Spatio-temporal correlations after a quantum quench in the Bose-Hubbard model

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Outline

Motivation
- Review of recent work on BHM and related experiments
- Why is it important to study correlations?

Theoretical formalism
- Schwinger-Keldysh technique → to describe out-of-equilibrium phenomena
- 2PI effective action approach → to calculate Green’s functions
- Effective theory of BHM
- Low-frequency approximation

Numerical results
- Focus on quenches within MI regime
- Quasi-momentum distribution after quench
- Light-cone spreading of correlations
Bosons in an optical lattice

- Can be described by the BHM [1]
  - Simplest model for interacting bosons on a lattice

- Optical lattices are versatile systems
  - Tune parameters in real-time [2,3]
  - Dynamically traverse quantum phase transitions

- Transition between superfluid and Mott-insulator observed via quench [4]
  - SF: Phase-coherent state
  - MI: Localised state

- Example of out-of-equilibrium dynamics in interacting quantum system

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Correlation functions

Cheneau et al. (2012)

Quenches from SF $\rightarrow$ MI by changing lattice potential (varies hopping in BHM)
- Slow relaxation times after quench from SF $\rightarrow$ MI [1],[2]
- Presence of harmonic trap $\rightarrow$ MI regions impede relaxation via mass transport [3]

Quenches within MI regime:
- Light-cone-like spreading of correlations in 1D systems [4],[5]
- Similar predictions for higher dimensions [6]

Suggests further study of correlations in BHM:
- Numerical approaches limited to 1D systems $\rightarrow$ require analytical approach
- Formalism needs to be accurate in both phase regimes

[1] PRA 84, 033620 (2011)
2PI Formalism

We want to calculate the equations of motion of the two-point Green’s function $G_{ij}(t_1, t_2) \equiv -i\langle \hat{\Phi}_i(t_1)\hat{\Phi}^\dagger_j(t_2) \rangle$

- Standard approach: 2PI-formalism
- Advantage: treats the calculation of $G$ and the mean field $\phi_i(t) \equiv \langle \hat{\Phi}_i(t) \rangle$ (i.e. superfluid order parameter) on equal footing

Apply the Schwinger-Keldysh technique in conjunction with the 2PI approach

- Schwinger-Keldysh: Allows for the description of out-of-equilibrium phenomena
- 2PI formalism: Used to calculate equations of motion for $G$ and $\phi$
- Price to pay - doubling of fields $\{\hat{\Phi}_{i+}(t), \hat{\Phi}_{i-}(t)\}$ in theory
2PI Equations of motion

Cornwall et al. [1] showed that $G$ and $\phi$ satisfy the following equations of motion

\[
\frac{\delta S[\phi]}{\delta \phi_a} + \frac{1}{2} \left[ i \frac{\delta [D^{-1}]_{bc}}{\delta \phi_a} G_{bc} \right] + \frac{\delta \Gamma_2}{\delta \phi_a} = 0
\]

\[
i [G]^{-1}_{ab} = i [D]^{-1}_{ab} - i \Sigma_{ab} \quad \text{(Dyson’s Equation)}
\]

- Action (describing theory): $S[\Phi]$
- Inverse “bare” propagator: $iD^{-1}_{ab} \equiv \frac{\delta^2 S[\phi]}{\delta \phi_a \delta \phi_b}$
- Self energy: $\Sigma_{ab} \equiv 2i \frac{\delta \Gamma_2}{\delta G_{ab}}$

\[
\Gamma_2 = \sum_{\text{dots}} + \sum_{\text{solid lines}} + \sum_{\text{dotted lines}} + \ldots
\]

- Dots: interaction vertices in $S[\Phi]$
- Solid lines: $G$
- Dotted lines: $\phi$

Real-time strong-coupling approach

- Generalize imaginary time strong-coupling approach to obtain an effective theory of BHM

\[
S[\Phi, \Phi^*] = \int dt_1 dt_2 \sum_{ij} \Phi^*_{i\alpha}(t_1) \left( \delta_{ij} [G]^{-1}_{i\alpha \beta}(t_1, t_2) + 2\delta(t_1 - t_2) J_{ij}(t_1) \sigma^1_{\alpha \beta} \right) \Phi_{j\beta}(t_2)
\]

\[
+ \frac{1}{4} \int dt_1 dt_2 dt_3 dt_4 \sum_i u_{i,\alpha_1 \alpha_2 \alpha_3 \alpha_4}(t_1, t_2, t_3, t_4) \Phi^*_{i\alpha_1}(t_1) \Phi^*_{i\alpha_2}(t_2) \Phi_{i\alpha_3}(t_3) \Phi_{i\alpha_4}(t_4)
\]

- Real time: M.P. Kennett and D. Dalidovich, PRA 84, 033620 (2011)

- Expected to provide accurate description of quench dynamics in both SF & MI regimes

- \(G\): Local \((J = 0)\) 2-point Green’s function

- \(u_{i,\alpha_1 \alpha_2 \alpha_3 \alpha_4}(t_1, t_2, t_3, t_4)\): \((J = 0)\) 4-point vertex
  - Local in space – non-local in time

- Our work: apply 2PI formalism to this effective theory of BHM
Low-frequency approximation

- To second order in $u$, equations of motion contain 7 time-integrals
  - Require further approximations to be amenable to numerics

Low-frequency approximation

- Focus on low-frequency dynamics
- Ignore $\mu/U \approx \text{integer}$, i.e. away from degenerate MI states
- Low-frequency limit of vertices: $u_{i,\alpha_1\alpha_2\alpha_3\alpha_4}(\omega_1, \omega_2, \omega_3, \omega_4) \to u_1$ or $u_2$, depending on Keldysh indices $\{\alpha\}$
- Net result: 7 time-integrals $\to$ 1 time-integral
Numerical Results

- Focus on special case of quenches within MI regime
  \[ \langle \hat{\Phi}^{(\dagger)}_{r_1} (t_1) \rangle = \langle \hat{\Phi}_{r_1, \alpha_1} (t_1) \hat{\Phi}_{r_2, \alpha_2} (t_2) \rangle = \langle \hat{\Phi}^{(\dagger)}_{r_1, \alpha_1} (t_1) \hat{\Phi}^{(\dagger)}_{r_2, \alpha_2} (t_2) \rangle = 0 \]

- Quasi-momentum distribution \( n_{\vec{k}}(t) = \langle \hat{\Phi}_{\vec{k}}^{\dagger}(t) \hat{\Phi}_{\vec{k}}(t) \rangle \)
- (a) 18 site chain; varied \( \vec{k} \); fixed quench timescale \( \tau_Q \)
  \( \tau_Q = 50 \)
  \( \mu = 0.25, T = 0, (J/U)_i = 0, (J/U)_f = 0.07 \)
(b-c) $(6 \times 6)$ square system

- (b) Fixed $\tau_Q$; varied $\vec{k}$
- (c) Fixed $\vec{k}$; varied $\tau_Q$
- $\mu = 0.25, T = 0, (J/U)_i = 0, (J/U)_f = 0.035$

General remark: $\vec{k} = 0$ grows the most after quench

- Small depletion in higher momentum states
- Consistent with a growing correlation length as SF phase is approached
Numerical Results

- Single-particle correlations in real-space \( \rho_{\Delta x}^{(F)}(t, t) = \langle \hat{\Phi}^\dagger_{\Delta x}(t)\hat{\Phi}_0(t) \rangle \)

![Graph showing single-particle correlations in real-space for different values of \( \Delta x \).]

- (a) \( \rho_{\Delta x}^{(F)}(t, t) \); 18 site chain
  - \( \mu = 0.25, T = 0, \tau_Q = 5, (J/U)_i = 0, (J/U)_f = 0.07 \)

- (b) Zoom-in of (a)
  - Propagation of correlation front, i.e. moving correlation ”wave-packet”
(c) Position of correlation front vs. time

- Light-cone spreading of correlations
- Propagation speed: $v_{1D} \simeq 3.7 \frac{Ja}{\hbar}$

Compared to literature:

- Barmettler et al. [1]: $v_{1D} = 6 \frac{Ja}{\hbar}$ for $U = \infty$,
  \hspace{1cm} $v_{1D} = 4 \frac{Ja}{\hbar}$ for $U = 0$ (exact results)
- Natu and Mueller [2]: $v_{1D} = 3.7 \frac{Ja}{\hbar}$ for weak interactions

(8 x 8) square system
- $\mu = 0.25$, $T = 0$, $\tau_Q = 5$, $\langle J/U \rangle_i = 0$, $\langle J/U \rangle_f = 0.03725$

(a) Light-cone effect persists; $v_{2D} = 2.9 \frac{Ja}{\hbar}$
- Navez and Schützhold result [1]: $6.2 \frac{Ja}{\hbar}$
- Our result expected to be more accurate with increasing dimensionality

(b) Long-time single-particle correlations $\rho_{\Delta r}^{(F)} (t_\infty, t_\infty)$
- Exponential decay with increasing separation distance $\Delta r$
- Inset: Correlation length $\xi$ grows with final hopping ratio $\langle J/U \rangle_f$

Conclusion

- Derived 2PI Schwinger-Keldysh action for the Bose-Hubbard model at strong coupling allowing for time dependent hopping
  - Obtained 2PI equations of motion for correlations and the order parameter during a quantum quench

- Observed light-cone spreading of correlations in MI quenches
  - We observed a 20% decrease in $v_d$ from $d = 1 \rightarrow 2$
  - Potentially large enough to be verified by experiment

- Future work and challenges:
  - Study other types of quenches, e.g. SF \(\rightarrow\) MI
  - Extend to multi-component BHMs
  - Equations of motion for quenches involving SF phase can be quite stiff, numerical stability an issue
Equations of motion in MI regime

First equation of motion:

\[
0 = \sum_{r,r'} \left\{ \delta_{r_1 r'} \left( \nu - i\lambda \frac{\partial}{\partial t_1} - \kappa^2 \frac{\partial^2}{\partial t_1^2} \right) + 2J_{r_1 r'}(t_1) \right\} \rho^{(\rho)}_{r_r' r}(t_1, t_2) \\
- 2iu_1 \rho^{(F)}_{r_1 r_1}(t_1, t_1) \rho^{(\rho)}_{r_1 r_2}(t_1, t_2) - u_2 \int_{t_2}^{t_1} dt' \rho^{(\rho)}_{r_1 r_1}(t_1, t') \rho^{(\rho)}_{r_1 r_1}(t', t_2) \\
- 2iu_1^2 \int_{t_2}^{t_1} dt' \sum_{r,r'} \left\{ 8\rho^{(F)}_{r_1 r}(t_1, t') \rho^{(\rho)}_{r_1 r_1}(t_1, t') \rho^{(F)}_{r_1 r_1}(t', t_1) \\
+ \left( 4 \left[ \rho^{(F)}_{r_1 r}(t_1, t') \right]^2 - \left[ \rho^{(\rho)}_{r_1 r_1}(t_1, t') \right]^2 \right) \rho^{(\rho)}_{r_r' r_1}(t', t_1) \right\} \rho^{(\rho)}_{r_r' r_2}(t', t_2)
\]

- Spectral function: \( \rho^{(\rho)}_{r_1 r_2}(t_1, t_2) \equiv i \left( \langle \Phi_{r_1}(t_1) \Phi_{r_2}^\dagger(t_2) \rangle - \langle \Phi_{r_2}(t_2) \Phi_{r_1}^\dagger(t_1) \rangle \right) \)
- Statistical propagator: \( \rho^{(F)}_{r_1 r_2}(t_1, t_2) \equiv \frac{1}{2} \left( \langle \Phi_{r_1}(t_1) \Phi_{r_2}^\dagger(t_2) \rangle + \langle \Phi_{r_2}(t_2) \Phi_{r_1}^\dagger(t_1) \rangle \right) \)
- \( \nu, \lambda, \kappa, u_1, u_2 \) are nontrivial functions of \( U \) and \( \mu \)
Equations of motion in MI regime

Second equation of motion:

\[ 0 = \sum_{r'} \left\{ \delta_{r_1 r'} \left( v - i\lambda \frac{\partial}{\partial t_1} - \kappa^2 \frac{\partial^2}{\partial t_1^2} \right) + 2J_{r_1 r'}(t_1) \right\} \rho^{(F)}_{r' r_2}(t_1, t_2) \]

\[ - 2iu_1 \rho^{(F)}_{r_1 r_1}(t_1, t_1) \rho^{(F)}_{r_1 r_2}(t_1, t_2) \]

\[ - u_2 \left( \int_{-\infty}^{t_1} dt' \rho^{(\rho)}_{r_1 r_1}(t_1, t') \rho^{(F)}_{r_1 r_1}(t', t_2) + \int_{-\infty}^{t_2} dt' \rho^{(F)}_{r_1 r_1}(t_1, t') \rho^{(\rho)}_{r_1 r_1}(t', t_2) \right) \]

\[ - \frac{1}{2} iu_1^2 \int_{-\infty}^{t_1} dt' \sum_{r'} \left\{ 2\rho^{(F)}_{r_1 r_1}(t_1, t') \rho^{(\rho)}_{r_1 r_1}(t_1, t') \rho^{(F)}_{r_1 r_1}(t', t_1) \right\} \rho^{(F)}_{r' r_2}(t', t_2) \]

\[ + \left( 4 \left[ \rho^{(F)}_{r_1 r_1}(t_1, t') \right]^2 - \left[ \rho^{(\rho)}_{r_1 r_1}(t_1, t') \right]^2 \right) \rho^{(\rho)}_{r_1 r_1}(t', t_1) \right\} \rho^{(F)}_{r_1 r_2}(t', t_2) \]

\[ - \frac{1}{2} iu_1^2 \int_{-\infty}^{t_2} dt' \sum_{r'} \left\{ 2\rho^{(F)}_{r_1 r_1}(t_1, t') \rho^{(\rho)}_{r_1 r_1}(t_1, t') \rho^{(F)}_{r_1 r_1}(t', t_1) \right\} \rho^{(F)}_{r_1 r_2}(t', t_2) \]

\[ + \left( 4 \left[ \rho^{(F)}_{r_1 r_1}(t_1, t') \right]^2 - \left[ \rho^{(\rho)}_{r_1 r_1}(t_1, t') \right]^2 \right) \rho^{(\rho)}_{r_1 r_1}(t', t_1) \right\} \rho^{(F)}_{r_1 r_2}(t', t_2) \]