Artificial entropy viscosity applied to relativistic hydrodynamics

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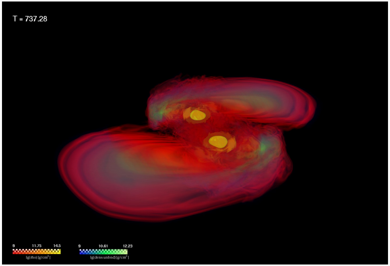
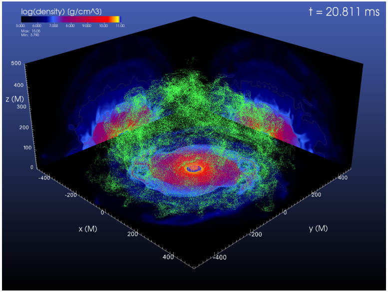
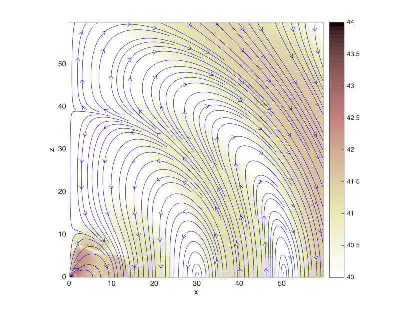
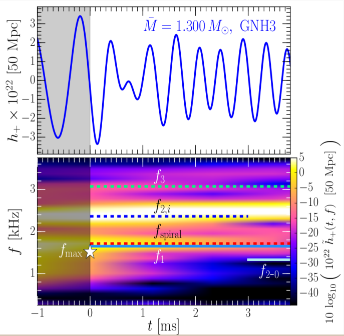
Issues in large numerical simulations









Issues in large numerical simulations

We need accurate and computationally efficient numerical methods

They should ideally be:

- accurate
- fast
- parallelizable
- scalable

. . . and hopefully easy to implement.

Euler equations

$$\nabla_\mu(\rho u^\mu) = 0$$

$$u^\mu \nabla_\mu u_\nu + \frac{1}{\rho h} P^\mu{}_\nu \nabla_\mu p = 0$$

$$u^\mu \nabla_\mu e + \rho h \nabla_\mu u^\mu = 0$$

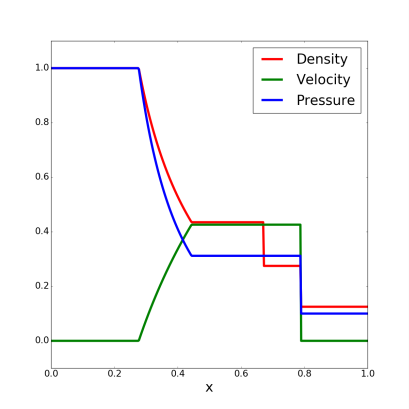
$$h = \frac{e+p}{\rho} \quad P_{\mu\nu} = g_{\mu\nu} + u_\mu u_\nu$$

$$\partial_t \mathbf{U} + \partial_i \mathbf{F}^i(\mathbf{U}) = \mathbf{S}(\mathbf{U})$$

"Valencia formulation"

(Banyuls et al., AstrophysJ 178 (1997))

Several issues with these equations, most importantly the generation of shocks

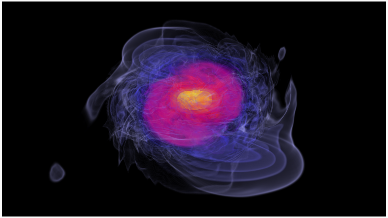


WhiskyTHC

WhiskyTHC: general relativistic Templated Hydrodynamics Code.

It includes:

- finite-differences and finite-volumes methods
- several flux limiting schemes (MP5, WENO, Lim03, MinMod)
- a positivity preserving limit
- ...



and sports the highest convergence order achieved in BNS simulations (3.2 using MP5)

Finite differences

In the case of a scalar conservation law in one dimension:

$$\partial_t U + \partial_x F(U) = 0$$

Define $h(x)$ so that $F[U(x_i)] = \dfrac{1}{\Delta x}\displaystyle\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} h(x')dx'$

Then $\partial_t U = -\left.\dfrac{\partial F}{\partial x}\right|_{x_i} = -\dfrac{h\left(x_{i+\frac{1}{2}}\right) - h\left(x_{i-\frac{1}{2}}\right)}{\Delta x}$

Given the point values of F, which are the cell averages of h, the point values of h are computed by a odd order finite-differences stencil.

The time evolution is then taken care of by a Runge-Kutta method (usually strong stability preserving and 3rd order accurate).

High Resolution Shock Capturing methods

To deal with shocks HRSC methods (High Resolution Shock Capturing) are generally used, such as:

- Monotonicity preserving 5th order method (MP5)
  Limit the numerical flux into a solution-dependent interval to tame shocks without compromising accuracy
- (Weighted) Essentially non-oscillatory methods ((W)ENO)
  Reconstruct the fluxes on every possible stencil of a given order, then weight the contributions with solution-dependent weights

Such methods perform quite well (especially MP5) but they have some drawbacks: they are somewhat slow, overly diffusive, difficult to implement and to generalize.

Artificial viscosity

An additional term in the Right Hand Side

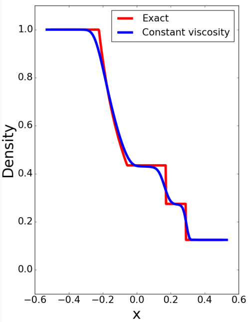
$$\Downarrow$$

$$\partial_t \mathbf{U} = RHS_{Euler}(\mathbf{U}) - \partial_i \left(\nu \partial^i \mathbf{U}\right)$$

mimicking similar terms in the Navier-Stokes equations.

How to choose ν?

- constant ν
- active in case of compression ($\partial_t \rho > 0$)
    - Severe smearing of shocks
- bulk ν proportional to $\nabla_i v^i$
    - Instabilities associated with strong couplings to the velocity for high W



Entropy viscosity

All systems of conservation laws admit an entropy functional.

For the Euler equations: the physical entropy S, *e.g.*
$S \simeq \rho \log\left(\frac{\epsilon}{\rho^{\gamma-1}}\right)$
This satisfies:

$$R = \nabla_\mu (S u^\mu) \geq 0$$

Entropy production

$R = \delta(x^\mu - a^\mu)$ *where* a^μ *is the location of shocks*

Therefore $\nu = min(c_e\ \Delta x^2\ R, c_{max}\ \Delta x\ \lambda)$ should be a good choice for the viscosity.

where Δx is the grid spacing, $\lambda = 1$ the maximum wave speed and $c_{e/max}$ are tunable constants with typical values of 1 and 0.5 (first-order upwind scheme)

Entropy viscosity

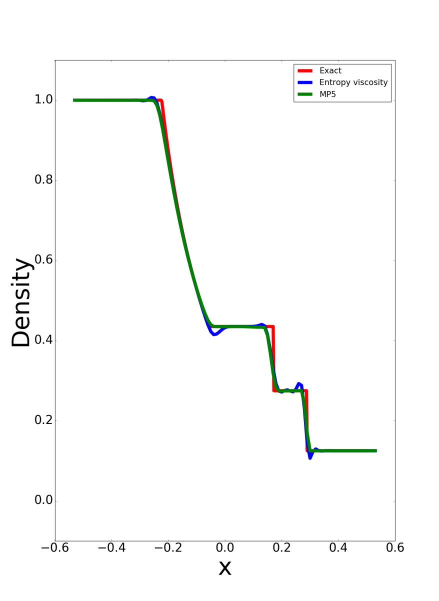
Reworking R and casting it in 3+1 formalism:

$$R = \rho u^\mu \nabla_\mu s = \left(\frac{\rho W}{\alpha}\right)\left[\partial_t s + (\alpha v^i - \beta^i)\partial_i s\right]$$

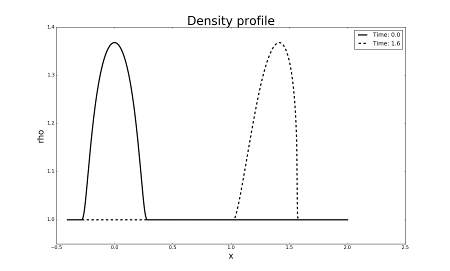
The derivatives are evaluated via finite differences

Smoothing:	Low density boost:
$\bar{\nu}_i = \frac{1}{4}(\nu_{i-1} + 2\nu_i + \nu_{i+1})$	$\rho_{ijk} < \rho_{threshold} \;\Rightarrow\; \nu_{ijk} = \nu_{max}$
To mitigate large gradients and account for the time lag of the viscosity.	To quench chaotic low-density dynamics.

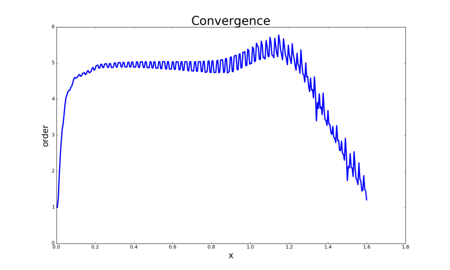
1D Tests



Exact
Entropy viscosity
MP5
Density
x

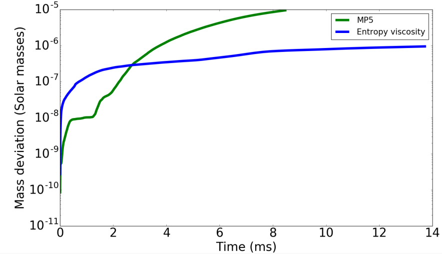


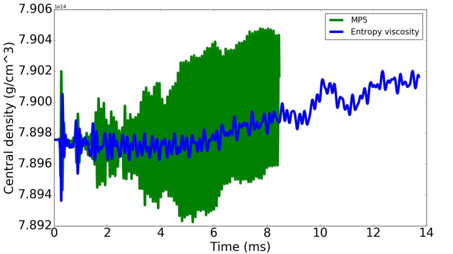
Density profile
Time: 0.0
Time: 1.6
rho
x

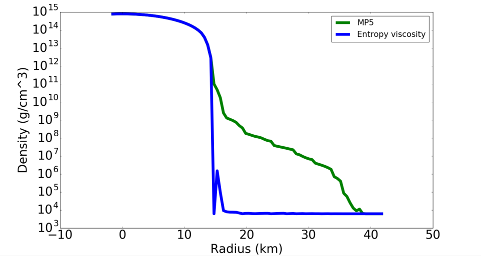


Convergence
order
x

Applications: Isolated neutron star

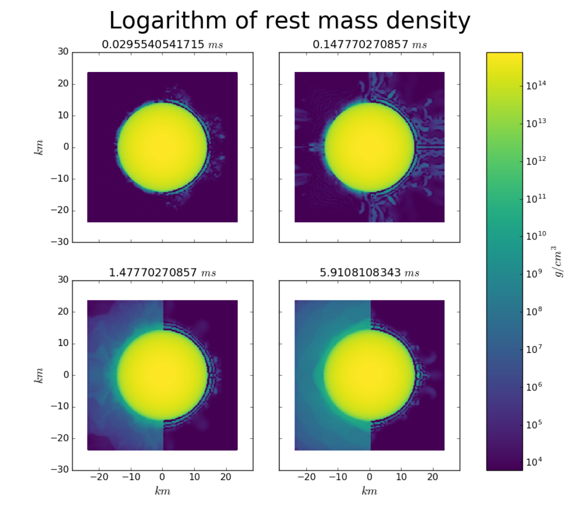




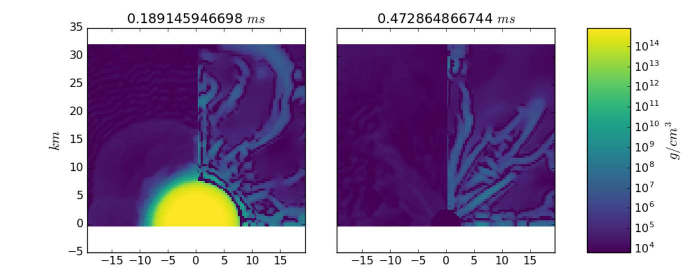


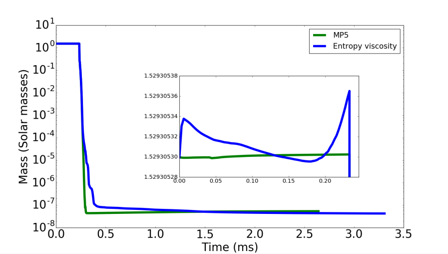
Applications: Isolated neutron star

Left: MP5 Right: Entropy viscosity



Applications: Unstable star collapse





In these tests there is no tuning of coefficients: $c_e = 1$ and $c_{max} = 0.5$.

Conclusions

An artificial viscosity scheme with the entropy viscosity prescription is a viable alternative to commonly used methods in numerical relativistic hydrodynamics. It satisfies the requirements of:

- Accuracy
- Performance
- Parallel implementation
- Method independence

The next goal: combine entropy viscosity with discontinuous Galerkin methods. This would address the issue of Scalability.

Conclusions

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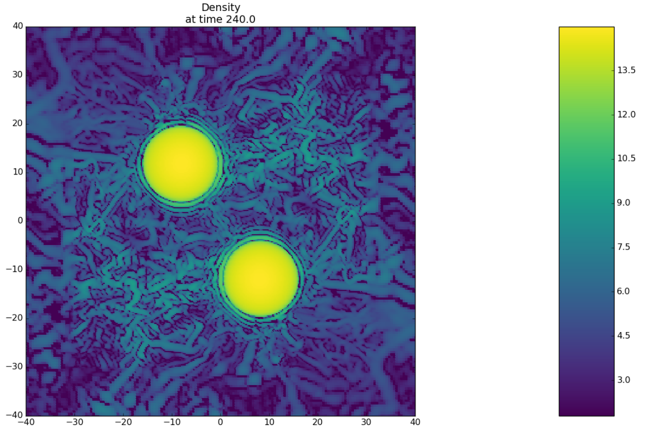
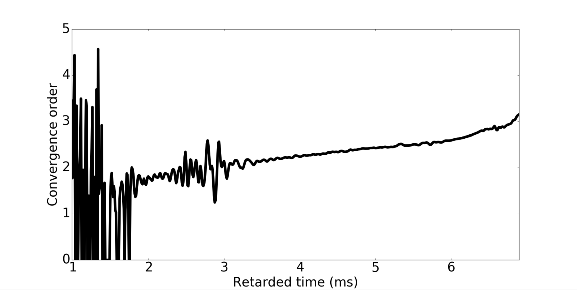
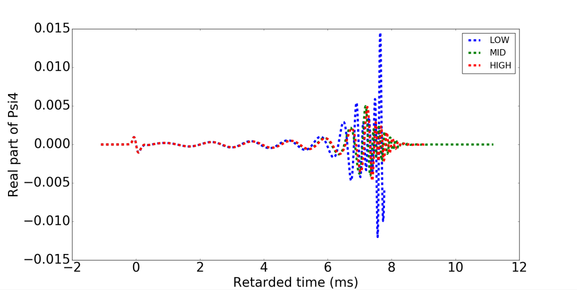
Thank you for your attention

Applications: Binary neutron stars



Applications: Binary neutron stars





Applications: Binary neutron stars