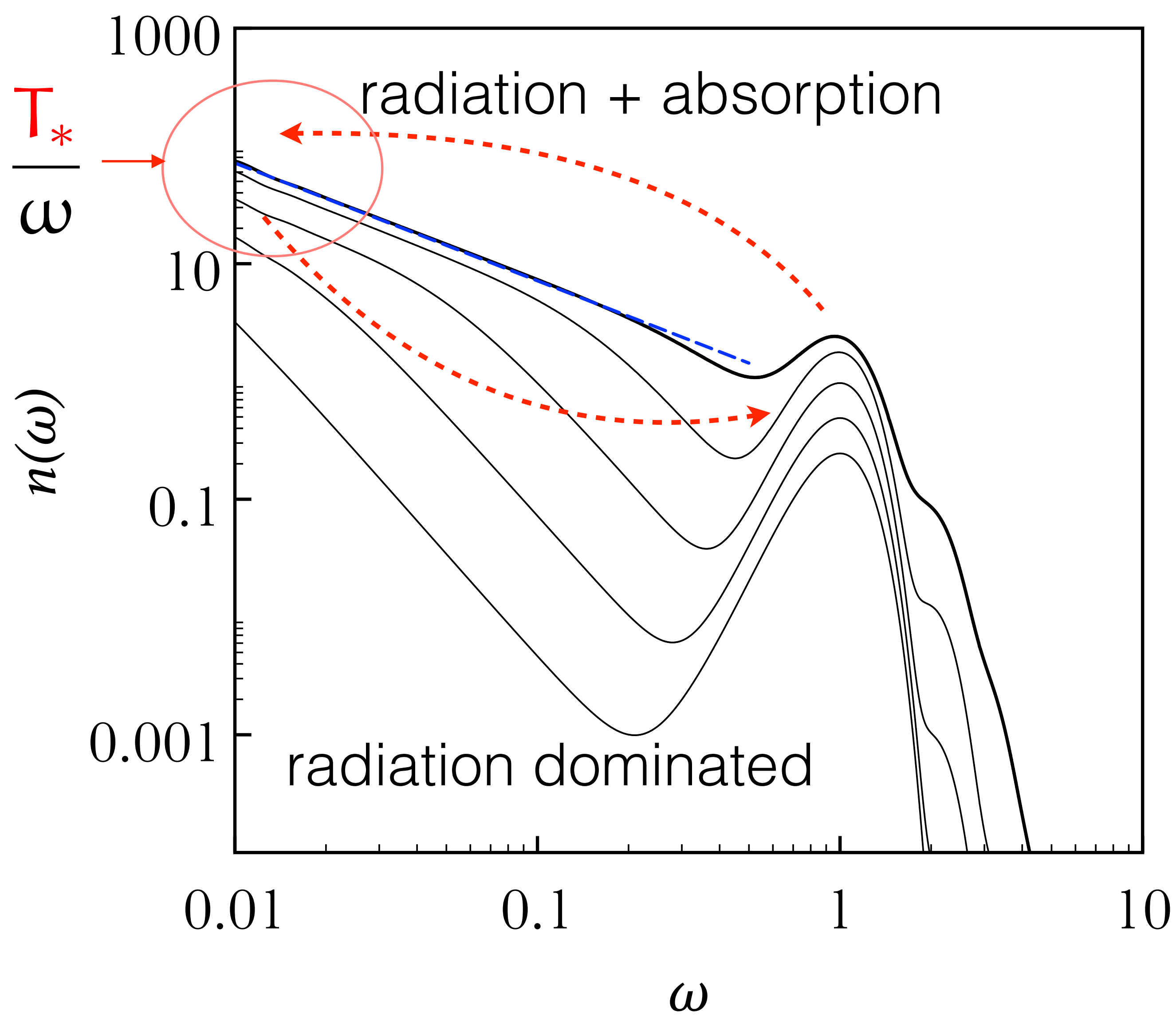
- The solution exhibits an essential singularity (instantaneous thermalization of the zero mode)

$$n(k) = n_{eq}(k) \left[1 - \exp\left(-\frac{t}{\tau_{rel}} \right) \right] \quad \rightarrow \frac{T_*}{k} \quad \text{at late times}$$

- Below the forcing the system thermalizes rapidly **(no fluxes)**

Thermalization of the soft sector ($k \ll k_f$)

The early times dynamics (Numerical simulation of the kinetic equation in the BH regime)

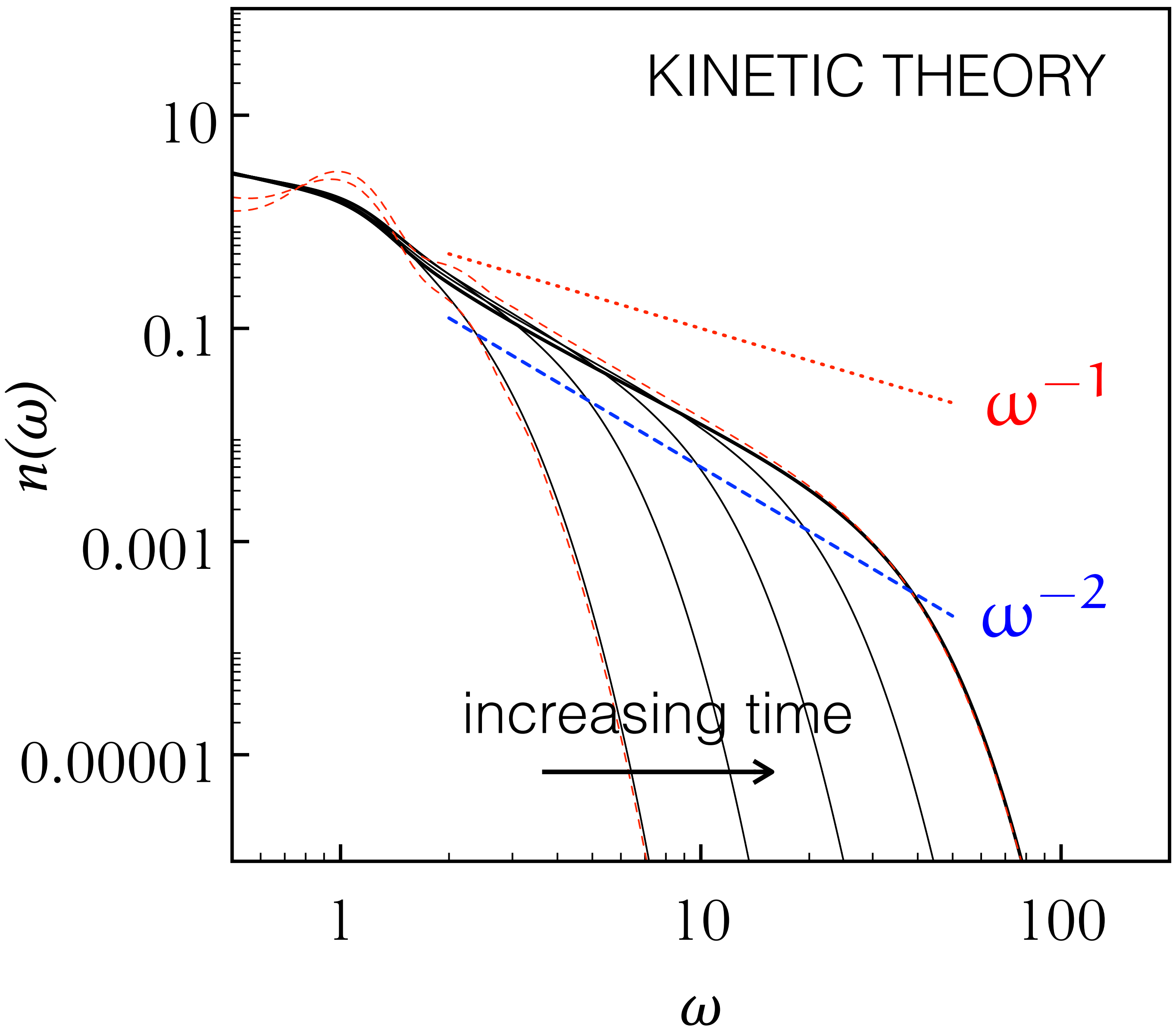


Below $\omega_*(t) \equiv \sqrt{\hat{q}t}$ the system is in thermal equilibrium although it is constantly driven out of equilibrium

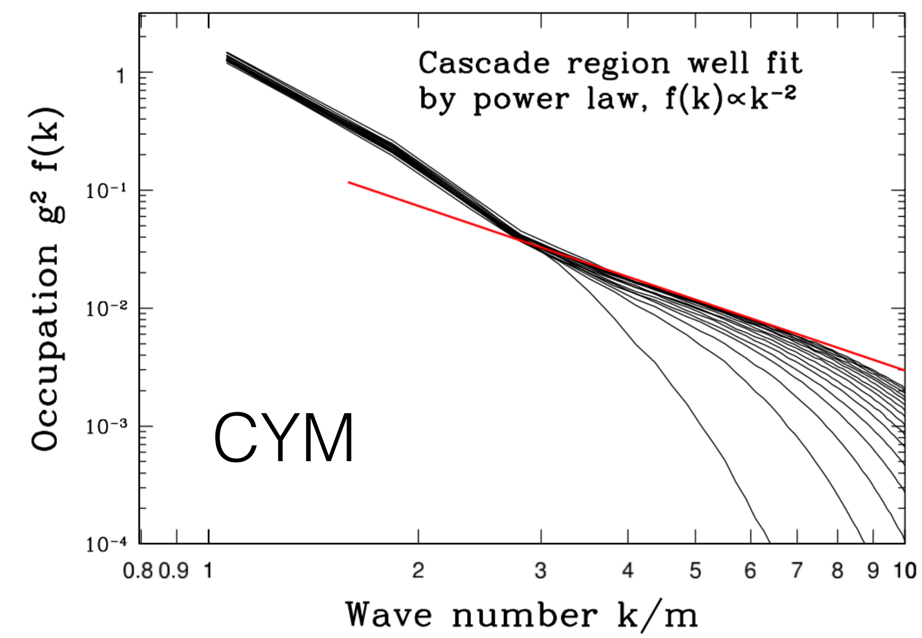
$(\omega = k)$

The thermal bath interacts mainly with the source: $\hat{q} \sim \alpha^2\, \omega_f^3\, n^2(\omega_f)$

Interplay between elastic and inelastic processes (I)



P.Arnold, G. D. Moore (2005)



- In the presence of elastic processes ω^{-2} spectrum persists.

Interplay between elastic and inelastic processes (II)

- **Heuristic analysis:** for a spectrum falling faster than ***1/k*** one can neglect the drag term in the elastic part. Then, the collision integral in the UV reads

$$\frac{\partial}{\partial t} n(k) \simeq \frac{\hat{q}_{inel}}{4k^3} \frac{\partial}{\partial k} k^3 \frac{\partial}{\partial k} n(k) + \frac{\hat{q}_{el}}{4k^2} \frac{\partial}{\partial k} k^2 \frac{\partial}{\partial k} n(k)$$

$$\equiv \frac{B}{4k^{3-\beta}} \frac{\partial}{\partial k} k^{3-\beta} \frac{\partial}{\partial k} n(k)$$

where $B = \hat{q}_{inel} + \hat{q}_{el}$ and $\beta = \frac{2}{1 + \hat{q}_{inel}/\hat{q}_{el}}$

- **Steady state solution:** $n(k) \sim \frac{1}{k^{2-\beta}}$ $\quad 0 < \beta < 1$

- k^{-2} no longer a fixed point. One could expect that at late times the spectrums flattens toward k^{-1}

Summary and outlook

- Wave turbulence in QCD is different from scalar theories. It is dominated by **nonlocal interactions in momentum space**: Kolmogorov-Zakharov spectra are not physically relevant
- **Inelastic processes dominates** the dynamics with a direct energy cascade
- To the right of the forcing scale: we find a **quasi steady state spectrum** $\sim \mathrm{k}^{-2}$ (in the LPM and BH regimes) in agreement with Classical Yang-Mills simulations
- Outlook: mass corrections, anisotropic fluxes, strong turbulence in the presence of strong fields (on the lattice): different exponents?

Backup

Wave Turbulence (III): example, NLS equation

- **Classical:** non-linear wave equation (e.g. Nonlinear Schrödinger eq)

$$i\frac{\partial}{\partial t}\Psi = -\frac{\nabla_x^2}{2m}\Psi + \lambda|\Psi|^2\Psi$$

- Randomness in initial condition $\Psi(x, t = 0)$
- Observable: **occupation number**

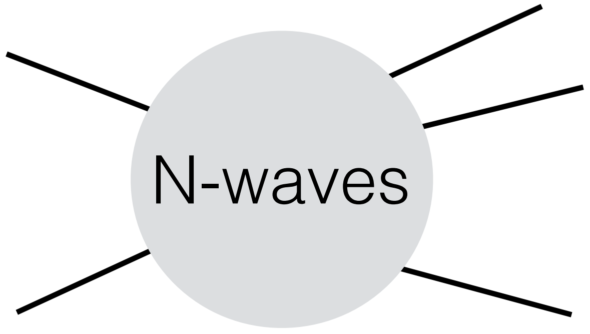
$$\langle a^*(k)a(k')\rangle \equiv n(k)\,\delta(k-k')$$

- Dispersion relation

$$\omega(k) = k^2/2m$$

- **Weak turbulence:** **(1)** NonLinear/Linear << 1 **(2)** Random-Phase-Approximation ⇒ **Kinetic description**

$$\frac{\partial}{\partial t}n(k) \equiv I_{coll}[n] \sim n^{N-1}$$



V. E. Zakharov, V. S. L'vov, G. Falkovich (Springer- Verlag, 1992)

Dual cascade: Fjørthoft argument (1953)

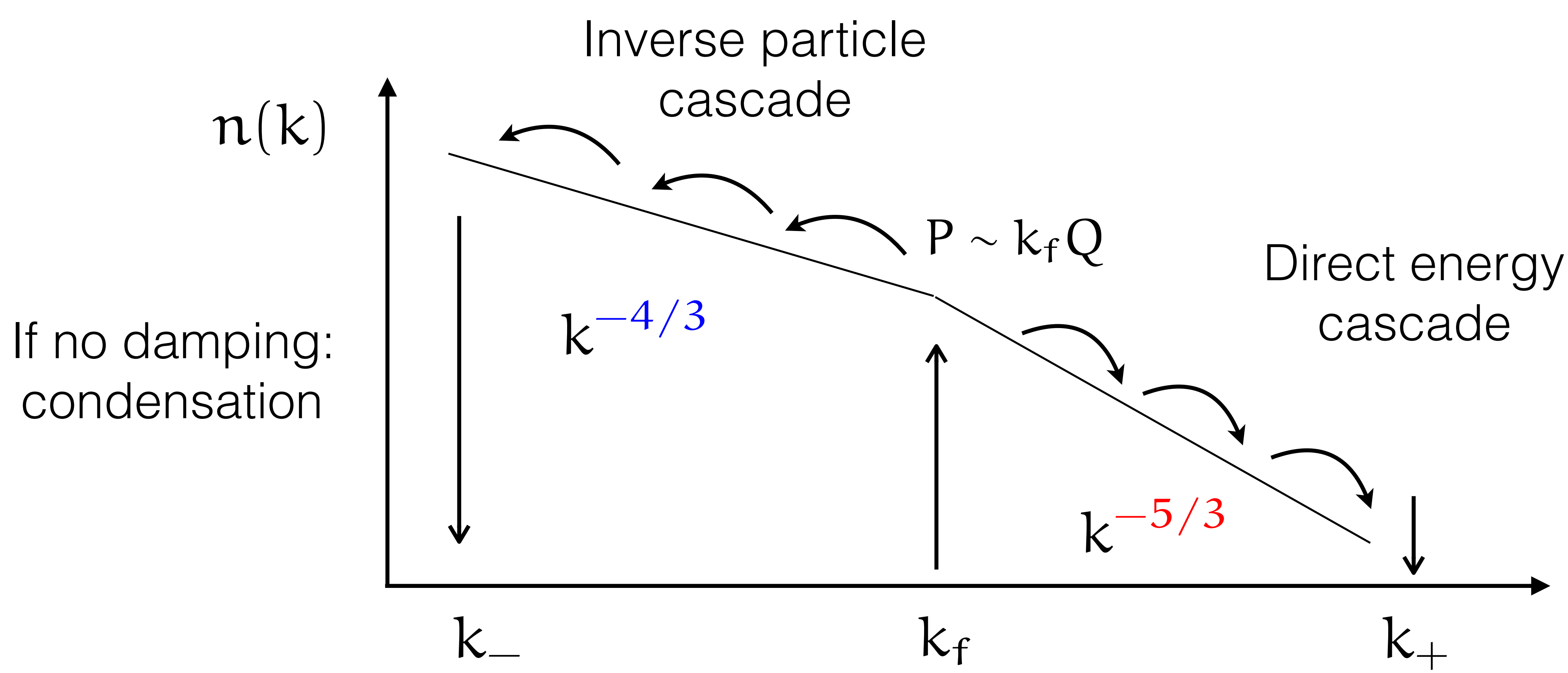
- Q: Direction of fluxes? Injection of energy at k_f and dissipating at

$$k_- \ll k_f \ll k_+$$

- *Reductio ad absurdum:* If energy was dissipating at low momenta then particles would dissipate faster than the pumping rate! $\Longrightarrow$ **Direct energy cascade**

$$Q_- \sim \frac{P}{k_-} \gg \frac{P}{k_f} \sim Q$$

absurd



Thermalization of the soft sector