Relativistic Stars in Starobinsky gravity with matched asymptotic expansion

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Outline

Starobinsky Model

The models where the one replaces *Einstein-Hilbert* Lagrangian, $\mathcal{R}$, with a function of scalar curvature terms, are called as $f(\mathcal{R})$ theories of gravity.

Starobinsky model is $f(\mathcal{R}) = \mathcal{R} + \alpha\mathcal{R}^2$.

- Starobinsky introduced this model in 1979 motivated by supergravity.
- It does not contain ghost-like modes.
- It is consistent with solar system tests.
- It provides *inflation* in early universe.

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  - Ganguly et al. [2014] showed that the trace equation poses a *singular perturbation problem* if the second term in Lagragian is assumed perturbed.
  - By using matched asymptotic expansions method and perturbative approach, we find a solution that can match with Schwarzschild's solution; Arapoğlu, Çıkıntoğlu, and Ekşi [2016]

The Field Equations in Starobinsky Model

In general relativity ($f(\mathcal{R}) = \mathcal{R}$) the field and the trace equations are

$$\mathcal{R}_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\mathcal{R} = 8\pi T_{\mu\nu}, \qquad \mathcal{R} = -8\pi T$$

and in Starobinsky model the field and the trace equations are

$$(1 + 2\alpha\mathcal{R})\left(\mathcal{R}_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\mathcal{R}\right) + \frac{\alpha}{2}g_{\mu\nu}\mathcal{R}^2 - 2\alpha\left(\nabla_\mu\nabla_\nu - g_{\mu\nu}\Box\right)\mathcal{R} = 8\pi T_{\mu\nu},$$
$$6\alpha\Box\mathcal{R} - \mathcal{R} = 8\pi T$$

where $\Box = \nabla^\mu\nabla_\mu$.

Hydrostatic equations in Starobinsky Model

We use spherically symmetric and static metric,

$$\mathrm{d}s^2 = -\exp\left(2\Phi\right)\mathrm{d}t^2 + \exp\left(2\lambda\right)\mathrm{d}r^2 + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2\right).$$

Perfect-fluid approximation; $T^{\mu}_{\nu} = \mathrm{diag}\left(-\rho, P, P, P\right)$.
Mass is defined as, $\exp\left(-2\lambda\right) \equiv 1 - 2m\left(r\right)/r$.
Parameters are made dimensionless as

$$x = \frac{r}{R_*},\ \epsilon = \frac{\alpha}{R_*^2},\ \bar{\mathcal{R}} = R_*^2\mathcal{R},\ \bar{P} = R_*^2 P,\ \ \bar{\rho} = R_*^2\rho\ ,\ \ \bar{m} = \frac{m}{R_*}$$

where R_* is the radius of the star. Boundary conditions,

$$\bar{m}(0) = 0, \qquad \left.\frac{\mathrm{d}\bar{\mathcal{R}}}{\mathrm{d}x}\right|_{x=0} = 0, \qquad \bar{P}(1) = 0, \qquad \bar{\mathcal{R}}(1) = 0.$$

Hydrostatic equations in Starobinsky Model

Finally, the dimensionless hydrostatic equations are obtained as

$$\begin{aligned}
\left(1+2\epsilon\bar{\mathcal{R}}+\epsilon\bar{\mathcal{R}}'x\right)\frac{d\bar{m}}{dx} =& \frac{x^2}{12}\left[48\pi\bar{P}+\left(2+3\epsilon\bar{\mathcal{R}}\right)\bar{\mathcal{R}}+32\pi\bar{\rho}\right] \\
&+\frac{\epsilon\bar{\mathcal{R}}'}{6\left(1+2\epsilon\bar{\mathcal{R}}\right)}\left[x^3\left(\bar{\mathcal{R}}+3\epsilon\bar{\mathcal{R}}^2+16\pi\bar{\rho}\right)-6\bar{m}\left(1+2\epsilon\bar{\mathcal{R}}\right)\right] \\
&+\epsilon^2 x\left(x-2\bar{m}\right)\frac{2}{\left(1+2\epsilon\bar{\mathcal{R}}\right)}\bar{\mathcal{R}}'^2 \\
\frac{d\bar{P}}{dx} =& -\frac{\bar{\rho}+\bar{P}}{4x\left(x-2\bar{m}\right)\left(1+2\epsilon\bar{\mathcal{R}}+\epsilon x\bar{\mathcal{R}}'\right)}\times \\
&\left[16\pi x^3\bar{P}+4\bar{m}+8\epsilon\bar{m}\bar{\mathcal{R}}-\epsilon x^3\bar{\mathcal{R}}^2-8\epsilon x\left(x-2\bar{m}\right)\bar{\mathcal{R}}'\right] \\
\epsilon\left(1+2\epsilon\bar{\mathcal{R}}\right)\bar{\mathcal{R}}'' =& \left(-8\pi\bar{\rho}+24\pi\bar{P}+\bar{\mathcal{R}}\right)\frac{1}{6}\left(\frac{x}{x-2\bar{m}}\right)\left(1+2\epsilon\bar{\mathcal{R}}\right) \\
&+\frac{1}{6}\frac{\epsilon}{x-2\bar{m}}\left[\left(1+2\epsilon\bar{\mathcal{R}}\right)12\bar{m}x^{-1}-12\left(1+2\epsilon\bar{\mathcal{R}}\right)\right]\bar{\mathcal{R}}' \\
&+\frac{1}{6}\frac{\epsilon}{x-2\bar{m}}\left[3\epsilon x^2\bar{\mathcal{R}}^2+x^2\bar{\mathcal{R}}+16\pi x^2\bar{\rho}\right]\bar{\mathcal{R}}'+2\epsilon^2\bar{\mathcal{R}}'^2.
\end{aligned}$$

Singular Perturbation Problem

Consider

$$y'' + \epsilon y' + y = 0, \qquad y(0) = A \qquad y(1) = B \qquad 0 < x < 1 \qquad (1)$$

where $\epsilon \to 0$. The above equation can be solved by introducing solution as a perturbed expansion

$$y(x) = \sum_{n=0} \epsilon^n y_n(x). \qquad (2)$$

Yet, for a similar differential equation like

$$\epsilon y'' + y' + y = 0 \qquad (3)$$

there is no solution in the form as Eq. (2) which could satisfy both boundary conditions. Such equations pose *singular perturbation problems*. A well-known method for solving singular perturbation problems is *matched asymptotic expansions* [Bender and Orszag, 1978].

Solutions of Uniform Density Stars

The dimensionless mass, pressure and Ricci scalar for uniform density are obtained as

$$\bar{m}(x) = \frac{4}{3}\pi x^3 \bar{\rho} - \epsilon^{1/2}\frac{4}{3}\pi\bar{\rho}\beta \exp\left(-\frac{1-x}{\epsilon^{1/2}\beta}\right)$$

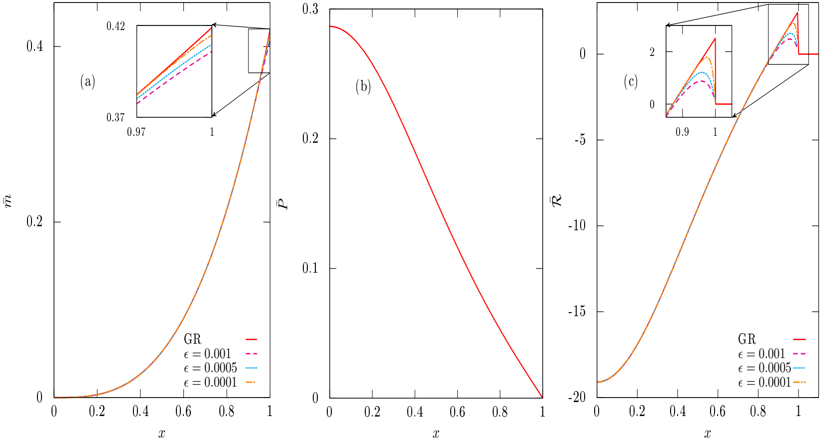
$$\bar{P}(x) = \bar{\rho}\frac{\sqrt{1-2\bar{M}^{\text{out}}} - \sqrt{1-2\bar{M}^{\text{out}}x^2}}{\sqrt{1-2\bar{M}^{\text{out}}x^2} - 3\sqrt{1-2\bar{M}^{\text{out}}}}$$

$$\begin{aligned}\bar{\mathcal{R}}(x) =& 16\pi\bar{\rho}\frac{2\sqrt{1-2\bar{M}^{\text{out}}x^2} - 3\sqrt{1-2\bar{M}^{\text{out}}}}{\sqrt{1-2\bar{M}^{\text{out}}x^2} - 3\sqrt{1-2\bar{M}^{\text{out}}}} \\ &+ D_0\left[1 + \frac{1-2\pi\bar{\rho}}{1-2\bar{M}^{\text{out}}}(1-x) + \frac{8\pi\bar{\rho}}{\epsilon^{1/2}\beta^3}(1-x)^2\right]\exp\left(-\frac{1-x}{\beta\epsilon^{1/2}}\right) \\ &+ \epsilon^{1/2}\left[\frac{D_0^2}{\beta}\exp\left(-2\frac{1-x}{\beta\epsilon^{1/2}}\right) + D_1\exp\left(-\frac{1-x}{\beta\epsilon^{1/2}}\right)\right].\end{aligned}$$

where

$$D_0 = -8\pi\bar{\rho}, \qquad D_1 = -\frac{(8\pi\bar{\rho})^2}{\sqrt{6\left(1-2\bar{M}^{\text{out}}\right)}}, \qquad \bar{M}^{\text{out}} = \frac{4}{3}\pi\bar{\rho}$$

Solutions of Uniform Density Stars



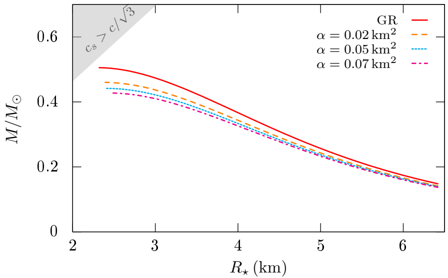
Here panels (a), (b) and (c) represent composite solutions for dimensionless mass $\bar{m}$, pressure $\bar{P}$ and Ricci scalar $\bar{\mathcal{R}}$. For all figures $\bar{\rho} = 0.1$.

Mass-Radius Relation of Uniform Density Stars

$$M(\mu) = 2M_L \frac{6\sqrt{\pi}}{5\sqrt{10}} \mu^{3/2} g(\mu) \times \left\{ 1 - \sqrt{\alpha \frac{4\rho_L}{9\pi} \mu^3 \left(\frac{20\pi}{\mu^2 (g(\mu))^{2/3}} - 16\pi \right)} \right\}$$

$$R(\mu) = R_L \left(\frac{2}{5\mu} \right)^{1/2} \left(\frac{9\pi}{4} g(\mu) \right)^{1/3}$$

The mass and the radius of the star (M-R) relation with different value of α are shown in the figure. Here M, $M_\odot$, R_* are the stellar mass, the Sun mass and radius of the star, respectively. In the gray region, the causality is violated since the sound speed, $c_s = \sqrt{dP/d\rho}$, is higher than $\sqrt{3}$ of the light speed, c.



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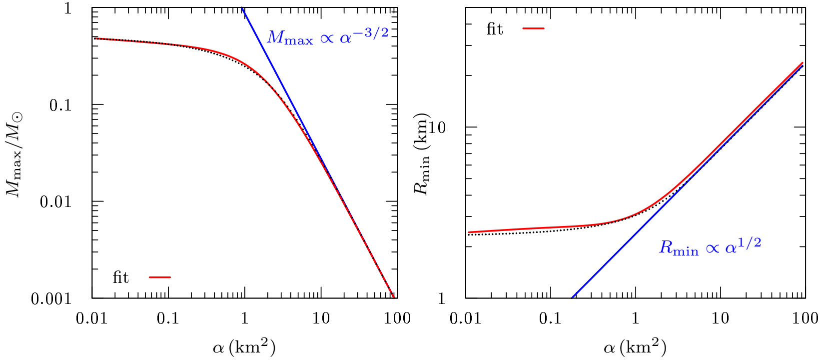

The left figure shows relation of the maximum stellar mass, $M_{\rm max}$ with α. The right figure shows relation of the radius of the star, $R_{\rm min}$ which corresponds to the maximum stellar mass with α. Here, $M_\odot$ is the Sun mass.

Solutions of Tolman VII Density Stars

Tolman-VII density distribution: $\bar{\rho}(x) = \bar{\rho}_c\left(1 - x^2\right)$

The dimensionless mass, pressure and Ricci scalar for Tolman VII density are obtained as

$$\begin{aligned} \bar{m}(x) =& \frac{4}{15}\pi x^3 \bar{\rho}_c \left(5 - 3x^2\right) \\ \bar{P}(x) =& f(x) \tan\left[g(x)\right] + h(x) \\ \bar{R}(x) =& 8\pi \left[\frac{\bar{\rho}_c}{5}\left(10 - 8x^2\right) - 3f(x)\tan\left[g(x)\right]\right] \end{aligned}$$

where

$$f(x) = \sqrt{\frac{\bar{\rho}_c}{10\pi}} \exp(-\lambda), \qquad h(x) = -\frac{\bar{\rho}_c}{15}\left(5 - 3x^2\right)$$

$$g(x) = -\frac{1}{2}\ln\left[\frac{\sqrt{\frac{8\pi\bar{\rho}_c}{5}}\left(x^2 - \frac{5}{6}\right) + \exp(-\lambda)}{\frac{1}{6}\sqrt{\frac{8\pi\bar{\rho}_c}{5}} + \sqrt{1 - \frac{16\pi}{15}\bar{\rho}_c}}\right] + \tan^{-1}\left(\frac{2\bar{\rho}_c}{15\sqrt{\frac{\bar{\rho}_c}{10\pi}}\left(1 - \frac{16\pi}{15}\bar{\rho}_c\right)}\right).$$

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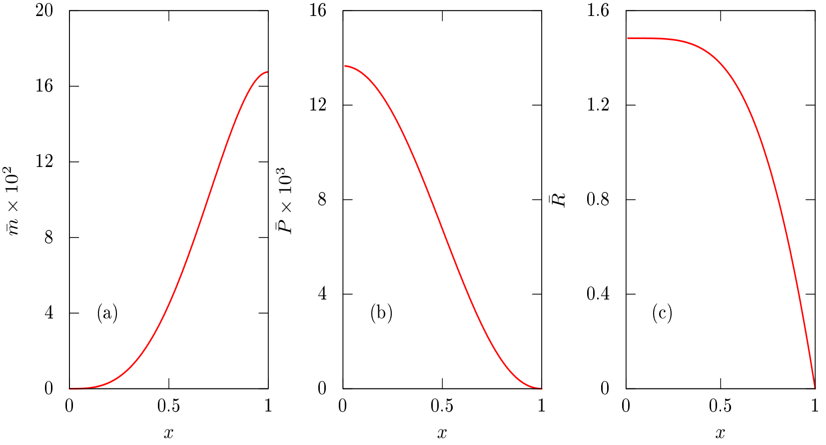

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Summary

- The field equations pose a singular perturbation problem if the modified term is assumed perturbative.
- An appropriate method for handling such problems is the MAE.
- We obtained the solutions for uniform density and Tolman-VII density distributions with MAE method in Starobinsky model.
- Tolman-VII type solutions we obtained in Starobinsky gravity are the same with the solutions in general relativity.
- In both cases, we found that the solutions can match to the Schwarzschild's solution at the surface of the star in Starobinsky model.
- In both cases, the solutions provide stable star configurations.