Shape dependence of two-cylinder Rényi entropies for a free boson lattice field theory

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Motivation

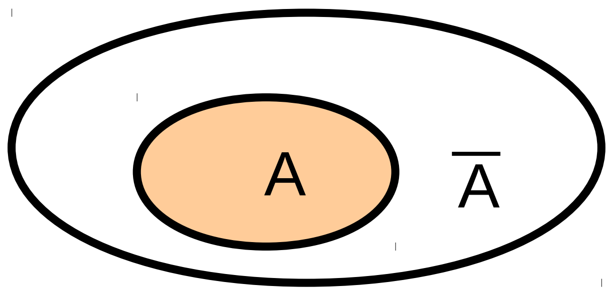
• Entanglement entropy (EE) measures quantum mechanical correlations in many-body systems and field theories.

• The problem of EE is completely solved for $z = 1$ quantum critical points in $D = 1$ (system is described by CFT), but poorly understood in $D > 1$.

• Certain scaling terms in EE are universal providing us with a powerful tool to characterize phases and phase transitions in a variety of theories.

– This makes important to confirm numerically the theoretical predictions for universal parameters entering EE.

Entanglement Entropy (EE)



$$\rho = |\psi\rangle\langle\psi|$$

$$\rho_A = \mathrm{Tr}_{\bar{A}}\rho$$

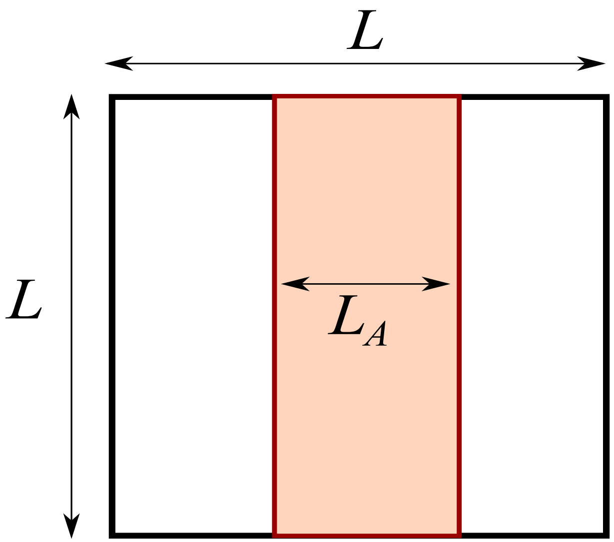
$\rho_A = \sum_i p_i |i\rangle\langle i|$ – mixed state

p_i – entanglement spectrum

- Von-Neumann and Renyi ($\alpha > 1$) entanglement entropies:

$$S_1(A) = -\mathrm{Tr}\left(\rho_A \log \rho_A\right), \qquad S_\alpha(A) = \frac{1}{1-\alpha}\log\left(\mathrm{Tr}\rho_A^\alpha\right)$$

Geometry: a torus cut into two cylinders



An $L \times L$ square lattice divided in region A and its complement, forming two cylinders. Each boundary may be periodic or anti-periodic.

In two dimensions and at the critical point the Rényi entropies are expected to scale with the size of region A (area law) as

$$S_\alpha = a\frac{L}{\delta} + c\gamma(u) + \cdots$$

[L. Bombelli, et.al., Phys. Rev. D **34**, 373 (1986)]

[M. Srednicki, Phys. Rev. Lett. **71**, 666 (1993)]

– δ is the lattice cutoff, a and c are non-universal and $u = L_A/L$

– $\gamma(u)$ is expected to reflect universality!

Two-cylinder scaling functions I

(1) $(1+1)$-dimensional CFT scaling function (postulated heuristically for a critical resonating-valence bond wavefunction on a square lattice)

[H.Ju, A. Kallin, M. B. Hastings, R.G. Melko, Phys. Rev. B $\mathbf{85}$, 165121 (2012)]

$$\gamma_{\mathrm{1D}}(u) = \ln\sin(\pi u), \qquad u = L_A/L$$

(2) $z = 2$ quantum Lifshitz model (QLM) (motivated by the study of dimer RVB wavefunctions)

[J.-M. Stephan, H. Ju, P. Fendley, R.G. Melko, New J. Phys. $\mathbf{15}$, 015004 (2013)]

$$\gamma_{\mathrm{QLM}}(u) = \ln\left(\frac{\theta_3(2iu)\theta_3(2i(1-u))}{\eta(2iu)\eta(2i(1-u))}\right), \qquad u = L_A/L$$

θ_3 is the Jacobi theta-function, η is the Dedekind eta-function.

Two-cylinder scaling functions II

(3) AdS/CFT correspondence function (derived in $3+1$ dimensions using the AdS soliton metric)

[X.Chen, G.Cho, T.Faulkner, E. Fradkin, J. Stat. Mech. **2015**, P02010 (2015)]

$$\gamma_{\mathrm{AdS}}(\chi) = \chi^{-1/3}\left(\int_0^1 \frac{d\zeta}{\zeta^2}\left(\frac{1}{\sqrt{P(\chi,\zeta)}} - 1\right) - 1\right), \quad P(\chi,\zeta) = 1 - \chi\zeta^3 - (1-\chi)\zeta^4$$

$$u(\chi) = \frac{3\chi^{1/3}(1-\chi)^{1/2}}{2\pi}\int_0^1 \frac{d\zeta\,\zeta^2}{(1-\chi\zeta^3)}\frac{1}{\sqrt{P(\chi,\zeta)}}, \qquad u = L_A/L$$

(4) Extensive mutual information (EMI) model

[W. Witczak-Krempa, L. Hayward Sierens, R. G. Melko, arXiv:1603.02684]

$$\gamma_{\mathrm{EMI}}(u) = \frac{\cot^{-1}(2u)}{u} + \frac{\cot^{-1}(2(1-u))}{1-u}, \qquad u = L_A/L$$

Free bosons on the square lattice

- Consider a free scalar field ϕ of mass m on a lattice in $\mathrm{d}+1$ dimensions

$$H = \frac{1}{2}\sum_{x,y=1,1}^{L,L}\left[\pi_{x,y}^2 + (\phi_{x+1,y}-\phi_{x,y})^2 + (\phi_{x,y+1}-\phi_{x,y})^2 + m^2\phi_{x,y}^2\right].$$

- Upon Fourier transforming

$$H = \frac{1}{2}\sum_{\mathbf{k}}\left[\pi_{\mathbf{k}}\pi_{-\mathbf{k}} + \omega_{\mathbf{k}}^2\phi_{\mathbf{k}}\phi_{-\mathbf{k}}\right], \qquad \omega_{\mathbf{k}} = \sqrt{4\sin^2(k_x/2) + 4\sin^2(k_y/2) + m^2}.$$

- Working on a lattice, we need to specify the boundary conditions

  – For anti-periodic boundary condition (APBC) in i-th direction

$$\phi(x_i,\ldots) = -\phi(x_i+L,\ldots), \qquad k_i = (2n_i+1)\pi/L, \qquad n_i = 0,1,\ldots,L-1.$$

  – For periodic boundary condition (PBC) in i-th direction

$$\phi(x_i,\ldots) = \phi(x_i+L,\ldots), \qquad k_i = 2n_i\pi/L, \qquad n_i = 0,1,\ldots,L-1.$$

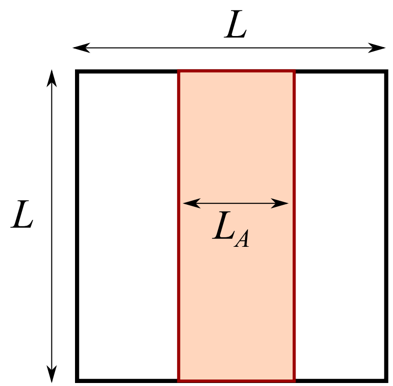
We are interested in EE for free critical bosons, but small $m \neq 0$ is required for regularization if PBC in both directions.

Entanglement entropies for free bosons

- The key quantities

$$(X_A)_{\mathbf{x}\mathbf{x}'} = \langle \phi_{\mathbf{x}} \phi_{\mathbf{x}'} \rangle = \frac{1}{2L^2} \sum_{\mathbf{k}} \frac{\cos\left[k_x \left(x - x'\right)\right] \cos\left[k_y \left(y - y'\right)\right]}{\omega_{\mathbf{k}}},$$

$$(P_A)_{\mathbf{x}\mathbf{x}'} = \langle \pi_{\mathbf{x}} \pi_{\mathbf{x}'} \rangle = \frac{1}{2L^2} \sum_{\mathbf{k}} \omega_{\mathbf{k}} \cos\left[k_x \left(x - x'\right)\right] \cos\left[k_y \left(y - y'\right)\right].$$



$\mathbf{x}, \mathbf{x}' \in A$

[H. Casini, M. Huerta, J. Phys. A: Math. and Theor. **42**, 504007 (2009)]

[I. Peschel, J. Phys. A: Math. Gen. **36**, L205 (2003)]

- If ν_ℓ are the eigenvalues of the matrix $C_A = \sqrt{X_A P_A}$

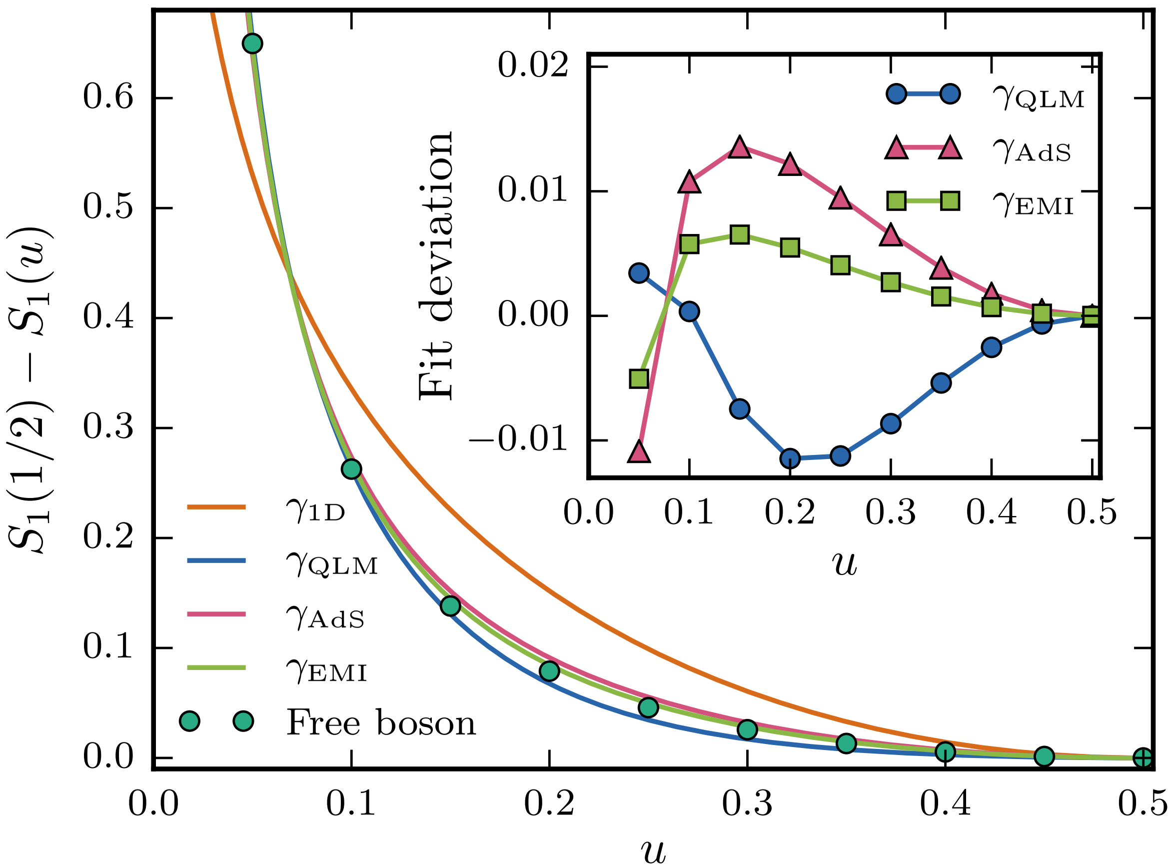
$$S_1(A) = \sum_\ell \left[\left(\nu_\ell + \frac{1}{2}\right) \log\left(\nu_\ell + \frac{1}{2}\right) - \left(\nu_\ell - \frac{1}{2}\right) \log\left(\nu_\ell - \frac{1}{2}\right) \right],$$

$$S_\alpha(A) = \frac{1}{\alpha - 1} \sum_\ell \log\left[\left(\nu_\ell + \frac{1}{2}\right)^\alpha - \left(\nu_\ell - \frac{1}{2}\right)^\alpha \right].$$

Fitting entanglement entropy S_1

- $S_\alpha(u)$ is maximal at $u = L_A/L = 0.5$ and obeys the property $S_\alpha(u) = S_\alpha(1-u)$.

- Plot $S(1/2) - S(u)$ and perform the least-squares fits of it the form $c\,[\gamma(1/2) - \gamma(u)]$ with c being the sole fitting parameter.

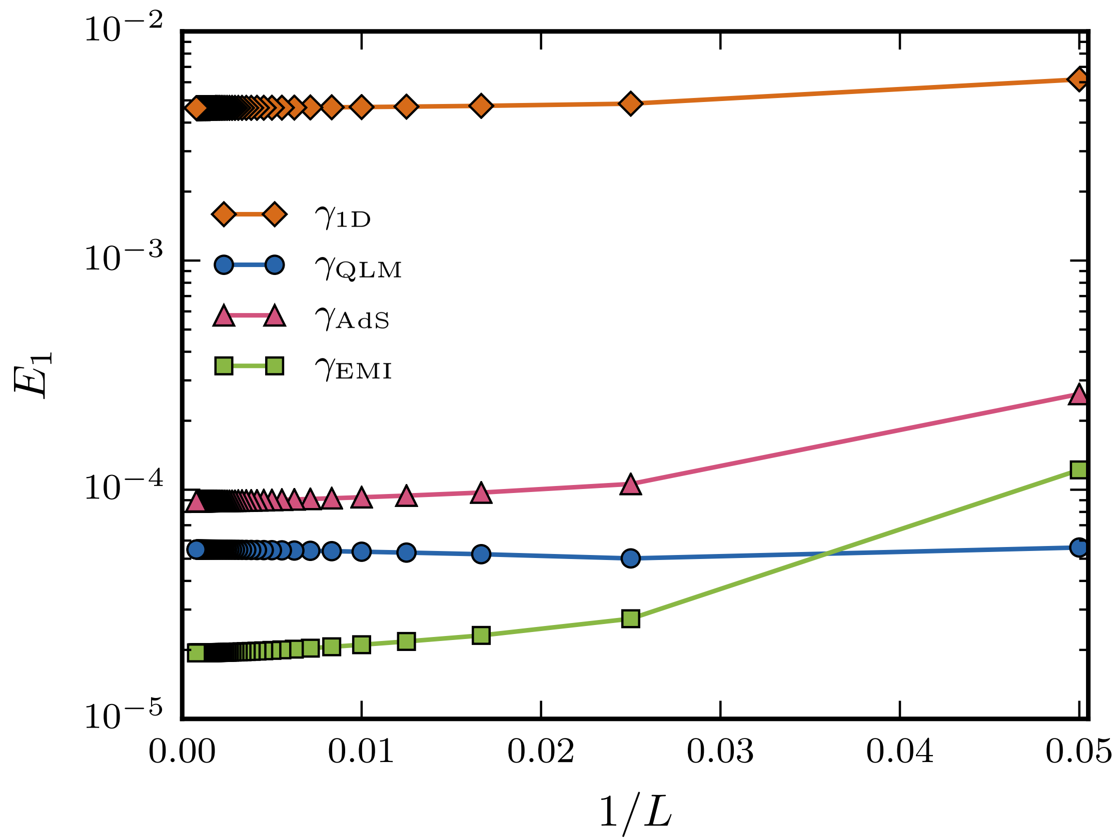


Shape dependence of the von Neumann entanglement entropy for an $L = 240$ free boson system, along with fits corresponding to the four candidate functions.

Fitting error for the von Neumann entropy S_1

- To quantify the quality of each fit, we sum the squared residuals and normalize (n_u is the number of points used to fit), defining the fitting error by

$$E_1 = \frac{1}{n_u - 2} \sum_{i=1}^{n_u} \Big(c\big[\gamma\,(1/2) - \gamma(u_i)\big] - \big[S_1\,(1/2) - S_1(u_i)\big] \Big)^2 . \qquad (1)$$

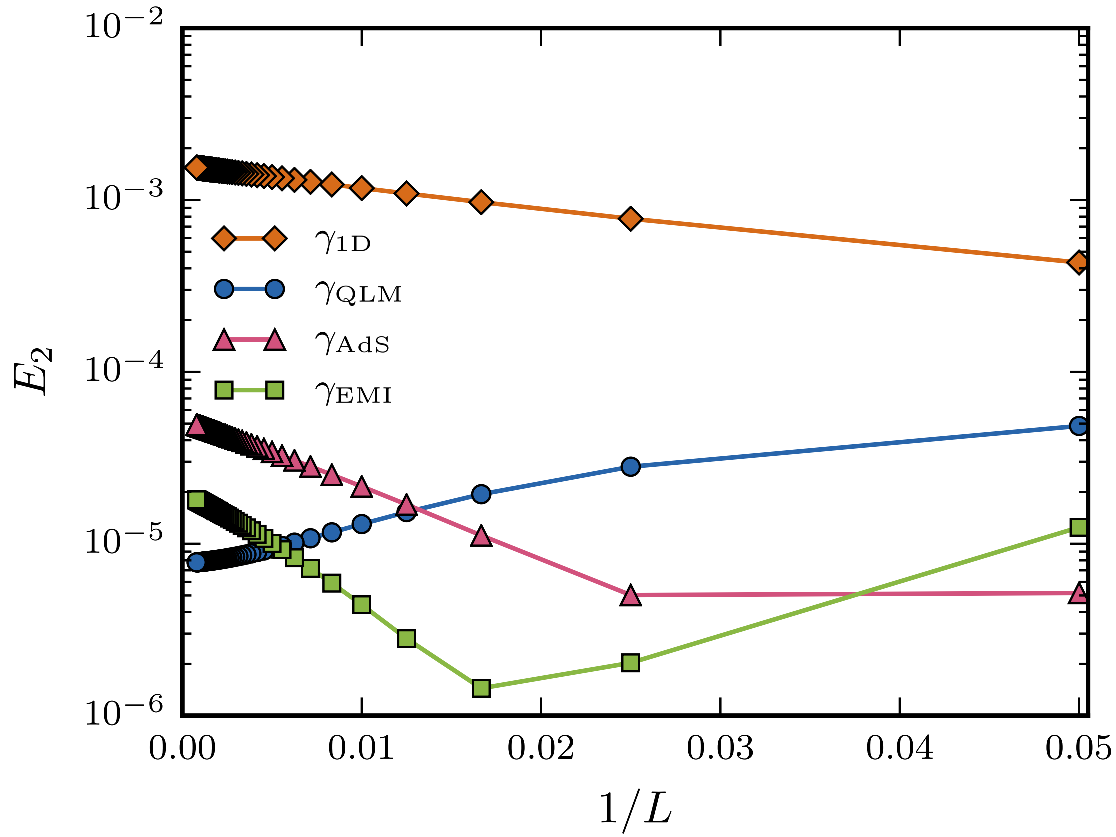


The fitting errors corresponding to the four candidate functions for γ as a function of $1/L$ for the von Neumann entropy. The errors are measured using resolution $\Delta u = 0.05$ on lattices of size $L = 20, 40, \ldots, 1240$. PBC are along x and APBC along y directions.

Fitting error for the second Rényi entropy S_2

- To quantify the quality of each fit, we sum the squared residuals and normalize (n_u is the number of points used to fit), defining the fitting error by

$$E_2 = \frac{1}{n_u - 2} \sum_{i=1}^{n_u} \Big(c\big[\gamma\,(1/2) - \gamma(u_i)\big] - \big[S_2\,(1/2) - S_2(u_i)\big] \Big)^2 . \qquad (2)$$



The fitting errors corresponding to the four candidate functions for γ as a function of $1/L$ for the von Neumann entropy. The errors are measured using resolution $\Delta u = 0.05$ on lattices of size $L = 20, 40, \ldots, 1240$. PBC are along x and APBC along y directions.

Entanglement Entropies depend on the boundary conditions

Fitting error	Boundary conditions	1D	QLM	AdS	EMI
E_1	PBC along x, APBC along y	4.6301×10^{-3}	5.4829×10^{-5}	8.9039×10^{-5}	1.9398×10^{-5}
	APBC along x, PBC along y	4.6000×10^{-3}	4.2347×10^{-5}	1.1226×10^{-4}	3.2742×10^{-5}
	APBC along x, APBC along y	5.0523×10^{-3}	2.8521×10^{-5}	3.1028×10^{-4}	1.6915×10^{-4}
	PBC along x, PBC along y	3.1851×10^{-3}	4.1407×10^{-3}	1.5626×10^{-3}	2.2611×10^{-3}
E_2	PBC along x, APBC along y	1.5416×10^{-3}	7.7936×10^{-6}	4.9223×10^{-5}	1.7987×10^{-5}
	APBC along x, PBC along y	1.4844×10^{-3}	9.9809×10^{-6}	4.4994×10^{-5}	1.5967×10^{-5}
	APBC along x, APBC along y	1.6467×10^{-3}	1.9580×10^{-5}	1.3172×10^{-4}	8.0410×10^{-5}
	PBC along x, PBC along y	9.0657×10^{-4}	2.6372×10^{-3}	1.2070×10^{-3}	1.6109×10^{-3}

- The table summarizes the fitting errors for $L = 1240$ computed using resolution $\Delta u = 0.05$ for various boundary conditions. A small mass of $m = 10^{-6}$ is employed for the case of PBC along both the x and y directions.
  - The values of the best fit are given in red
  - The values of the second best fit are given in green

Conclusions and Future work

• We have computed numerically the von Neumann and second Rényi entanglement entropies for a system of free massless bosons in $2+1$ dimensions, focusing on the universal term $\gamma(u = L_A/L)$. Several candidate functions have been used to fit the numerical data.

– The fits depend on the boundary conditions and non of them are perfect.

– γ_{1D} gives the worst fits except for the case of PBC in both directions.

– γ_{EMI} and γ_{QLM} give the best fits when APBC is at least in one direction.

– The success of the quantum Lifshitz model (γ_{QLM}) is surprising because QLM is a non-conformally invariant theory with dynamical exponent $z = 2$.

• As a future work, it will be interesting to do a similar study for the case of interacting bosons at criticality in $2+1$ dimensions.