Force-free magnetospheres

Taner Akgün

(U. Alicante, Spain)

Collaborators

José Pons *(U. Alicante, Spain)*
Juan Antonio Miralles *(U. Alicante, Spain)*
Pablo Cerdá-Duran *(U. Valencia, Spain)*

Motivation

Neutron star magnetospheres (with rotation): Goldreich & Julian (1969); Michel (1973; 1974)...

Force-free approximation: plasma pressure & inertia negligible compared to the electromagnetic field.

Pulsar equation: (Michel 1973; Contopoulos, Kazanas & Fendt 1999; Spitkovsky 2006)...

Caveat: force-free may not be satisfied everywhere (Beloborodov & Thompson 2007).

Motivation

Magnetars have relatively slow rotation, so we neglect it.

The pulsar equation reduces to the Grad-Shafranov equation.

Important in astrophysics & plasma physics, but limited analytic solutions available...

Numerical solutions: Glampedakis, Lander & Andersson (2014); Pili, Bucciantini & Del Zanna (2015).

Motivation

The presence of twist (toroidal field) is suggested by observations (magnetar spectra).

Possibly maintained by helicity transfer from the interior, implying that the magnetosphere is not current-free.

May be of importance for long-term evolution.

In our case:
Interior is determined by the evolution (core+crust).
Exterior adjusts (instantaneously) to the surface.

Introduction

Notation,

$$\boldsymbol{B} = \boldsymbol{\nabla} P \times \boldsymbol{\nabla} \phi + T \boldsymbol{\nabla} \phi$$

$$\frac{4\pi \boldsymbol{J}}{c} = \boldsymbol{\nabla} \times \boldsymbol{B} = -\triangle_{\mathrm{GS}} P \boldsymbol{\nabla} \phi + \boldsymbol{\nabla} T \times \boldsymbol{\nabla} \phi$$

In a static axisymmetric fluid, the Lorentz force cannot have an azimuthal component, so T must be a function of P. Then,

$$4\pi \boldsymbol{f} = (\boldsymbol{\nabla} \times \boldsymbol{B}) \times \boldsymbol{B} = -\frac{\triangle_{\mathrm{GS}} P + G(P)}{\varpi^2} \boldsymbol{\nabla} P$$

Grad-Shafranov equation

In a barotropic fluid, the force density must further be expressible as the gradient of a potential. Thus,

$$\triangle_{\mathrm{GS}} P + G(P) = \varrho\varpi^2 F(P)$$

Force-free F=0, current-free G=F=0.

Auxiliary definitions

Magnetic energy and helicity,

$$8\pi E = \int B^2\, dV = \oint B^2 (\boldsymbol{r}\cdot d\boldsymbol{S}) - 2\oint (\boldsymbol{r}\cdot\boldsymbol{B})(\boldsymbol{B}\cdot d\boldsymbol{S})$$

$$H = \int \boldsymbol{A}\cdot\boldsymbol{B}\, dV = 2\int A_\phi B_\phi\, dV$$

Twist,

$$\varphi \equiv \Delta\phi = \int_0^\ell \frac{B_\phi}{B_{\rm pol}\, r \sin\theta}\, d\ell$$

Multipole content,

$$a_l \equiv \frac{r^l A_l(r)}{R_\star^l}$$

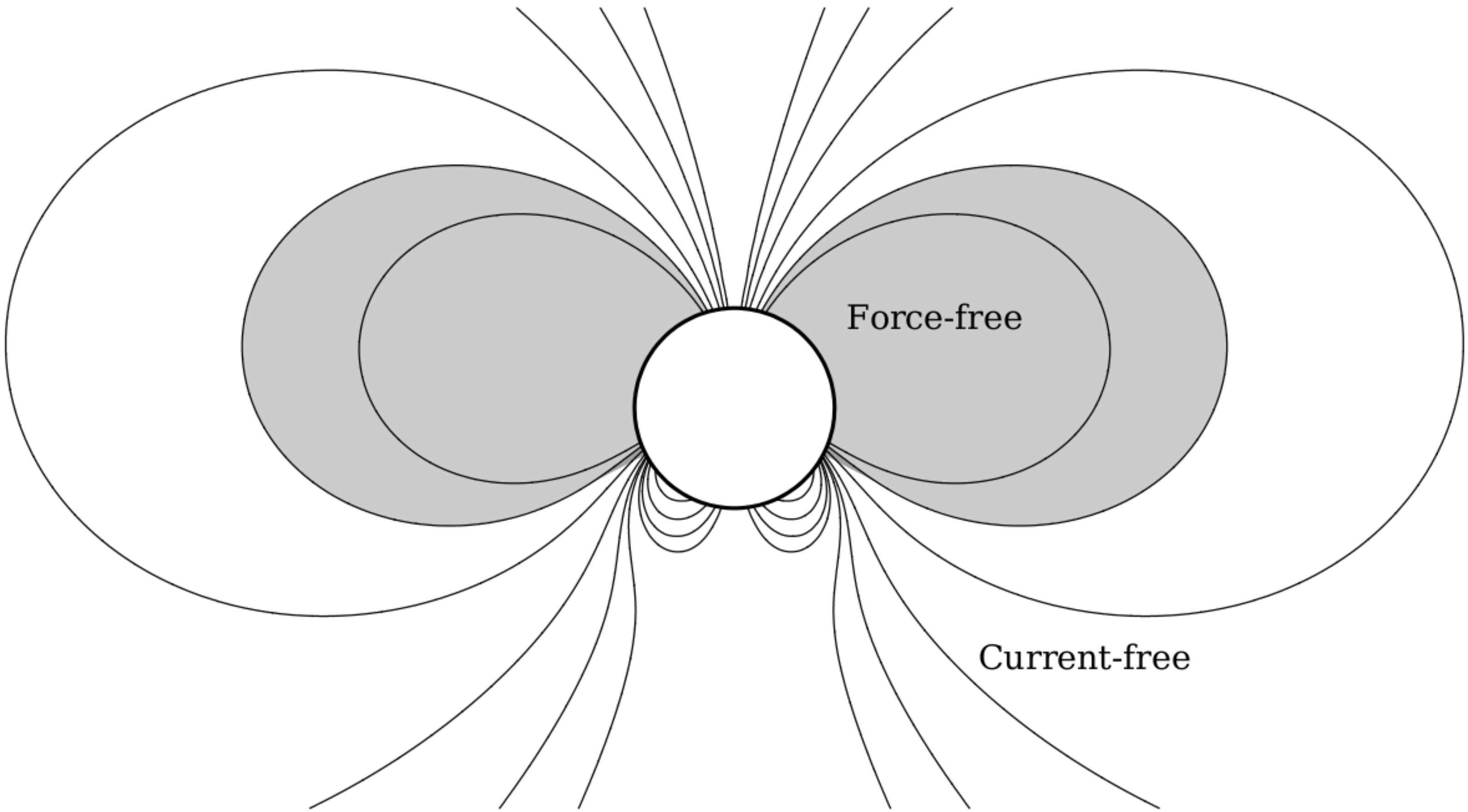
Force-free model

Toroidal function,

$$T(P) = \begin{cases} s(P - P_c)^\sigma & \text{for} \quad P \geqslant P_c \ , \\ 0 & \text{for} \quad P < P_c \ . \end{cases}$$

Three parameters: *s*, P_C and σ.

Ranges of the parameters:
- *s* (controls the magnitude of the toroidal field), positive;
- P_C (controls the volume of the toroidal field), between 0 and 1;
- σ, must be 1 or larger.

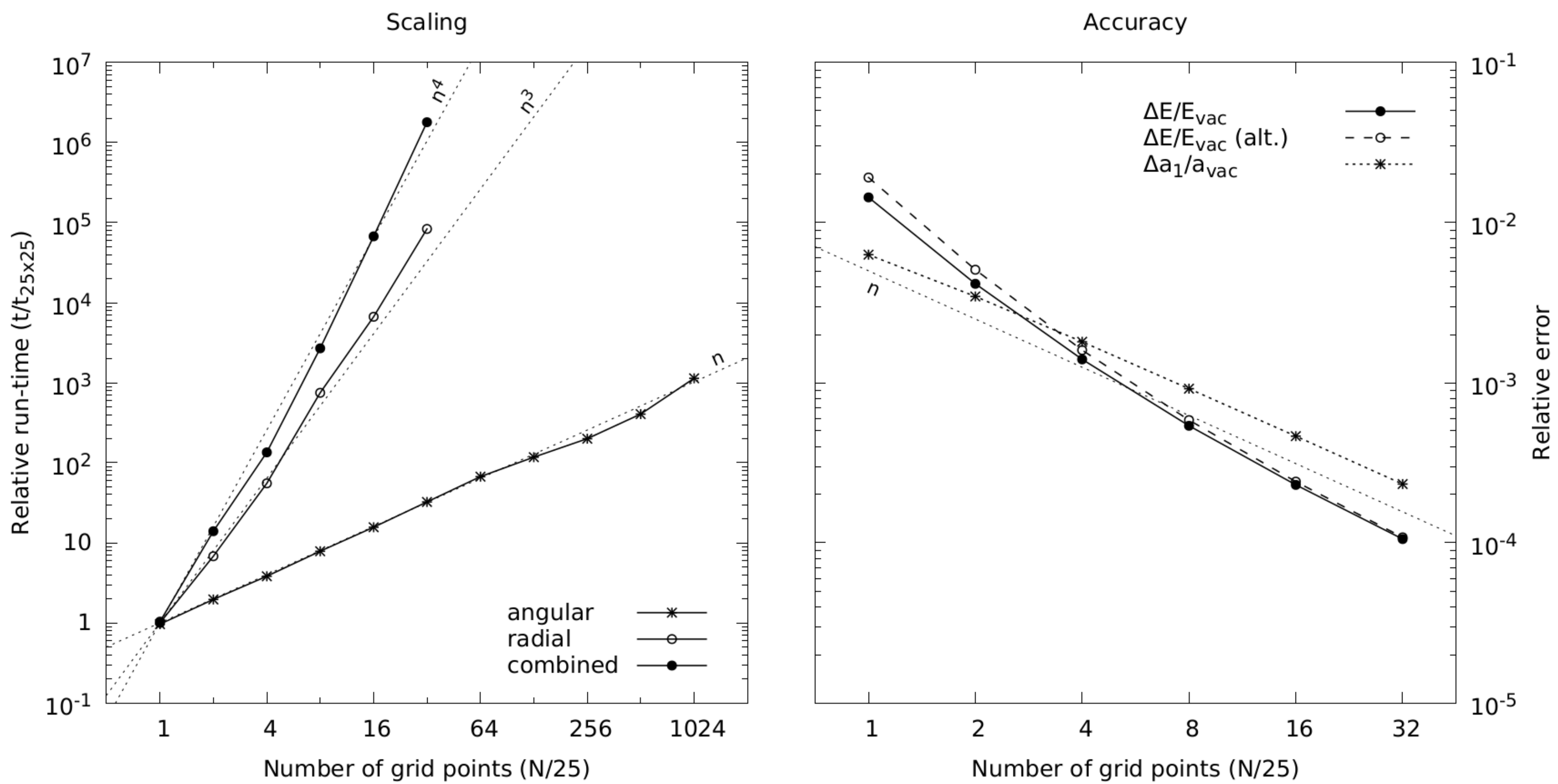

Force-free
Current-free

Numerical implementation

Discretize and solve numerically the linear system for the unknown function *P*.

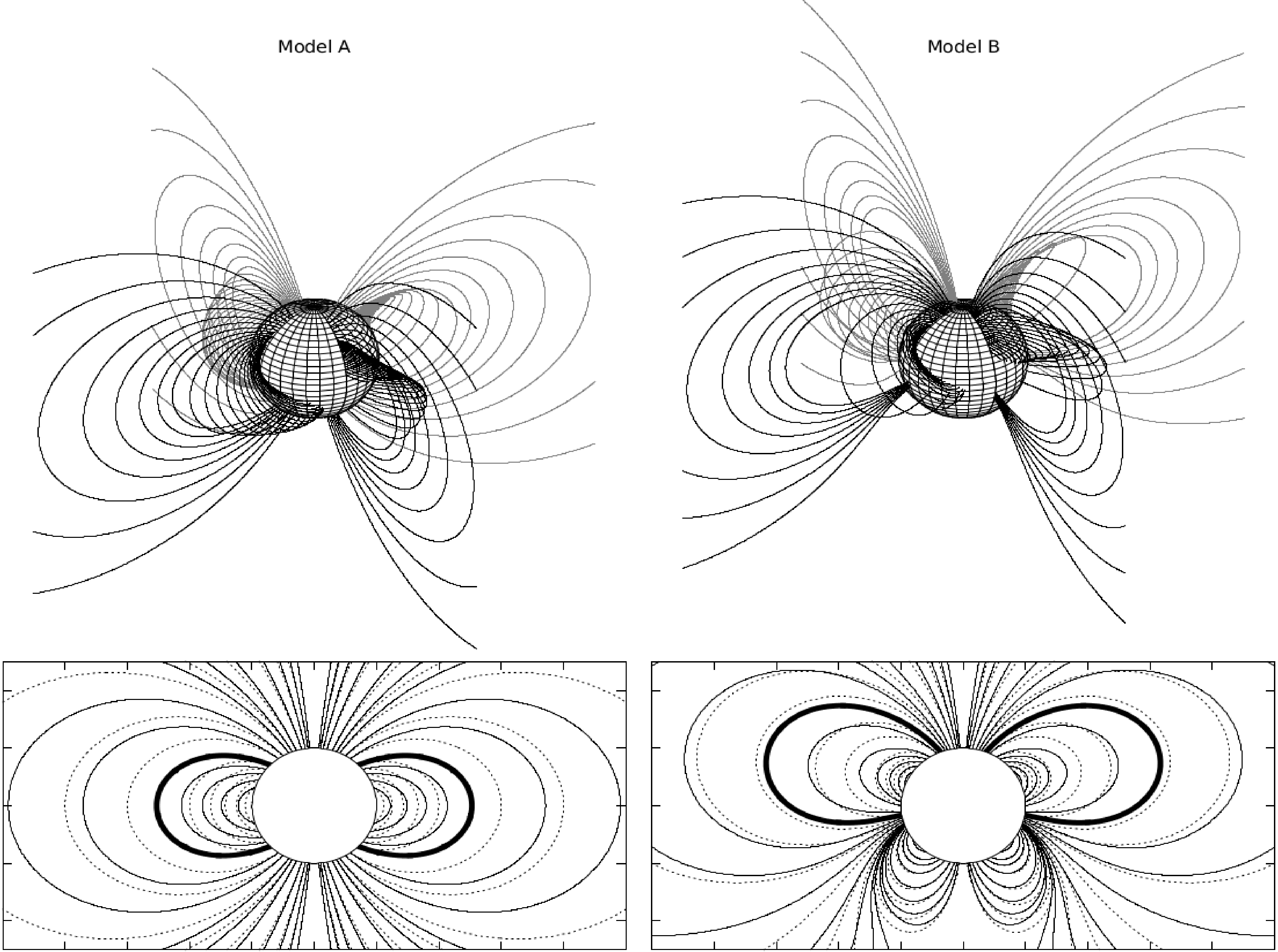
Boundary conditions:
- current-free at the surface (different from Glampedakis, Lander & Andersson 2014 and Pili, Bucciantini & Del Zanna 2015),
- along the axis,
- and at an external radius.

Iterations are needed since *T* is a non-linear (step) function of *P*, until convergence is achieved.



Scaling
Relative run-time (t/t_{25x25})
10^7
10^6
10^5
10^4
10^3
10^2
10
1
10^{-1}
n^4
n^3
n
angular
radial
combined
1
4
16
64
256
1024
Number of grid points (N/25)
Accuracy
$\Delta E/E_{vac}$
$\Delta E/E_{vac}$ (alt.)
$\Delta a_1/a_{vac}$
n
Relative error
10^{-1}
10^{-2}
10^{-3}
10^{-4}
10^{-5}
1
2
4
8
16
32
Number of grid points (N/25)

Results

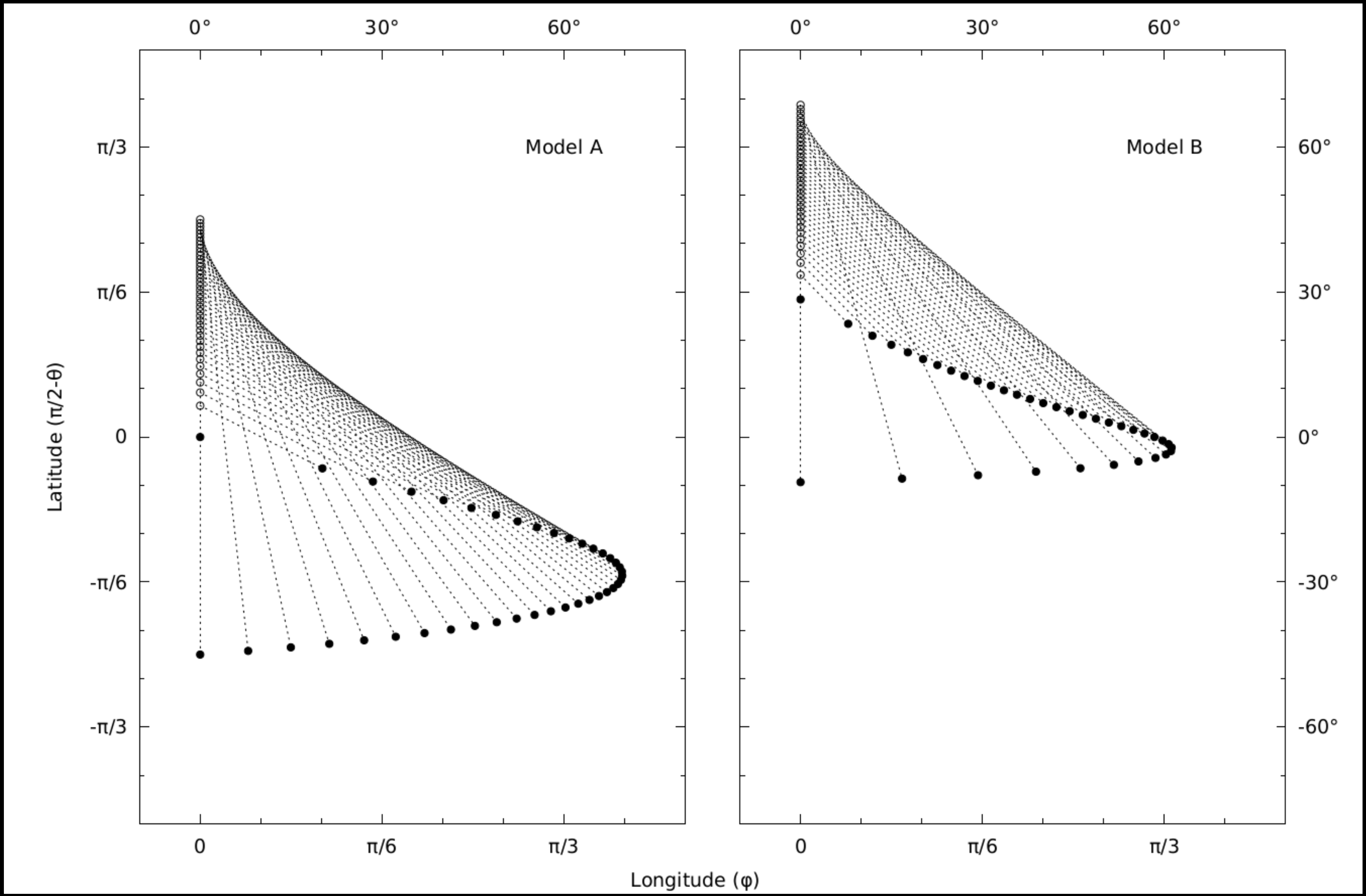


Model A
Model B

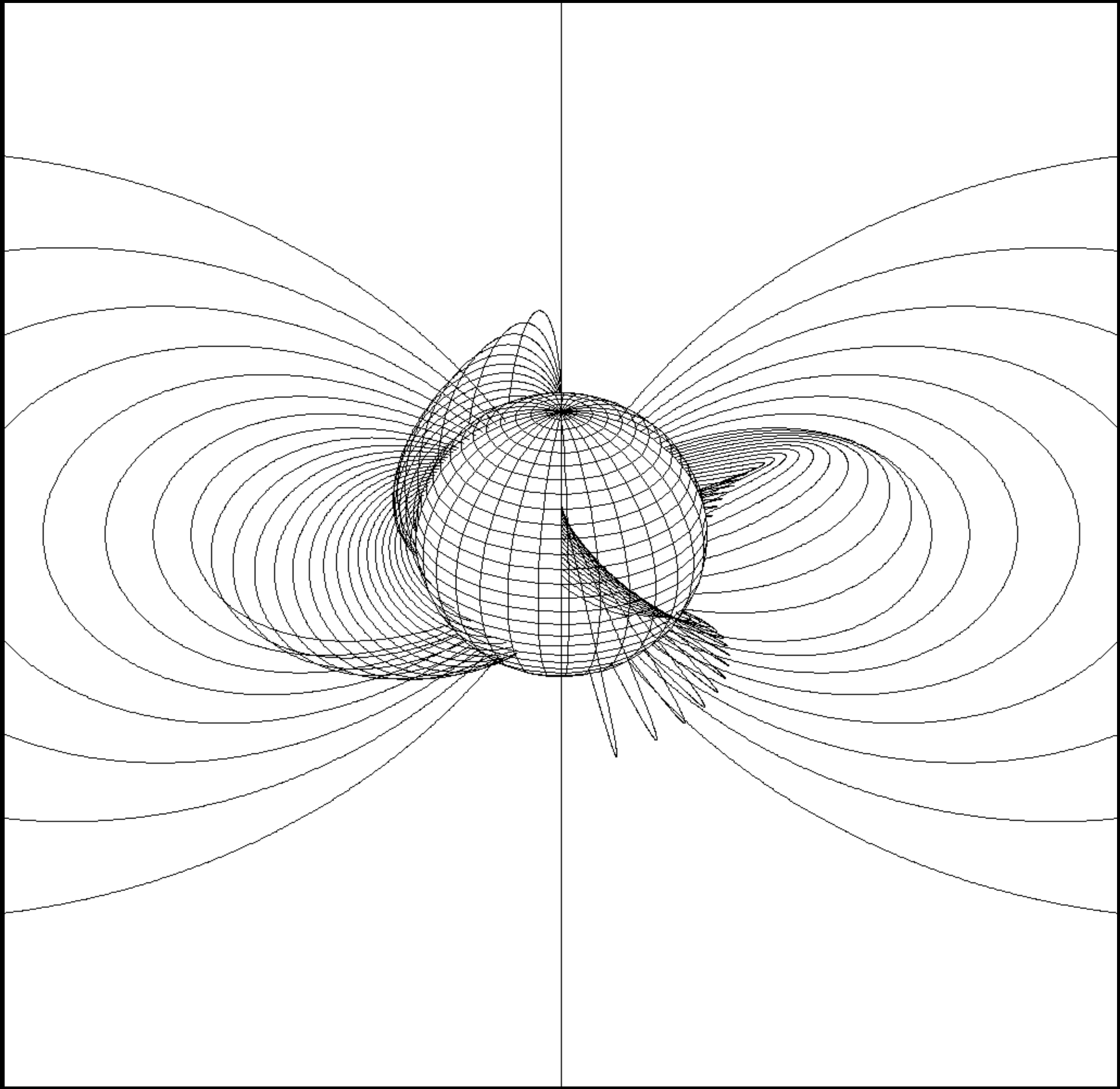
Parameters

	model A	model B
Parameters:		
w	1	0.5
s	1.6	1.15
P_{c}	0.5	0.25
σ	1	1
Derived quantities:		
Energy, E	0.393	0.432
Energy, E/E_{vac} (%)	118%	113%
Helicity, H	4.19	3.24
Maximum twist, φ_{max}	1.22	1.07
Dipole strength, a_1	1.22	0.699
Quadrupole strength, a_2	-	0.655

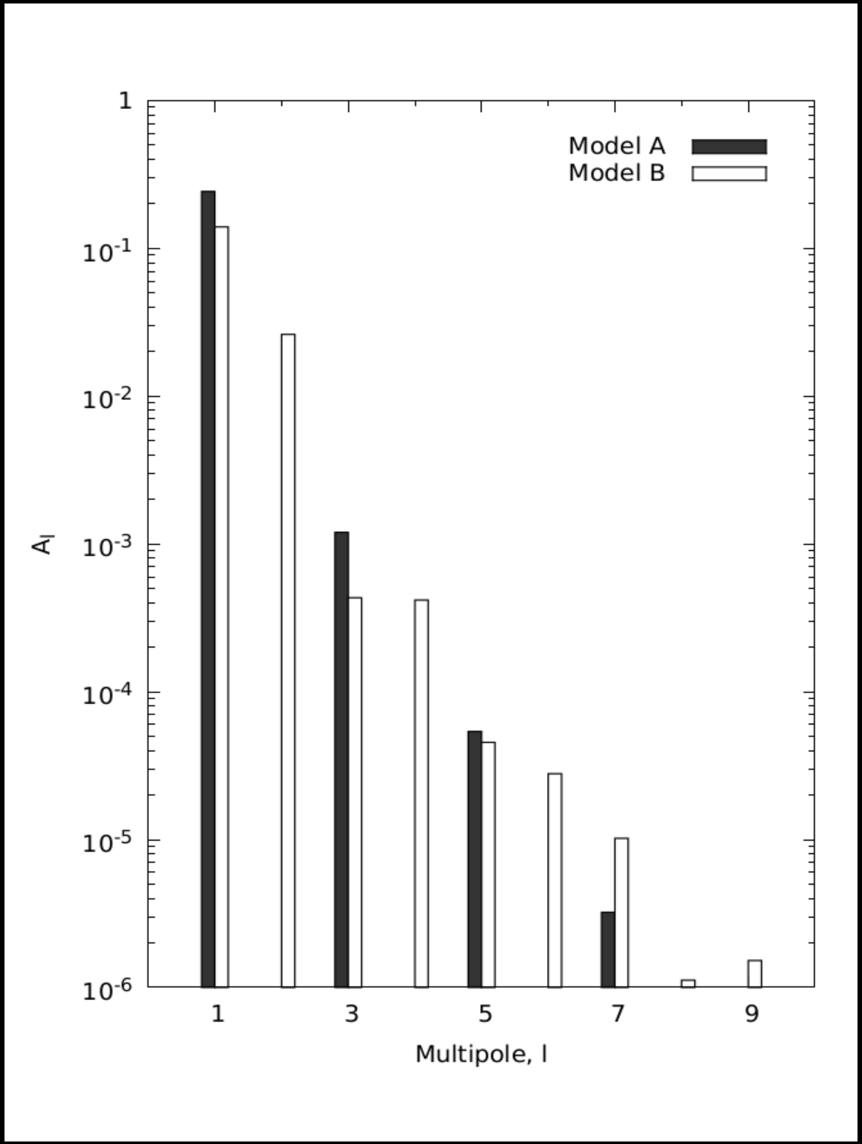
Twist map



0°
30°
60°
0°
30°
60°
Model A
Model B
π/3
π/6
0
-π/6
-π/3
60°
30°
0°
-30°
-60°
Latitude (π/2-θ)
0
π/6
π/3
0
π/6
π/3
Longitude (φ)



Multipole content



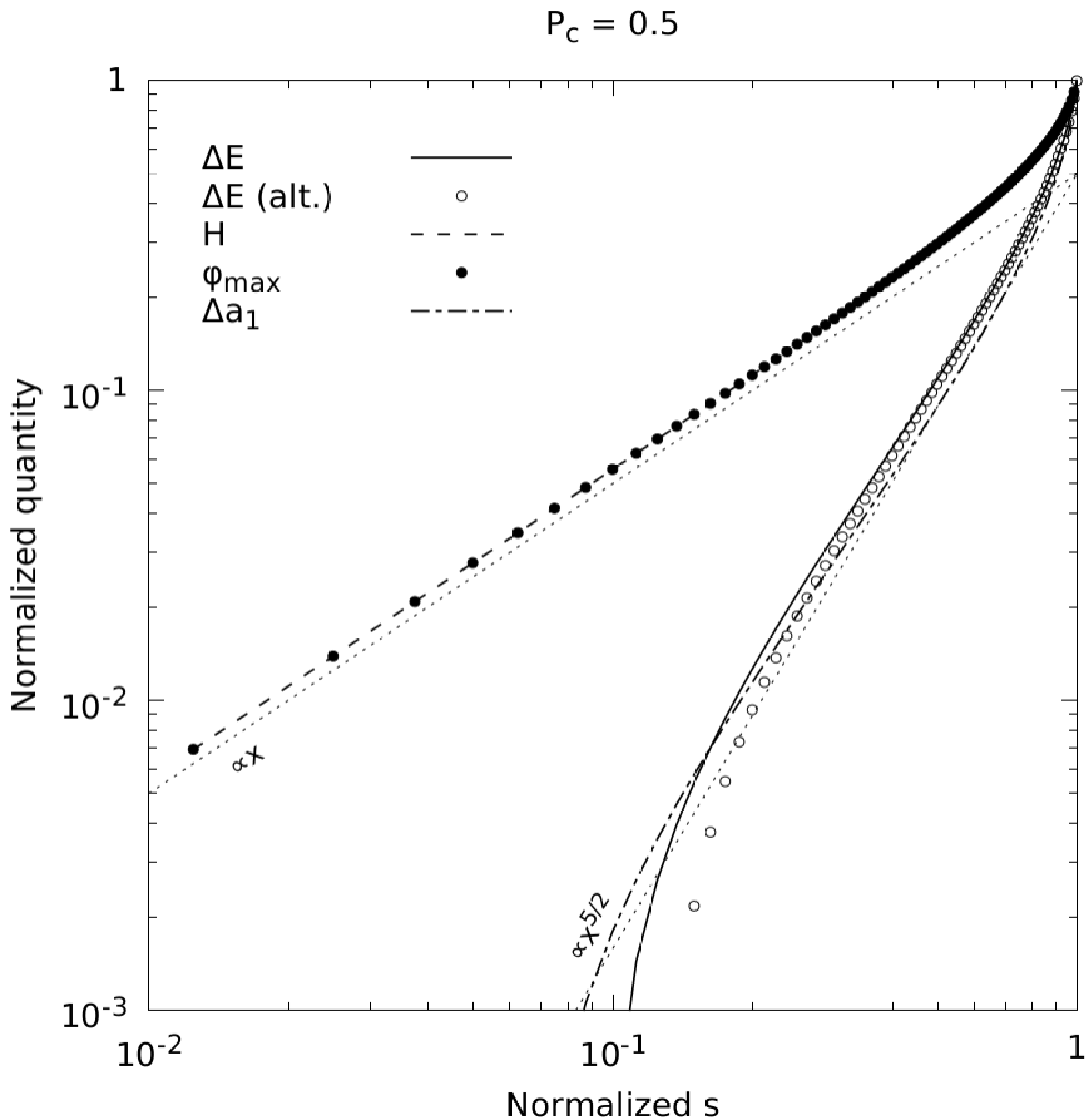
Results

Some observations:

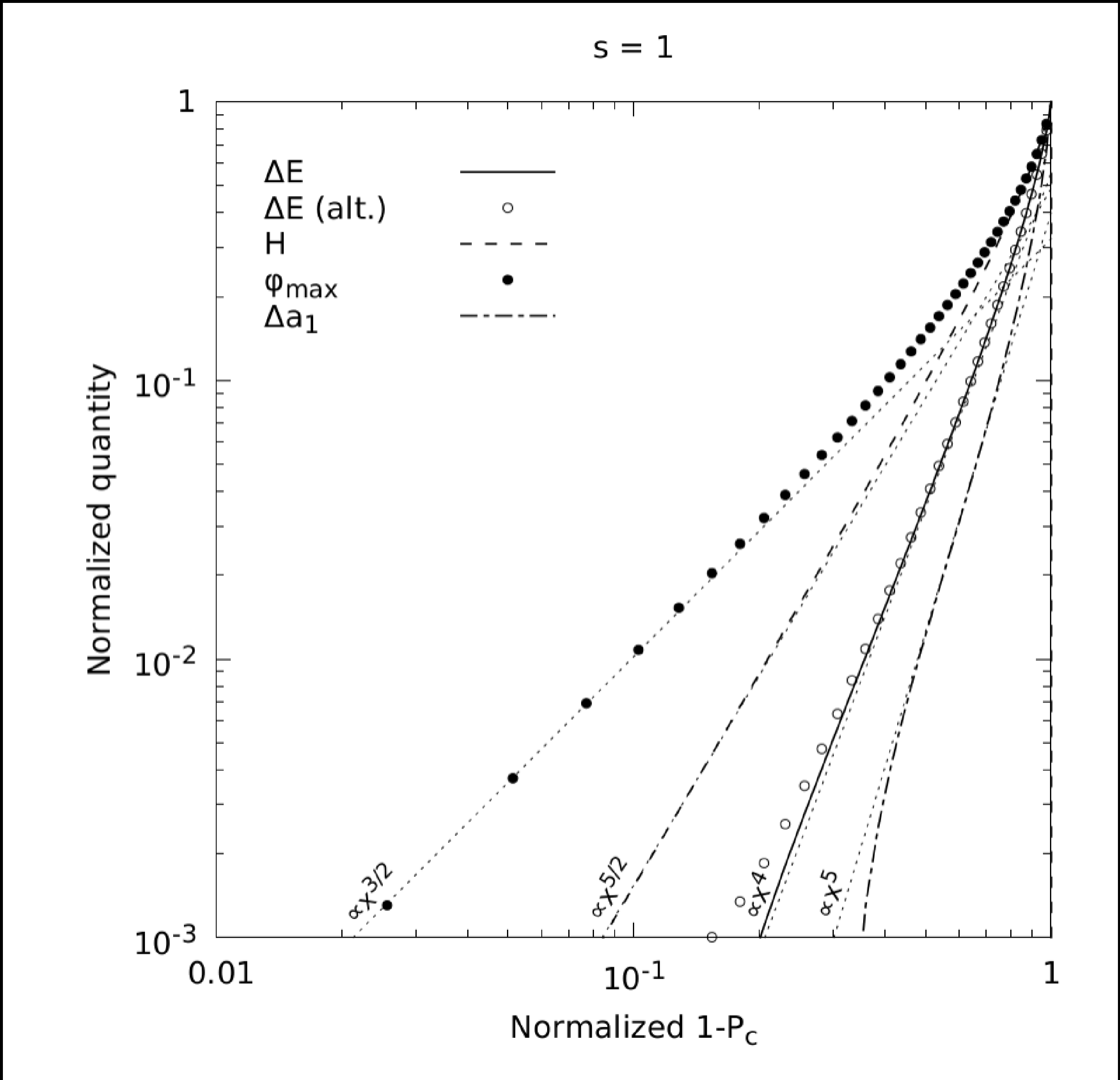
- field lines expand with increasing *s*;
- the dipole content increases compared to the surface by up to ~20%;
- there is a maximum *s*;
- the maximum twist is nearly constant for the maximum *s*.

Relations between parameters

… quantities as functions of s and of P_C, for σ=1.



$P_c = 0.5$
1
10^{-1}
10^{-2}
10^{-3}
10^{-2}
10^{-1}
1
Normalized quantity
Normalized s
ΔE
ΔE (alt.)
H
φ_{max}
Δa_1
$\propto x$
$\propto x^{5/2}$



s = 1
ΔE
ΔE (alt.)
H
φmax
Δa1
∝x3/2
∝x5/2
∝x4
∝x5
Normalized quantity
Normalized 1-Pc
1
10-1
10-2
10-3
0.01
10-1
1

Mathematical construct

Assume a dipole field,

$$P(r,\theta) = \frac{\sin^2\theta}{r}$$

… and we can calculate the helicity, twist and maximum twist (as functions of s and P_C).

Limiting values for P_C near 1,

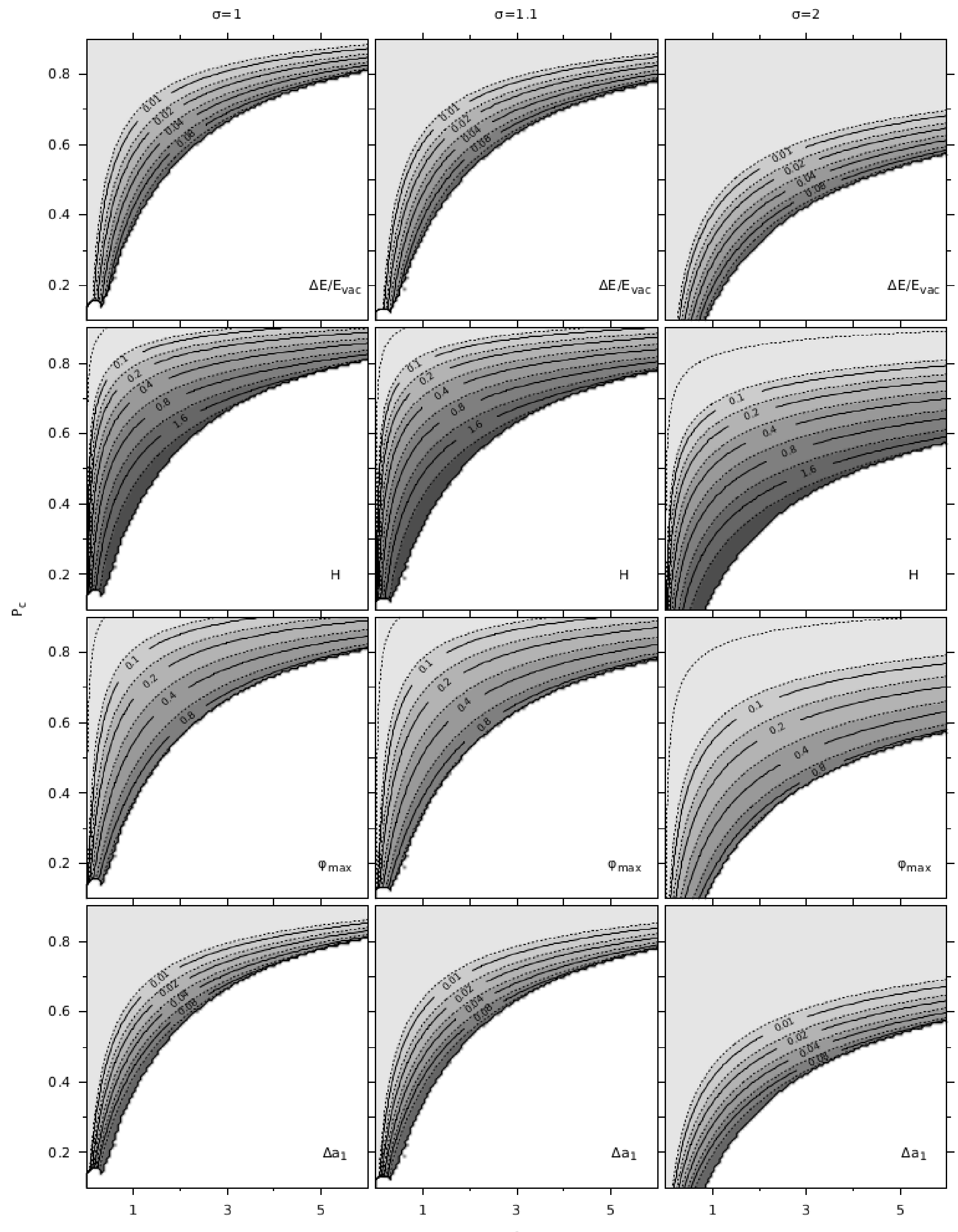
$$H \to \frac{32\pi s\epsilon^{5/2}}{15} + \mathcal{O}(\epsilon^{7/2})$$

$$\varphi_{\max} \to \frac{4\sqrt{3}s\epsilon^{3/2}}{9} + \mathcal{O}(\epsilon^{5/2})$$

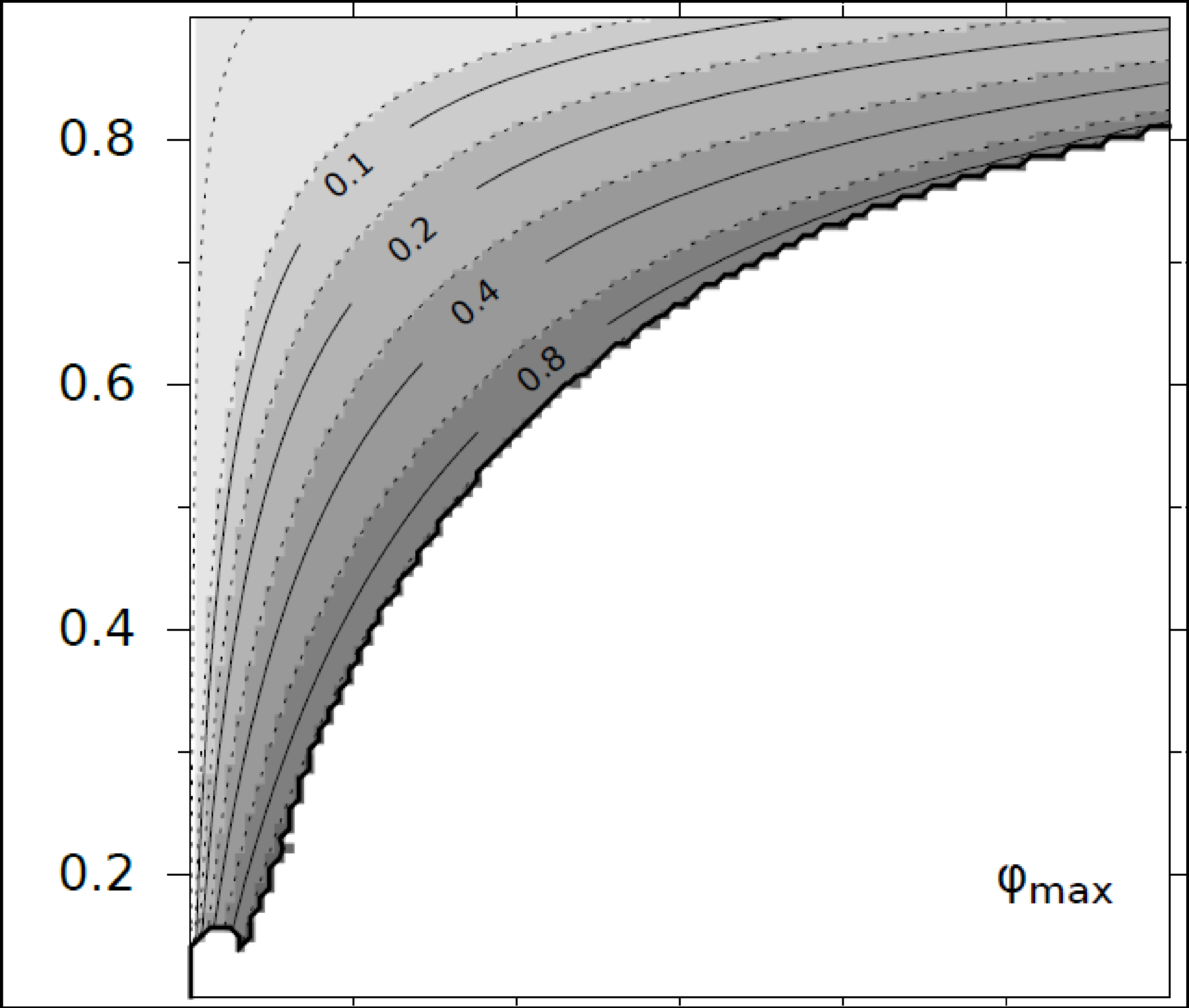
Plots as functions of two parameters

Plots of,
- relative energy (up to ~20%),
- helicity,
- maximum twist (up to around ~1.2 rad),
- and dipole content,

as functions of the three parameters.



σ=1
σ=1.1
σ=2
P_c
s
ΔE/E_vac
H
φ_max
Δa_1
0.01
0.02
0.04
0.08
0.1
0.2
0.4
0.8
1.6
0.2
0.4
0.6
0.8
1
3
5



0.8
0.6
0.4
0.2
0.1
0.2
0.4
0.8
φmax

Maximum twist

We are unable to find solutions when the maximum twist reaches a certain critical value (~1.2 to 1.4 rad).

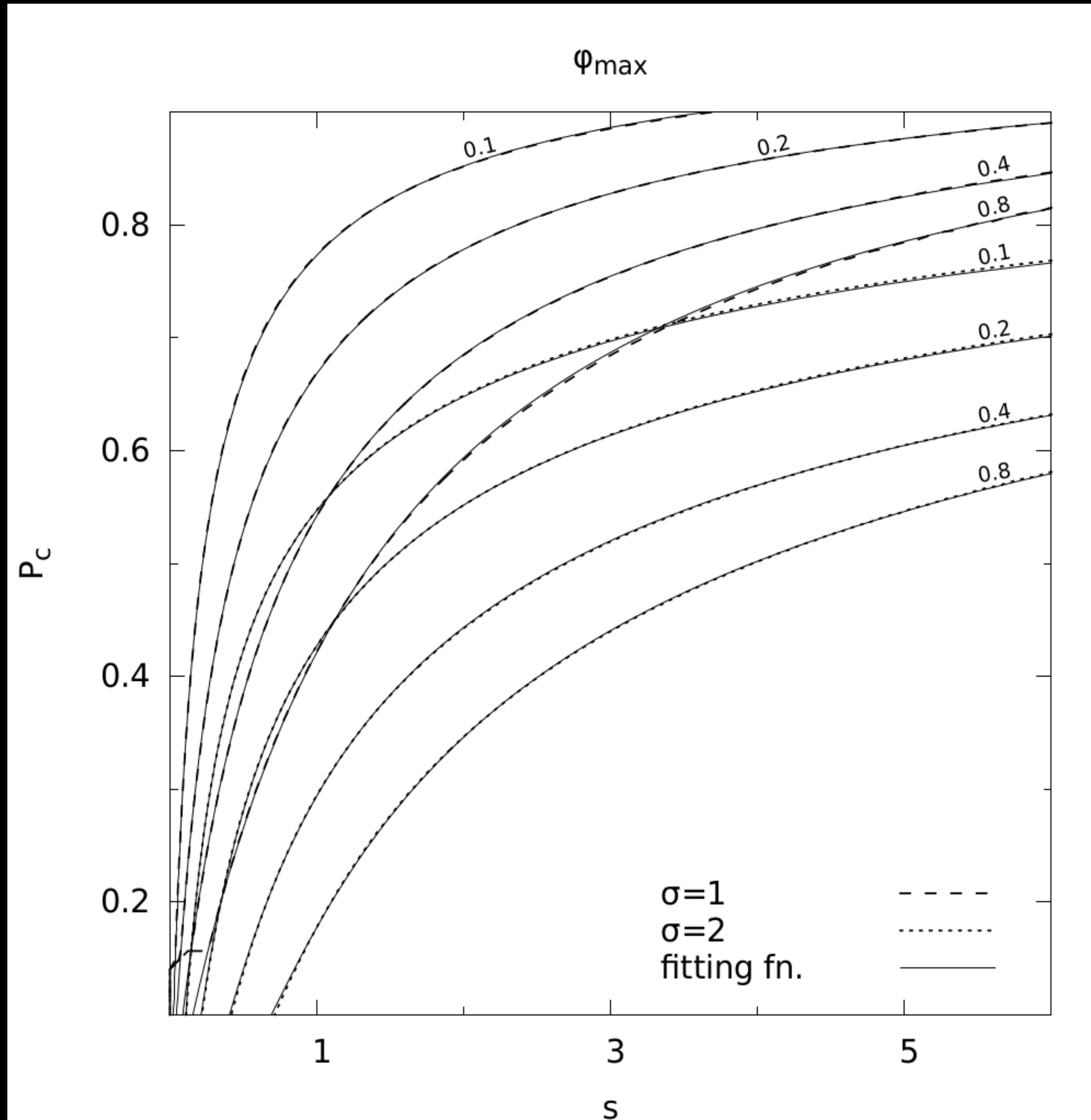
This holds for a wide range of parameters.

A similar conclusion (~1.6) was reached by Mikic & Linker (1994) through simulations of resistive MHD applied to the disruption of coronal arcades.

Fit to the contours

$$s = \frac{\gamma P_{\mathrm{c}}^{m}}{(1 - P_{\mathrm{c}})^{n}} \ .$$

	$\sigma = 1$			$\sigma = 2$		
φ_{max}	γ	m	n	γ	m	n
0.1	0.156	0.284	2.56	0.155	0.892	1.41
0.2	0.310	0.280	2.53	0.315	0.903	1.38
0.4	0.621	0.296	2.41	0.661	0.937	1.26
0.8	1.13	0.309	2.12	1.35	0.983	1.00


φmax
0.1
0.2
0.4
0.8
0.1
0.2
0.4
0.8
0.8
0.6
0.4
0.2
Pc
1
3
5
s
σ=1
σ=2
fitting fn.

Concluding remarks

We take a particular choice for the boundary condition at the surface.

Our model allows for transfer or energy & helicity from the interior to the exterior.

We don't model the dynamics of the magnetosphere.

We find a maximum twist (~1.2 to 1.4); comparable to the result of ~1.6 obtained by Mikic & Linker (1994) in a very different model.

Concluding remarks

Implications of the critical twist:

- as helicity is transferred from the interior to the magnetosphere the twist increases;
- there appears to be a critical (maximum) value beyond which force-free solutions no longer exist;
- increasing further the twist might result in a sudden disruption of the magnetospheric loops and might explain SGRs or X-ray bursts.

Future work:

- impose this magnetosphere as a boundary condition to the long-term evolution of the interior.