Turnaround radius in an accelerated universe for Einstein and for modified gravity

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Outline

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2. Turnaround radius in scalar-tensor gravity
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Turnaround radius with Hawking mass

Part of a larger program aiming at applying quasilocal mass in cosmology. Already used to test whether Newtonian $N$-Body simulations of large scale structures are reliable (VF, Prain & Lapierre-Léonard, PRD 2015).

Consider present accelerated era of the universe and the largest bound objects in the sky. The turnaround radius was suggested as a possible way to test dark energy (Roupas et al. 2014, PRD 89, 083002; Pavlidou & Tomaras 2014, JCAP 09, 020; Pavlidou, Tetradis, Tomaras 2014, JCAP 05, 017)

but the concept of TR is older (Souriau 1981; Stuchlik 1983; Stuchlik et al. 1989-2005; Mizony & Lachiéze-Rey 2005; Blau & Rollier 2008, ...)

Consider an accelerated FLRW universe with one spherical inhomogeneity; massive test particles with zero radial initial velocity cannot collapse if they are outside a critical radius $R_c$ (turnaround radius), but can only expand.
For $R < R_c$, outer layers of dust reach zero radial acceleration and collapse under self-gravity. If you cross outside $R_c$ in geodesic motion, you will never fall back.

TR studied in Schwarzschild-de Sitter, Lemaître-Tolman-Bondi, and McVittie spacetimes.

**SdS (heuristic):**
\[
ds^2 = - \left(1 - \frac{2M}{R} - H^2 R^2 \right) dt^2 + \frac{dR^2}{\left(1 - \frac{2M}{R} - H^2 R^2 \right)} + R^2 d\Omega_2^2
\]

\[
H = \sqrt{\Lambda/3}, \quad R_c = \left(\frac{3GM}{\Lambda}\right)^{1/3}
\]
Turnaround radius with Hawking mass in GR

Turnaround radius in scalar-tensor gravity

Conclusions
Radial timelike geodesics obey $R(\tau) = (R^3 - R_c^3) H^2 / R^2$

LTB models (dust) Pavlidou, Tetradi & Tomaras 2014 have

$$ds^2 = -dt^2 + \frac{R'(t, r)}{1 + f(r)} dr^2 + R^2(t, r) d\Omega^2_{(2)}$$

with $' \equiv d/dr$, $f(r)$ related to initial density profile. Radial timelike geodesics obey

$$\ddot{R} = - \frac{G\mathcal{M}(r)}{R^2} + \frac{\Lambda R}{3}$$

and the turnaround radius is $R_c = \left( \frac{3G\mathcal{M}(r_c)}{\Lambda} \right)^{1/3}$ where

$\mathcal{M}(r) = \int_0^R dR R^2 \rho$ Lemaître mass.
More realistic: post-FLRW space (1st order)

\[ ds^2 = a^2(\eta) \left[ - (1 + 2\phi) \, d\eta^2 + (1 - 2\phi) \left( dr^2 + r^2 d\Omega^2_{(2)} \right) \right] \]

Pavlidou, Tetradi & Tomaras find timelike radial geodesics obey

\[ \ddot{R} = -\frac{4\pi}{3} \left( \rho_{DE} + 3P_{DE} \right) R - \frac{GM(r)}{R^2} = \frac{\dot{a}}{a} - \frac{GM(r)}{R^2} \]

where it is suggested (but not written down)

\[ \mathcal{M}(r) = \int_0^R dR \, R^2 \rho_{\text{total}} \]

\[ \rightarrow \quad R_c = \left( \frac{3\mathcal{M}}{4(3w + 1)\pi \rho_{DE}} \right)^{1/3} \]

(reduces to SdS expression for \( w = -1 \)).
Questions: (not answered, nor posed)

- gauge-invariance;
- what is the “mass in a sphere of radius $R$”? Should it include $\rho_{DE}$? If not, why? Should it include only $\rho_{\text{perturbation}}$? Why?
Use Hawking-Hayward quasilocal energy (reduces to Misner-Sharp-Hernandez mass in spherical symmetry) and a new splitting of it. Assumptions:

- GR is valid
- 1st order in metric perturbations; spherical symmetry \( \phi = \phi(r) \) (consequences of \( \phi \neq \phi(r) \) discussed in Barrow & Saich 1993, MNRAS 262, 717)
- FLRW background, spatially flat, accelerated by DE with \( \rho_{DE}, P_{DE} = w\rho_{DE} \)
Physical mass is the Hawking quasilocal energy\(^1\)
Idea: total mass in a region bounded by a surface \(S\) is measured by behaviour of null geodesics at \(S\)

\[
S = \text{closed spacelike orientable 2-surface}
\]

\[
\mathcal{R} = \text{induced Ricci scalar on } S
\]

\[
\theta(\pm) = \text{expansions of outgoing/ingoing null geodesic congruences from } S
\]

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General perturbations of FLRW

\[ \sigma_{ab}^{(\pm)} = \text{shear tensors of null congruences} \]

\[ \omega^a = \text{projection on } S \text{ of the commutator of null normal vectors to } S \text{ (anholonomicity)} \]

\[ \mu = \text{volume 2-form on } S \]

\[ A = \text{area of } S \]

\[ M_{HH} \equiv \frac{1}{8\pi} \sqrt{\frac{A}{16\pi}} \int_S \mu \left( R + \theta^{(+)} \theta^{(-)} - \frac{1}{2} \sigma_{ab}^{(+)} \sigma_{ab}^{(-)} - 2\omega_a \omega^a \right) \]
Compute for

$$d\tilde{s}^2 = a^2(\eta) \left[ - (1 + 2\phi_N) d\eta^2 + (1 - 2\phi_N) \left( dr^2 + r^2 d\Omega_{(2)}^2 \right) \right]$$

post-Newtonian

and attempt to decompose as $M_{HH} = (\text{local}) + (\text{cosmological})$

to first order (general pert.)
Final result (with two methods) is

\[ \tilde{M}_H = \Omega M_H - \frac{R\Omega,\eta \Omega,\eta}{4\pi} \int_S \mu \phi_N + \sqrt{\frac{R^3 \Omega^2,\eta}{2}} \]

local + cosmological

Prain, Vitagliano, VF & Lapierre-Léonard, *Class. Quantum Grav.*, in press

Now adapt to spherical symmetry →
Turnaround radius with Hawking mass in GR

\[ M_H = ma + \frac{H^2 R^3}{2} (1 - \phi) \approx ma + \frac{H^2 R^3}{2} \]

with \( m = \int d^3 \vec{x} \nabla^2 \phi \) Newtonian mass \( \sim \) comoving length scale \( ma \sim \) physical length scale. Criterion for a system on the verge of breaking down is now

local part \quad ma = \frac{H^2 R^3}{2} \quad \text{cosmological part } \rightarrow \quad R_c(t) = \left( \frac{2ma}{H^2} \right)^{1/3}

Now \( H^2 = 8\pi G \rho_{DE}/3 \rightarrow R_c(t) = \left( \frac{3ma}{4\pi \rho_{DE}} \right)^{1/3} \) and, if \( w = \text{const.}, \quad R_c = \left( \frac{3ma}{4\pi \rho_0} \right)^{1/3} a^{\frac{3w+4}{3}} \)

Compare with Pavlidou, Tetradis & Tomaras

\[ \frac{R_c}{R_c^{(PTT)}} = \left( \frac{|3w + 1|}{2} \right)^{1/3} \approx 1 \quad \text{if } w \approx -1 \]
but now

- no ambiguities in “mass inside a sphere of radius $R_c$”; 
- rigorous derivation of turnaround radius $R_c$
- important if you want to constrain $w$
Can express $R_c = R_c(z)$ and invert to obtain

$$\int dz \frac{w(z) + 1}{z + 1} = \ln \left[ \left( \frac{3ma}{4\pi \rho} \right)^{1/3} \frac{1}{R(z)} \right]$$

If $w = \text{const.}$ reduces to

$$w(z) = -1 + \frac{\ln \left[ \left( \frac{3ma}{4\pi \rho_0} \right)^{1/3} \frac{1}{R_c(z)} \right]}{\ln(z + 1)}$$

constrain $w$ if $ma$ and $R_c$ are known.
Turnaround radius in ST gravity

\[ ds^2 = a^2(\eta) \left[ - (1 + 2\psi) d\eta^2 + (1 - 2\phi) \left( dr^2 + r^2 d\Omega^2 \right) \right], \]

\[ \phi = \phi(r), \psi = \psi(r). \] Massive test particles follow timelike geodesics

\[ \frac{du^a}{d\tau} + \Gamma^a_{bc} u^b u^c = 0, \]

\[ u_c u^c = -1 \] and the geodesic eq. give

\[ \frac{du^0}{d\tau} + \frac{a_\eta}{a} (u^0)^2 + 2\psi' u^0 u^1 + \frac{a_\eta}{a} (1 - 2\phi - 2\psi) (u^1)^2 = 0 \]

\[ \frac{du^1}{d\tau} + \psi' (u^0)^2 + \frac{2a_\eta}{a} u^0 u^1 - \phi' (u^1)^2 = 0 \]
Areal radius is $R(t, r) = ar\sqrt{1 - 2\phi} \simeq ar\,(1 - \phi)$, further manipulation yields

$$\frac{d^2 R}{dt^2} = \left[ \ddot{a}r + \frac{\dot{a}u^1}{au^0} + \frac{1}{au^0} \frac{d}{d\tau} \left( \frac{u^1}{u^0} \right) \right] (1 - \phi)$$

Criterion locating the (unique) turnaround radius is $d^2 R/ dt^2 = 0$, which becomes

$$\ddot{a}r - \frac{\psi'}{a} = 0$$
In terms of the *areal* turnaround radius,

\[ R_c = a(t)r_c [1 - \phi(r_c)] \]

or, using the gravitational slip \( \xi \equiv (\phi - \psi) / \phi \),

\[ \ddot{a} R_c (1 + \phi_c) - \phi_c' (1 - \xi_c) + \phi_c \xi_c' = 0 \]
Conclusions

- Turnaround radius is an opportunity to test gravity and the $\Lambda$CDM model.
- Split $M_H$ for spherical perturbations of FLRW $\rightarrow$ rigorous derivation of $R_c$, small correction, much needed clarification of “mass”.
- In modified gravity, no accepted $M_H$, use criterion $\ddot{R} = 0 \rightarrow$ eq. for $R_c$ in ST gravity.
- Is it important? Astronomers claim that the upper bound on $R_c$ in GR is exceeded by far in two galaxy groups (Lee et al. 2015, Astrophys. J. 815, 43; Lee, arXiv:1603.06672). Wait and see!
THANK YOU