Reentrant phase transitions and van der Waals behaviour for hairy black holes

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Black hole chemistry

- There is a close relationship between the laws of thermodynamics and the laws of black hole mechanics [Hawking et al; 1973]

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- Geometric and scaling arguments suggest that the cosmological constant, Λ, should be considered as a thermodynamic parameter in the first law [Kastor et al; 0904.2765],

$$dM = TdS + VdP + \sum_i \Omega_i dJ_i + \Phi dQ$$

where

$$P = -\frac{\Lambda}{8\pi} = \frac{(d-1)(d-2)}{16\pi\ell^2} \quad (\text{here } \Lambda < 0)$$

$\Rightarrow$ mass is enthalpy.

- Why? The first law and Smarr formula are related via Eulerian scaling

$$(d-3)M = (d-2)TS - 2VP + (d-2)\sum_i \Omega_i J_i + (d-3)\Phi Q$$

Black hole chemistry: results

- Can write black hole equations of state which lead to precise physical analogies with thermal systems: van der Waals behaviour, triple points, (multiple)-reentrant phase transitions [Mann, Kubiznak; 1401.2586].
- Thermodynamically inspired black hole solutions [Mann, Kubiznak; 1408.1105].
- Here we study the “chemistry” of AdS black holes with conformal scalar hair for the first time [Hennigar, Mann; 1509.06798].

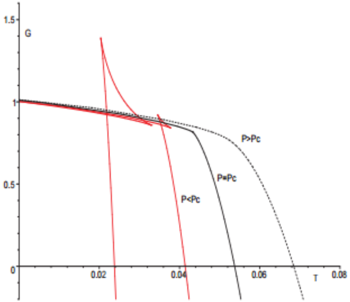

Figure: Gibbs free energy for the charged AdS black hole [1205.0559].

Black holes with conformal scalar hair

$$\mathcal{I} = \int d^d x \sqrt{-g} \left[\frac{1}{2\kappa} [R - 2\Lambda] - \frac{1}{2} (\nabla\phi)^2 + \frac{1}{2} \frac{d-2}{4(d-1)} R\phi^2 \right]$$

- For Einstein gravity conformally coupled to a scalar field in asymptotically flat spacetime there exist well-known no-hair theorems. "Black holes have no hair."
- In the presence of a cosmological constant, the no hair theorems can be evaded. Hairy black holes with conformal scalar hair are known to exist in $d = 3$ and $d = 4$ [Martinez et al; hep-th/0205319, hep-th/0406111].
- In $d > 4$ no go results had been reported $\Rightarrow$ a new a approach required. Idea: couple the scalar field to dimensionally extended Euler densities [Oliva, Ray; 1112.4112].

An aside: Lovelock gravity

- The focus here will be on a special class of hairy black holes where the scalar field is conformally coupled to higher curvature terms.
- Higher curvature gravity modifies the standard Einstein-Hilbert action through the addition of higher curvature terms,

$$\mathcal{I} = \int d^d x \sqrt{g} \left(c_0 + c_1 R + c_2 \mathcal{L}(\mathcal{R}^2) + c_3 \mathcal{L}(\mathcal{R}^3) + \cdots \right)$$

- Generically, bad things will happen e.g. the field equations will no longer be second order differential equations.
- Lovelock gravity is the most general higher curvature theory of gravity which maintains second order field equations,

$$\mathcal{L}^{(k)} = \frac{2k!}{2^k} \delta^{a_1}_{[c_1} \delta^{b_1}_{d_1} \cdots \delta^{a_k}_{c_k} \delta^{b_k}_{d_k]} R_{a_1 b_1}{}^{c_1 d_1} \cdots R_{a_k b_k}{}^{c_k d_k}$$

- Appear in perturbative approaches to quantizing gravity.

Hairy black holes

- Design a tensor,
$$S_{\mu\nu}{}^{\gamma\delta} = \phi^2 R_{\mu\nu}{}^{\gamma\delta} + \text{necessary terms}$$
which transforms as $S_{\mu\nu}{}^{\gamma\delta} \to \Omega^{-8/3} S_{\mu\nu}{}^{\gamma\delta}$ when $g_{\mu\nu} \to \Omega^2 g_{\mu\nu}$ and $\phi \to \Omega^{-1/3}\phi$.
- Can build terms where the scalar field is conformally coupled to the Euler densities [Oliva et al; 1112.4112, 1508.03780],
$$\mathcal{L}^{(k)}(\phi, \nabla\phi) = b_k \frac{2k!}{2^k} \phi^{3d-8k} \delta^{a_1}_{[c_1} \delta^{b_1}_{d_1} \cdots \delta^{a_k}_{c_k} \delta^{b_k}_{d_k]} S_{a_1 b_1}{}^{c_1 d_1} \cdots S_{a_k b_k}{}^{c_k d_k}$$
- Here [Hennigar, Mann; 1509.06798] we consider solutions of the theory,
$$\begin{aligned} \mathcal{I} &= \frac{1}{\kappa} \int d^5x \sqrt{-g} \Big[R - 2\Lambda - \frac{1}{4} F^2 + \kappa \Big(b_0 \phi^{15} + b_1 \phi^7 S_{\mu\nu}{}^{\mu\nu} \\ &\quad + b_2 \phi^{-1} (S_{\mu\gamma}{}^{\mu\gamma} S_{\nu\delta}{}^{\nu\delta} - 4 S_{\mu\gamma}{}^{\nu\gamma} S_{\nu\delta}{}^{\mu\delta} + S_{\mu\nu}{}^{\gamma\delta} S^{\nu\mu}{}_{\gamma\delta}) \Big) \Big] \end{aligned}$$

Metric & properties

- The metric is given by

$$ds^2 = -fdt^2 + \frac{dr^2}{f} + r^2 d\Sigma^2_{(k)3}\,, \quad f = k - \frac{m}{r^2} - \frac{q}{r^3} + \frac{e^2}{r^4} + \frac{r^2}{\ell^2}$$

with $k = -1, 0, 1$ while

$$q = \frac{64\pi}{5} k b_1 n^9\,, \quad n = \epsilon \left(-\frac{18 k b_1}{5 b_0}\right)^{1/6}\,, \quad \phi = \frac{n}{r^{1/3}}\,, \quad A = \frac{\sqrt{3}e}{r^2} dt\,, \quad F = dA\,,$$

here $\epsilon = -1, 0, 1$. The couplings have to obey the constraint $10 b_0 b_2 = 9 b_1^2$.

- Thermodynamic quantities which satisfy the first law & Smarr formula,

$$M = \frac{3\omega_{3(k)}}{16\pi} m\,, \quad Q = -\frac{\sqrt{3}\omega_{3(k)}}{16\pi} e\,, \quad S = \omega_{3(k)} \left(\frac{r_+^3}{4} - \frac{5}{8} q\right)\,, \quad V = \frac{\omega_{3(k)}}{4} r_+^4$$

$$T = \frac{1}{\pi \ell^2 r_+^4} \left[r_+^5 + \frac{k \ell^2 r_+^3}{2} + \frac{q \ell^2}{4} - \frac{e^2 \ell^2}{2 r_+}\right]\,, \quad \Phi = -\frac{2\sqrt{3}}{r_+^2} e$$

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Equation of state and Gibbs energy

- The equation of state is,

$$P = \frac{T}{v} - \frac{2k}{3\pi v^2} + \frac{512}{243}\frac{e^2}{\pi v^6} - \frac{64}{81}\frac{q}{\pi v^5}$$

where the specific volume is $v = 4r_+/3$.

- The Gibbs free energy is,

$$G = M - TS \quad = \quad \omega_{3(k)}\left[\frac{9kv^2}{256\pi} - \frac{27v^4P}{1024} + \frac{40q^2}{81\pi v^4} + \left(\frac{5Pv}{8} + \frac{5k-4}{12\pi v}\right)q + \left(\frac{5}{9\pi v^2} - \frac{320q}{243\pi v^5}\right)e^2\right].$$

Results I: van der Waals behaviour ($k = 1$, $e = 0$, $q < 0$)

- For $e = 0$, $q < 0$ and $k = 1$ there is a single critical point,

$$T_c = -\frac{3}{20\pi}\frac{(-5q)^{2/3}}{q}, \quad v_c = \frac{4}{3}(-5q)^{1/3}, \quad P_c = \frac{9}{200\pi}\left(-\frac{\sqrt{5}}{q}\right)^{2/3}$$

with mean field theory critical exponents, $\alpha = 0$, $\beta = \frac{1}{2}$, $\gamma = 1$, $\delta = 3$, $P_c v_c / T_c = 2/5$.

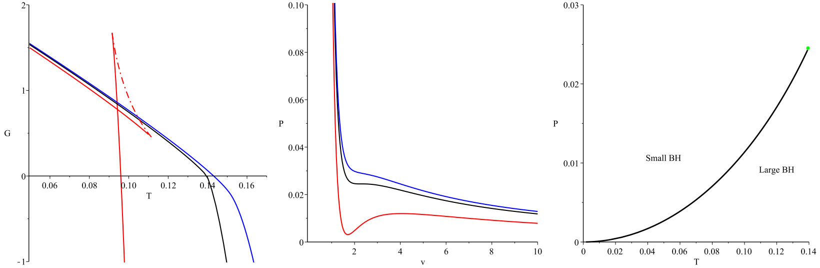

Figure: $q = -1$: G vs. T, P vs. v and P vs. T (left to right). red ⇒ less than critical value, black ⇒ critical value, blue ⇒ greater than critical value.

Results II: reentrant phase transition ($k = 1$, $e = q = 1$)

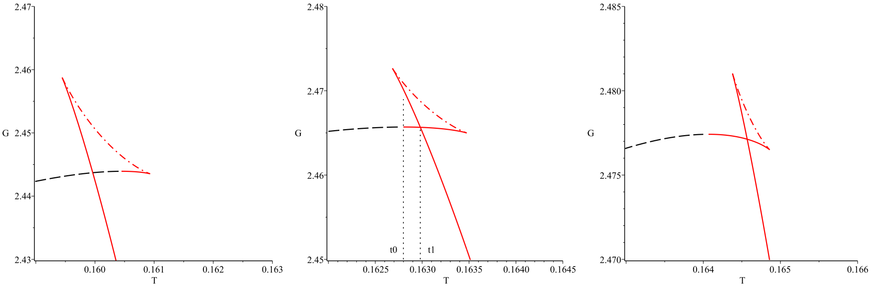

Figure: **Gibbs free energy**: $k = 1$, $e = q = 1$. Pressures $P = 0.031, 0.0313$, and 0.032 (left to right). The dashed black line corresponds to parameters yielding negative entropy. We see a large/small/large BH reentrant phase transition.

- **Reentrant phase transition**: A monotonic variation of a thermodynamic parameter yields two (or more) phase transitions, with the final state macroscopically similar to the initial state.

Results II: reentrant phase transition ($k = 1$, $e = q = 1$)

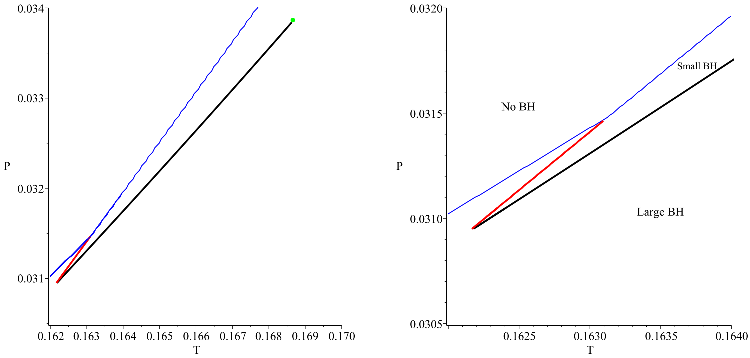

Figure: **Coexistence plots**: $k = 1$, $e = q = 1$. *Left*: $P - T$ coexistence plot showing zeroth and first order phase transitions (red and black curves, respectively). To the left of the blue line the black holes have negative entropy. *Right*: Zoomed version of the left plot.

Results III: zero entropy limit

$$S = \omega_{3(k)} \left(\frac{r_+^3}{4} - \frac{5}{8} q \right)$$

- Expanding the equation of state near the critical point in terms of $\omega = v/v_c - 1$, $\rho = P/P_c - 1$ and $\tau = T/T_c - 1$ gives

$$\rho \approx A\tau - B\omega\tau - C\omega^3$$

with $A, B, C > 0$. Solving for ω there are three real solutions:

$$\omega_1 = \frac{2}{3}\sqrt{\frac{-3B\tau}{C}} \quad \omega_2 = \omega_3 = -\frac{1}{3}\sqrt{-\frac{3B\tau}{C}}$$

- Then the entropy is,

$$q = \frac{27}{160} v_c^3 \Rightarrow S_i = \frac{27\pi^2 v_c^3}{128} \left((\omega_i + 1)^3 - 1 \right)$$

- Only one branch has positive entropy, so there are no phase transitions.

Results IV: summary

- No interesting results in the $k = 0, -1$ cases.

Case summary for $k = +1$		
	$q > 0$	$q < 0$
$e = 0$	no criticality	van der Waals
$e \neq 0$	reentrant phase transition	van der Waals

Table: q is the hair parameter; e is electric charge.

- Criticality ceases in the zero entropy limit.

Conclusions

- We considered, for the first time, the extended phase space thermodynamics of black holes with conformal scalar hair.
- In the hyperbolic and flat cases, there were no interesting results.
- In the spherical case, we see van der Waals behaviour and reentrant phase transitions
- In the case of zero entropy, the criticality ceases.

Thank you!

Example of “no criticality” behaviour

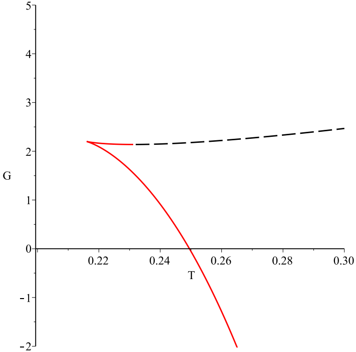

Figure: **Uncharged case: Gibbs free energy**: $k = 1, q = 1$. A representative G vs. T plot for $P = 0.05$ and $q = 1$. The red lines represent physical branches of the Gibbs energy, while the dashed black line corresponds to negative entropy black holes. The Gibbs free energy displays a cusp, the upper branch of which terminates at finite temperature due to enforcing positivity of entropy. At temperatures below the cusp, no black hole solutions exist.

Hairy black holes

- The tensor,

$$S_{\mu\nu}{}^{\gamma\delta} = \phi^2 R_{\mu\nu}{}^{\gamma\delta} - 12\delta^{[\gamma}_{[\mu}\delta^{\delta]}_{\nu]}\nabla_\rho\phi\nabla^\rho\phi - 48\phi\delta^{[\gamma}_{[\mu}\nabla_{\nu]}\nabla^{\delta]}\phi + 18\delta^{[\gamma}_{[\mu}\nabla_{\nu]}\phi\nabla^{\delta]}\phi$$

transforms as $S_{\mu\nu}{}^{\gamma\delta} \to \Omega^{-8/3} S_{\mu\nu}{}^{\gamma\delta}$ when $g_{\mu\nu} \to \Omega^2 g_{\mu\nu}$ and $\phi \to \Omega^{-1/3}\phi$.

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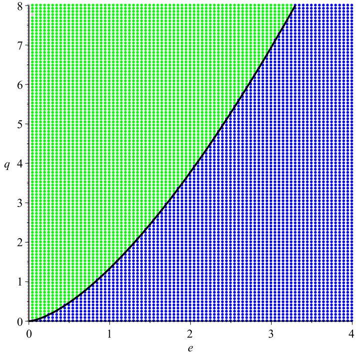

Figure: $k = 1$: q-e parameter space: blue dots indicate physical critical points, green dots indicate unphysical critical points (negative entropy). Black line: $q \approx 1.3375e^{3/2}$.