Open Dynamics under Rapid Repeated Interaction

Daniel Grimmer David Layden
Eduardo Martin-Martinez Robert Mann

University of Waterloo
Institute for Quantum Computing

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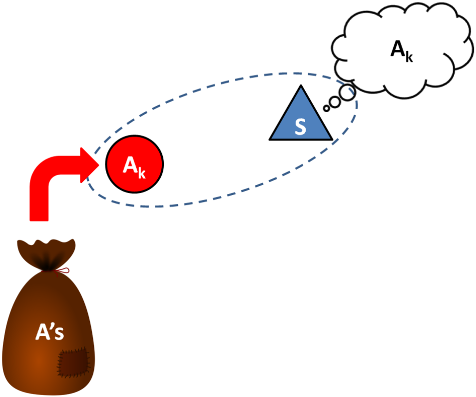
What happens in a single interaction?

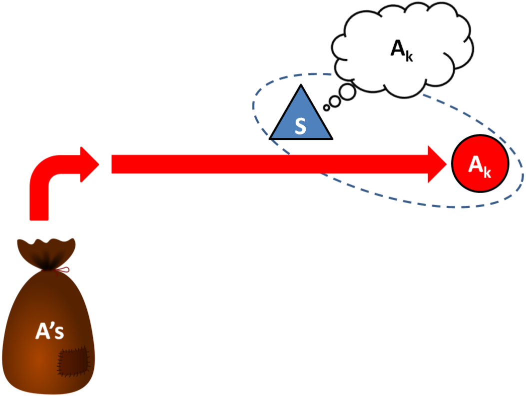
Some ancilla is picked (from an ensemble) and engages with the system:



$$\rho_{\mathrm{S}} \otimes \rho_{\mathrm{A}_k}$$

What happens in a single interaction?

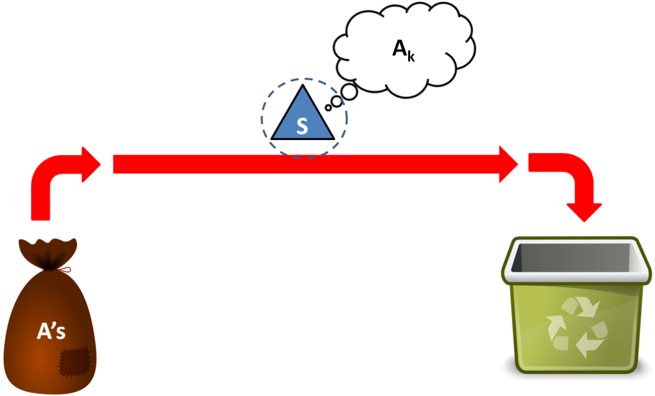
Then, depending on the ancilla chosen, the joint system evolves unitarily:



$$U_k(\delta t)(\rho_{\mathrm{S}} \otimes \rho_{\mathrm{A}_k})U_k(\delta t)^\dagger$$

What happens in a single interaction?

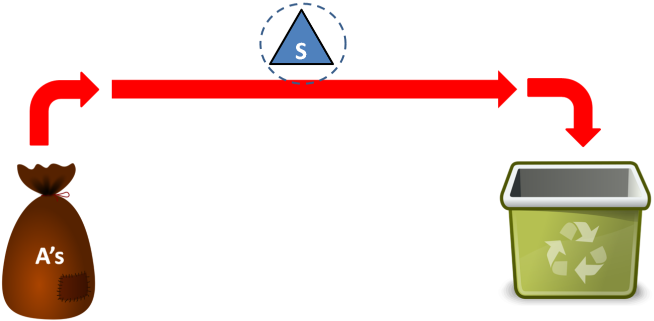
Following the interaction, the ancilla is discarded:



$$Tr_{A_k}\left(U_k(\delta t)(\rho_S \otimes \rho_{A_k})U_k(\delta t)^\dagger\right)$$

What happens in a single interaction?

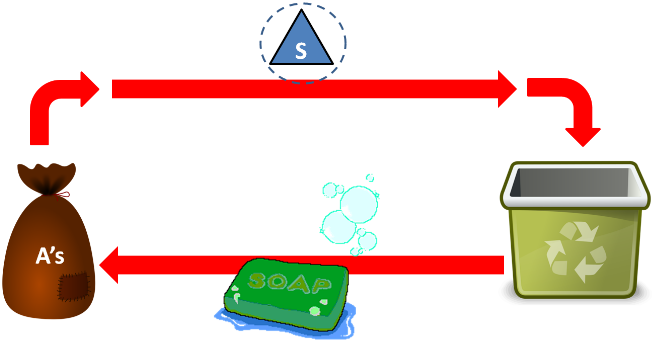
Finally we average over all ancillas which could have been chosen:



$$\bar{\phi}(\delta t) = \sum_k p_k \, Tr_{A_k}\big(U_k(\delta t)(\rho_S \otimes \rho_{A_k})U_k(\delta t)^\dagger\big)$$

What happens in a single interaction?

Optional: The ancillas can be reused if they are cleaned.



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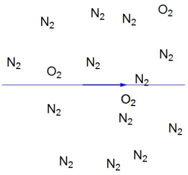
Some Applicable Scenarios

Through a Gas:

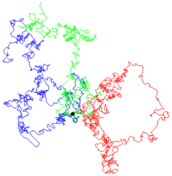


NMR: Nuclear spin interacting with electrons

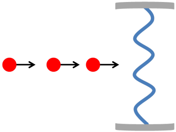
Gravitational Decoherence[1]

Atom bombarded by a series of atoms/light pulses:

In a Gas:



Entanglement Farming[2]
Cavity bombarded by atoms:



[1] D. Kafri, J.M. Taylor, G. J. Milburn; New Journal of Physics, Volume 16, June 2014
[2] E. Matrin-Martinez, E. Brown, W. Donnelly, A. Kempf; Phys. Rev. A 88, 052310 (2013)

Results

- In fast interaction limit ($\delta t \to 0$), evolution is unitary
- Decoherence related to classical/quantum ‘uncertainty’
- Applications
  - Decoherence in Media
  - Measurement Problem
  - Quantum Information Processing
  - Quantum Thermodynamics

Interpolation Scheme

Single interaction: $\bar{\phi}(\delta t) = \Sigma_k \, p_k \, \mathrm{Tr}_{\mathrm{A}_k}\big(U_{\delta t,k}(\delta t)(\,\cdot\,\otimes\rho_{\mathrm{A}_k})U_{\delta t,k}(\delta t)^\dagger\big)$

System evolves under repeated interactions, at $t = n\,\delta t$ we have,

$$\rho_S(n\;\delta t) = \bar{\phi}\big(\delta t\big)\big[\bar{\phi}\big(\delta t\big)\big[\ldots\bar{\phi}\big(\delta t\big)\big[\rho_S(0)\big]\ldots\big]\big] = \bar{\phi}\big(\delta t\big)^n\big[\rho_S(0)\big]$$

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Issue: Only know about discrete time points.

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Solution: We interpolate the system state as,

$$\rho_{\mathrm{S}}(t) = \Omega_{\delta t}(t)\big[\rho_{\mathrm{S}}(0)\big]$$

with exact matching at discrete time points, $\Omega_{\delta t}(n\,\delta t) = \bar{\phi}(\delta t)^n$.

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Issue: There are many choices for such a interpolation scheme.
Solution: Restrict to be Markovian, $\Omega_{\delta t}(t) = e^{\mathcal{L}_{\delta t}t}$. Yields unique

$$\mathcal{L}_{\delta t} = \frac{1}{\delta t}\log\left(\bar{\phi}(\delta t)\right)$$

Master Equation

This effective Liouvillian $\mathcal{L}_{\delta t}$ can be expanded as a series in δt generates time evolution for the interpolation scheme,

$$\frac{\mathrm{d}}{\mathrm{dt}}\rho_{\mathrm{S}}(t) = \mathcal{L}_{\delta t}[\rho_{\mathrm{S}}(t)] = \mathcal{L}_0[\rho_{\mathrm{S}}(t)] + \delta t\,\mathcal{L}_1[\rho_{\mathrm{S}}(t)] + \delta t^2\mathcal{L}_2[\rho_{\mathrm{S}}(t)] + \dots$$

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We take the general system-ancilla interaction Hamiltonian,

$$H_k(\xi) = H_{\mathrm{S}} \otimes \mathbf{1} + \mathbf{1} \otimes H_{\mathrm{A}_k} + H_{\mathrm{SA}_k}(\xi) \;\; \text{where} \;\; \xi = t/\delta t$$

and use it to explicitly find the forms of the coefficients $\mathcal{L}_0$ and $\mathcal{L}_1$.

Zeroth Order Liouvillian

To zeroth order the evolution is entirely unitary!

$$\mathcal{L}_0[\,\cdot\,] = \frac{-\mathrm{i}}{\hbar}[H_{\text{eff}}{}^{(0)},\,\cdot\,]$$

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where $H_{\text{eff}}{}^{(0)} = H_S + H^{(0)}$. Free evolution plus interaction effects,

$$H^{(0)} = \sum_k p_k \, \mathrm{Tr}_{\mathrm{A_k}}\left(\rho_{\mathrm{A}_k} \int_0^1 d\xi \, H_{\mathrm{SA}_k}(\xi)\right),$$

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The system and ancilla do not become entangled at leading order in δt.

Interpretation: Pushing vs. Talking
Ancillas push the system but *do not have time* to talk (entangle) with it.[3]

[3] D. Layden, E. Matrin-Martinez and A. Kempf; Phys. Rev. A 93, 040301(R) (2016)

First Order Liouvillian

First subleading dynamics introduce leading order dissipative effect as well as subleading unitary dynamics.

$$\mathcal{L}_1[\,\cdot\,] = \frac{-\mathrm{i}}{\hbar}[{H_{\text{eff}}}^{(1)},\,\cdot\,] + \frac{1}{2}\mathcal{D}[\,\cdot\,]$$

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$$\mathcal{L}_1[\,\cdot\,] = \frac{-\mathrm{i}}{\hbar}[{H_{\text{eff}}}^{(1)},\,\cdot\,] + \frac{1}{2}\mathcal{D}[\,\cdot\,]$$

The new subleading unitary term is ${H_{\text{eff}}}^{(1)} = H_1^{(1)} + H_2^{(1)} + H_3^{(1)}$ where

$$H_1^{(1)} = \sum_k p_k \left\langle G_1\left(\frac{-\mathrm{i}}{\hbar}[H_{\mathrm{SA}_k}(\xi), H_{\mathrm{S}}]\right)\right\rangle_k \qquad G_1(X) = \int_0^1 (\xi - 1/2)\, X(\xi)\, d\xi$$

$$H_2^{(1)} = \sum_k p_k \left\langle G_2\left(\frac{-\mathrm{i}}{\hbar}[H_{\mathrm{SA}_k}(\xi), H_{\mathrm{A}_k}]\right)\right\rangle_k \qquad G_2(X) = \int_0^1 \xi\, X(\xi)\, d\xi$$

$$H_3^{(1)} = \sum_k p_k \left\langle G_3\left(\frac{-\mathrm{i}}{\hbar}[H_{\mathrm{SA}_k}(\xi_1), H_{\mathrm{SA}_k}(\xi_2)]\right)\right\rangle_k \qquad G_3(Y) = \frac{1}{2}\int_0^1 d\xi_1 \int_0^{\xi_1} d\xi_2\, Y(\xi_1,\xi_2)$$

Leading Order Dissipative Terms

The leading order dissipation is

$$\mathcal{D}[\,\cdot\,] = \frac{1}{\hbar^2}[H^{(0)},[H^{(0)},\,\cdot\,]] - \frac{1}{\hbar^2}\sum_k p_k \mathrm{Tr}_{\mathrm{A_k}}\Big([G_0(H_{\mathrm{SA}_k}),[G_0(H_{\mathrm{SA}_k}),\,\cdot\,\otimes\rho_{\mathrm{A}_k}]]\Big)$$

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Dissipation is related to 'uncertainty' of interaction,

$$\mathcal{D}[\rho_S] = \sum_k p_k\,\mathrm{Tr}_{\mathrm{A_k}}\big(\mathrm{Var}(C)[\rho_{\mathrm{S}}\otimes\rho_{\mathrm{A}_k}]\big)$$

where

$$\begin{aligned}
\text{Variance:}\quad & \mathrm{Var}(C) = \langle\langle {C_k}^2\rangle\rangle - \langle\langle C_k\rangle\rangle^2 \\
\text{Generalized Average:}\quad & \langle\langle C_k\rangle\rangle[\rho_{\mathrm{SA}_k}] = \rho_{\mathrm{A}_k}\otimes\Sigma_l p_l\,\mathrm{Tr}_{\mathrm{A_l}}\big(C_l[\rho_{\mathrm{SA}_l}]\big) \\
\text{Diff. Evolution Op:}\quad & C_k[\rho_{\mathrm{SA}_k}] = (\mathrm{i}\hbar)^{-1}[G_0(H_{\mathrm{SA}_k}),\rho_{\mathrm{SA}_k}]
\end{aligned}$$

Dissipation as Uncertainty: Decoherence Rates

In some simple examples we see, decoherence rates are proportional to how much information we are ignoring in the relevant ancilla observable.

Qubit $\sigma_z\sigma_z$ Coupling:	$\Gamma = 2\,\delta t\,{J_0}^2\,\Delta^2_{\sigma_{\mathrm{A},Z}}$	$\Delta^2_X = \langle X^2\rangle - \langle X\rangle^2$
Qubit $\sigma_x\sigma_x$ Coupling:	$\Gamma = 2\,\delta t\,{J_0}^2\,\Delta^2_{\sigma_{\mathrm{A},X}}$	
Product Interaction: $(H_{\mathrm{SA}} = J_{\mathrm{S}} \otimes J_{\mathrm{A}})$	$\Gamma = \delta t\,\lvert J_{\mathrm{S}}\rvert^2\,\Delta^2_{J_{\mathrm{A}}}/\hbar^2$	

All decoherence rates bounded as $\Gamma \leq \langle M_k\rangle\,\delta t\,E^2/\hbar^2 + \mathcal{O}(\delta t^2)$ where $\langle M_k\rangle$ is the average ancilla dimension and E is the interaction energy scale.

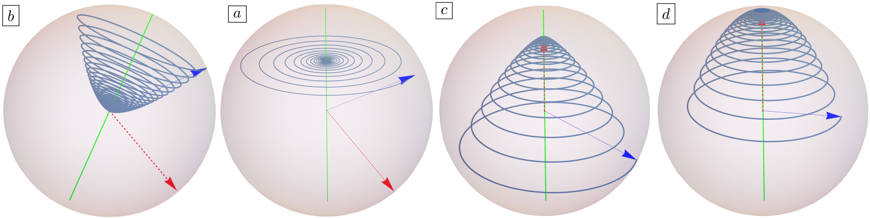
- Could be used to bound the dimension of environment's constituents.

Example: Qubit Qubit Interaction

As an example we consider a general interaction between qubits,

$$H(\xi) = \hbar\,\omega_{\mathsf{S}}\,\sigma_{\mathsf{S},z} + \hbar\,\omega_{\mathsf{A}}\,\sigma_{\mathsf{A},z} + \hbar\,\boldsymbol{\sigma}_{\mathsf{A}}\,\mathbf{J}(\xi)\,\boldsymbol{\sigma}_{\mathsf{S}}$$

with ancillas with bloch vectors, $\boldsymbol{R} = \mathrm{Tr}_{\mathsf{A}}\big(\rho_{\mathsf{A}}\,\boldsymbol{\sigma}_{\mathsf{A}}\big)$ (Red).



b) $\sigma_X\sigma_X$: Dephasing, a) $\sigma_Z\sigma_Z$: Projection, c,d) $\boldsymbol{\sigma}\boldsymbol{\sigma}$: Thermalizing/Purification
Green: $\boldsymbol{\omega}_{\text{eff}}$ axis, Red: Initial ancilla state, Blue: System state

Conclusion

General Model for Rapid Repeated Interactions:

- Ensemble of Ancilla/Coupling types
- No specific interaction Hamiltonians/ancilla state

In the continuum limit $\delta t \to 0$ the evolution is unitary:

- Pushing but no Talking
- Unitary Control

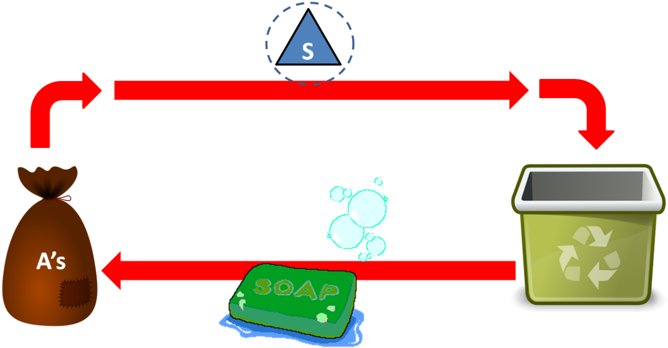
At small finite δt there are dissipative effects:

- Leading order information exchange
- Dissipation related to uncertainty

Qubit Examples:

- Dephasing: Decoherence in Media
- Projection: Measurement Problem
- Thermalization: Quantum Thermodynamics
- Purification: Pure State Initialization

Questions?



$$\bar{\phi}(\delta t) = \sum_k p_k Tr_{A_k}\left(U_k(\delta t)(\rho_S \otimes \rho_{A_k})U_k(\delta t)^\dagger\right)$$

Comparison to Other Schemes

- High Generality
  - Allow for ensemble of different interaction types.
  - No specific form chosen for ancillas
  - No specific form chosen for interaction Hamiltonian
  - Caves-Milburn repeated interaction model as a special case.
- Closed, analytic expression for master equation
- Finite interaction duration, δt
  - Do not take continuum limit ($\delta t \to 0$)
  - Necessary for information exchange at finite interaction strength.

Plan of Attack: Series of Series

$$
\begin{aligned}
\text{Model Inputs} &\to p_k,\ \rho_{\mathrm{A}_k},\ \text{and } H_k(t/\delta t)\\
\to \text{Evolve} &\to p_k,\ \rho_{\mathrm{A}_k},\ \text{and } U_{\delta t,k}(\delta t) = \mathcal{T}\exp\left(\int_0^{\delta t} d\tau\, H_k(\tau/\delta t)\right)\\
\to \text{Dyson Series} &\to p_k,\ \rho_{\mathrm{A}_k},\ \text{and } U_{\delta t,k}(\delta t) = \mathbf{1} + \delta t\ U_{k,1} + \delta t^2\ U_{k,2} + \ldots\\
\to \text{Average Int.} &\to \bar{\phi}(\delta t) = \Sigma_k\, p_k\, \mathrm{Tr}_{\mathrm{A}_k}\left(U_{\delta t,k}(\delta t)(\,\cdot\,\otimes\rho_{\mathrm{A}_k})U_{\delta t,k}(\delta t)^\dagger\right)\\
\to \text{Expand Series} &\to \bar{\phi}(\delta t) = \mathbf{1} + \delta t\ \bar{\phi}_1 + \delta t^2\ \bar{\phi}_2 + \ldots\\
\to \text{Interpolation} &\to \mathcal{L}_{\delta t} = \log\left(\bar{\phi}(\delta t)\right)/\delta t\\
\to \text{Expand Series} &\to \mathcal{L}_{\delta t} = \mathcal{L}_0 + \delta t\ \mathcal{L}_1 + \delta t^2\ \mathcal{L}_2 + \ldots
\end{aligned}
$$

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k, Labels for potential Ancilla's.
p_k, Probability for Ancilla k.
H_k, Hamiltonian for Ancilla k.

$U_{\delta t,k}(\delta t)$, Unitary for Ancilla k.
$\bar{\phi}(\delta t)$, Effective Discrete Updater.
$\mathcal{L}_{\delta t}$, Effective Liouvillian.