The ground state of the Blume-Capel-Haldane-Ising spin chain

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arXiv :1601.06195 [cond-mat.str-el]

June 14, 2016

The Blume-Capel-Haldane-Ising (BCHI) model

We study the 1-d model

$$H = \sum_{i=1}^{N} a(S_i^z)^2 + bS_i^z S_{i+1}^z,$$

with trivial eigenstates $|\{s_i\}\rangle$, where $s_i \in \{-s, -s+1, \ldots, s\}$.

- The case $a = 0$ and $s = \frac{1}{2}$ is the Ising model.
- The case $s = 1$ is the Blume-Capel model.
- It is the anisotropy part of the model studied by Haldane :

$$H_{\text{Haldane}} = |J| \sum_{i=1}^{N} \mathbf{S}_i \cdot \mathbf{S}_{i+1} + \mu(S_i^z)^2 + \lambda S_i^z S_{i+1}^z.$$

Haldane's model

$$H_{\text{Haldane}} = |J| \sum_{i=1}^{N} \mathbf{S}_i \cdot \mathbf{S}_{i+1} + \mu (S_i^z)^2 + \lambda S_i^z S_{i+1}^z$$

- Haldane studied the model for large spin with small anisotropy, and $0 < \mu < \lambda$.
- He observed that the $\mathbb{Z}_2$ symmetry is broken by the two classical ground-states (Néel ordered).
- He mapped the model to an integrable non-linear sigma model.
- He obtained the soliton interpolating between the degenerate ground-states.
- He conjectured that integer spins have a mass gap, whereas half-odd spins have a gapless spectrum.

Ground state and solitons in BCHI

For the much simpler model

$$H = \sum_{i=1}^{N} a(S_i^z)^2 + bS_i^z S_{i+1}^z$$

it is of relevance to find the ground state and the soliton (whenever $\mathbb{Z}_2$ is spontaneously broken) for all values of a and b.

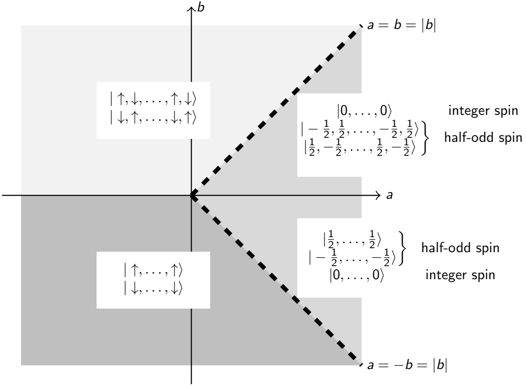
Our expressions will be :

- **Semiclassical**. $s_i \in [-s, s]$; the quantum states will be the closest states with integer or half-odd s_i ;
- **Exact**. No need to assume a slowly varying field.

Ground state for $S^z_{N+1} = S^z_1$, N even

This is the simplest case : no topological defect. One finds by induction or direct inspection :

FIGURE : Ground state for N even and arbitrary spin s



BCHI soliton

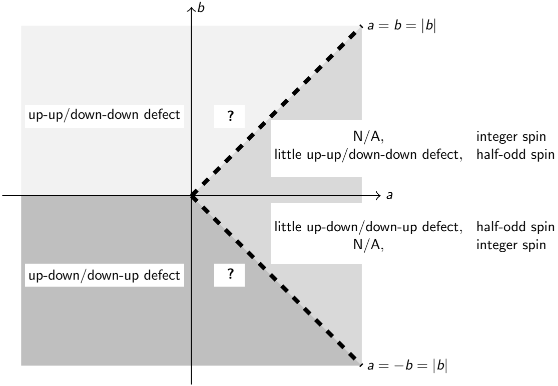


FIGURE : Soliton interpolating between doubly degenerate ground states (when applicable)

The non-trivial case is when $0 < a < |b|$.

Antiferromagnetic soliton when $0 < a < b$

- Set $s_0 = s$ and $s_{n+1} = (-1)^n s$, connecting the two Néel ground states, $n \geq 1$.
- The energy of the generic state $|s_0, \ldots, s_{n+1}\rangle$ is

$$E = \frac{1}{2}\mathbf{s}^{\mathsf{T}} B_n \mathbf{s} + 2as^2 + b\mathbf{s}^{\mathsf{T}}\mathbf{t} \ ,$$

  $\mathbf{s} = (s_1, s_2, \ldots, s_n)^{\mathsf{T}}$ and $\mathbf{t} = (s, 0, \ldots, 0, (-1)^n s)^{\mathsf{T}}$.
- The critical points are given by

$$B_n\mathbf{s} = -b\mathbf{t}.$$

- B_n is the Hessian matrix of E.
- If B_n is positive definite, hence invertible, the (unique) critical point $\mathbf{s} = -b(B_n)^{-1}\mathbf{t}$ minimizes E .

Eigenvalues of B_n

B_n is tridiagonal Toeplitz :

$$B_n = \begin{pmatrix} 2a & b & & & & & \\ b & 2a & b & & & & \\ & b & 2a & b & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & \ddots & \ddots & \ddots & \\ & & & & b & 2a & b \\ & & & & & b & 2a \end{pmatrix}_{n \times n}$$

Its eigenvalues are known :

$$\lambda_j = 2\left(a + b\cos\tfrac{j\pi}{n+1}\right), \qquad j = 1, 2, \ldots, n.$$

The Hessian is positive definite for $\cos\frac{\pi}{n+1} < \frac{a}{b}$.

The critical point

- The point $\mathbf{s} = -b(B_n)^{-1}\mathbf{t}$ solves the boundary problem $s_0 = s$, $s_{n+1} = (-1)^n s$ if $\cos\frac{\pi}{n+1} < \frac{a}{b}$.
- Explicitely

$$s_j = \bar{s}_j^n \stackrel{\text{def}}{=} \frac{(-1)^j s}{\sin(n+1)\theta/2}\ \sin\left(\frac{n+1}{2} - j\right)\theta \quad , \quad j = 1, \ldots, n,$$

where $\cos\theta = a/b$.

- $\bar{s}_j^n \in [-s, s]$.
- For large n (i.e. $a \lesssim b$),

$$\bar{s}_j^n \sim (-1)^j s \cos j\theta.$$

- It is a rotation of the (staggered) spin components by π over the sites $j = 1, \ldots, n$.

The antiferromagnetic soliton

- The $\bar{s}_j^n$ with integer n such that $\cos\frac{\pi}{n+1} < \frac{a}{b}$ solve the boundary problems $s_0 = s$, $s_{n+1} = (-1)^n s$, with corresponding energies E_n.
- The soliton must be one of them.
- Since the nth problem is subsumed in the $(n+1)$st, we have $E_n > E_{n+1}$.
- The soliton has maximal integer n such that $\cos\frac{\pi}{n+1} < \frac{a}{b}$.
- The soliton is

$$\bar{s}_j^n = \frac{(-1)^j s}{\sin(n+1)\theta/2}\ \sin\left(\frac{n+1}{2} - j\right)\theta \sim (-1)^j s \cos j\theta,$$

where $\cos\theta = a/b$ and $\cos\frac{\pi}{n+1} < \frac{a}{b} < \cos\frac{\pi}{n+2}$.

The ferromagnetic soliton when $0 < a < |b|$

- Locally defining the staggered spin operators $\bar{S}_j \equiv (-1)^j S_j$ provides a mapping $b \longleftrightarrow -b$, $S_j \longleftrightarrow \bar{S}_j$ between ferromagnetic and antiferromagnetic solitons.

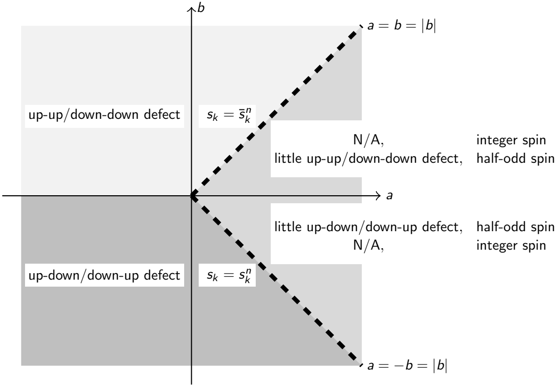
- The ferromagnetic soliton is

$$s_j^n = \frac{s}{\sin(n+1)\theta/2} \, \sin\left(\frac{n+1}{2} - j\right)\theta \sim s\cos j\theta,$$

where $\cos\theta = a/|b|$ and $\cos\frac{\pi}{n+1} < \frac{a}{|b|} < \cos\frac{\pi}{n+2}$.

BCHI soliton

FIGURE : Soliton interpolating between doubly degenerate ground states



The size n of the soliton depends only on $a/|b|$:

$$\cos \frac{\pi}{n+1} < \frac{a}{|b|} < \cos \frac{\pi}{n+2}.$$

Solitons as excitations

- The energy of excitation $\Delta(n)$ of the soliton of size n can be computed in closed form.
- In the large-n limit, $\cos\frac{\pi}{n} \sim \frac{a}{|b|} \sim 1$, and

$$\Delta(n) \sim |b|s^2 \, \frac{\pi^2}{2} \left(\frac{1}{n}\right),$$

  or equivalently

$$\Delta\left(\frac{a}{|b|}\right) \sim |b|s^2 \, \frac{\pi}{\sqrt{2}} \left(1 - \frac{a}{|b|}\right)^{1/2}.$$

- Solitons become massless as $a \to |b|$, with critical exponent $1/2$.
- Instability against pairs of soliton excitations as $a \to |b|$ destroys the possibility of a long range order in the ground state.

The periodic chains

- The BCHI model can be defined on the orientable closed chain (i.e. the trivial fiber bundle $[-s, s] \times S^1$), and on the non-orientable chain (i.e. the Möbius strip).
- When a defect is imposed by topology, it forces the two degenerate ground states to meet.
- As a consequence, the defect is just the soliton found above.
- This yields the ground state on any periodic chain, for all a and b.

TABLE : Number of allowed solitons on periodic chains of length $N \geq 2$.

	Orientable chain		Non-orientable chain	
	F ($b < 0$)	AF ($b > 0$)	F ($b < 0$)	AF ($b > 0$)
N even	Even	Even	Odd	Odd
N odd	Even	Odd	Odd	Even

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