Cluster of Excellence PRISMA
Precision Physics, Fundamental Interactions and Structure of Matter

Advanced methods for scattering amplitudes

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Scattering amplitudes

- important ingredients for cross sections
- for phenomenology, calculations are required for many processes
- calculations very challenging
- often, beautiful mathematical structures help with practical calculations

‘Ideal’ and ‘real’ scattering amplitudes

Is there some simpler version of QCD that allows to understand key properties of scattering amplitudes?

How can we obtain numerical results for cross sections at the LHC

This talk: tools for ‘real’ QCD coming from ‘ideal’ amplitudes

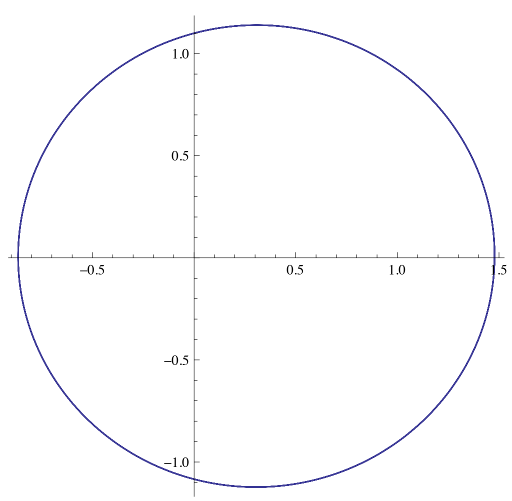
Idealized 'toy' theories: from Kepler to QFT

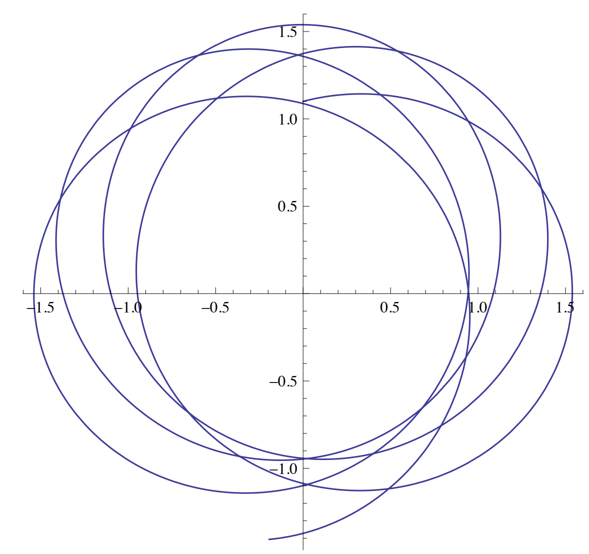
Idealized systems play an important role in physics

Often, (hidden) symmetries help to solve a problem

Example 1: Kepler problem

$$V = 1/r \qquad\qquad V = 1/r^{0.9}$$





- Laplace-Runge-Lenz (LRL) vector is conserved

$$\vec{A} = \frac{1}{2}\left(\vec{p} \times \vec{L} - \vec{L} \times \vec{p}\right) - \mu \frac{\lambda}{4\pi} \frac{\vec{x}}{|x|}$$

- consequence: orbits do not precess

Example 2: Hydrogen atom

- described by quantum mechanics
- hidden symmetry:
  Laplace-Runge-Lenz-Pauli vector

- gives elegant algebraic way to find spectrum

$$E_n = -\frac{mk^2}{2\hbar^2}\frac{1}{n^2} \qquad n = 1, 2, \ldots$$

- explains why there are n^2 states of energy E_n.

Is there a quantum field theory (preferably a gauge theory) that has the same symmetry?

Example 3: N=4 super Yang-Mills

- generalization of massless QCD
  - gluons, plus 4 complex fermions and 6 scalars in adjoint representation
  - masses can be added via Higgs mechanism
- conformal symmetry and (extended) supersymmetry
- has a hidden dual conformal symmetry

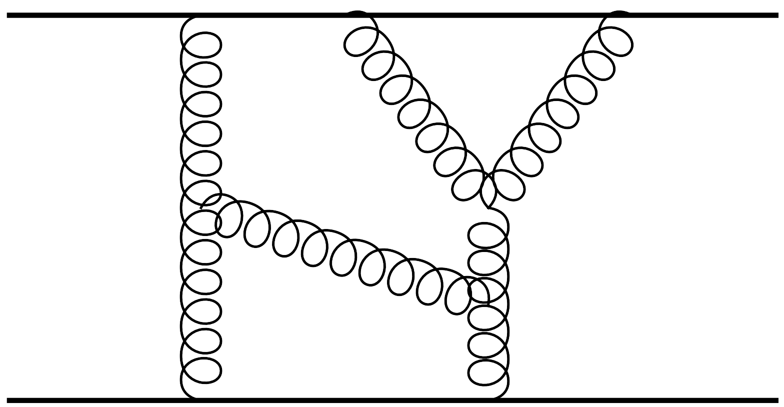
[Drummond, JMH, Korchemsky, Sokatchev, 2008]

[Yangian interpretation: Drummond, JMH, Plefka, 2009]

- this symmetry is a generalization of the LRL symmetry to a (planar) relativistic quantum field theory

[JMH and Caron-Huot, 2013]

e.g., extra symmetry governs spectrum of bound states of massive W bosons



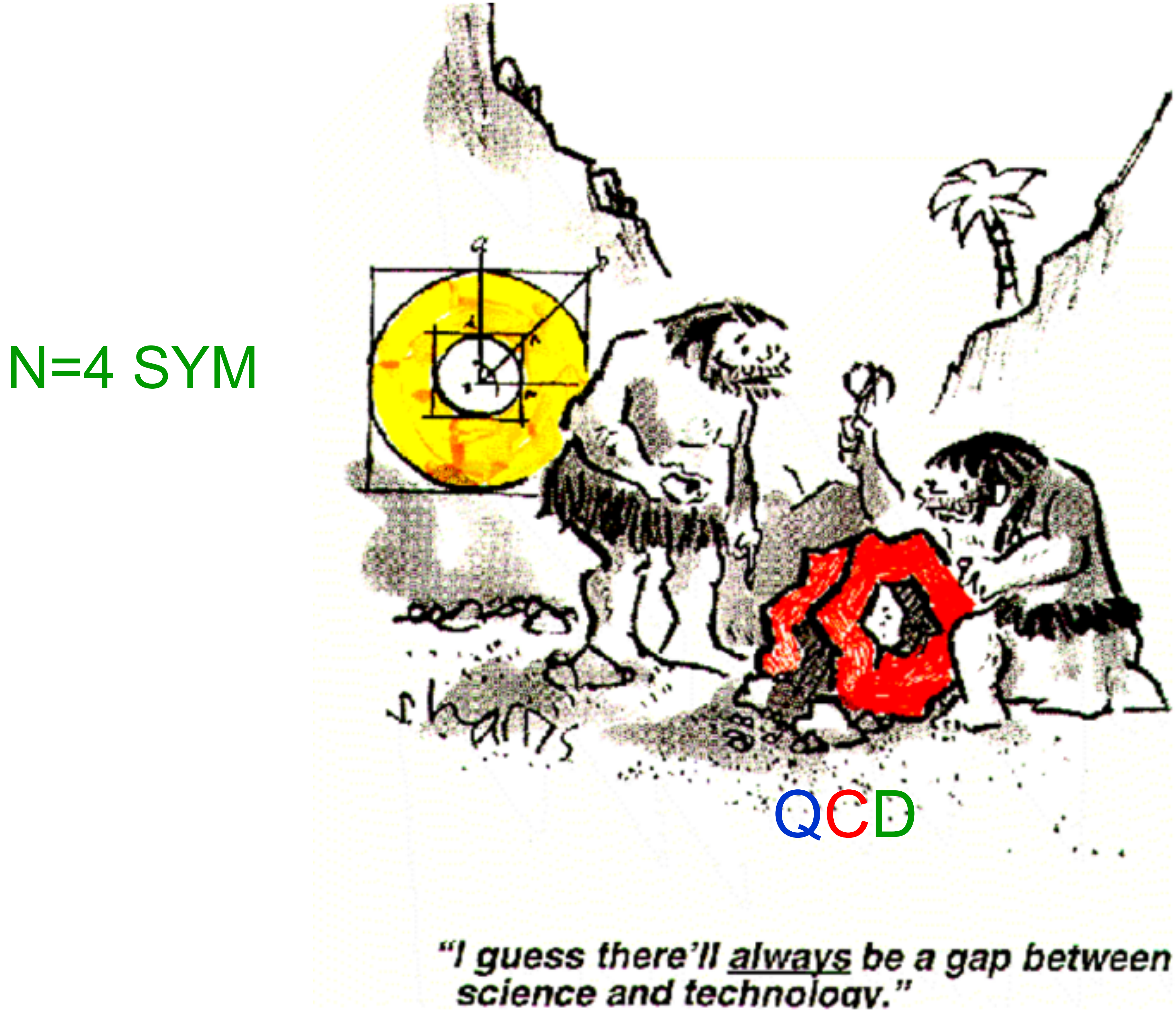
Laplace-Runge-Lenz symmetry

classical mechanics	Kepler problem
quantum mechanics	Hydrogen atom
quantum field theory	(planar) N=4 super Yang-Mills theory

Open questions:

- Is this the unique gauge theory with this property?
- Is there a generalization to the non-planar level?

From ‘science’ to ‘technology’



(slide from Lance Dixon’s talk at EPS HEP11 Grenoble)

On-shell techniques

- original idea: perturbative unitarity of S matrix
- on-shell recursions for tree amplitudes

[Britto, Cachazo, Feng, Witten, PRL 94 (2005)]

- construction of one-loop amplitudes

[Bern, Dixon, Dunbar, Kosower, Nucl. Phys. B425 (1994)]

[Anastasiou, Britto, Feng, Kunszt, Phys. Lett. B645 (2007)]

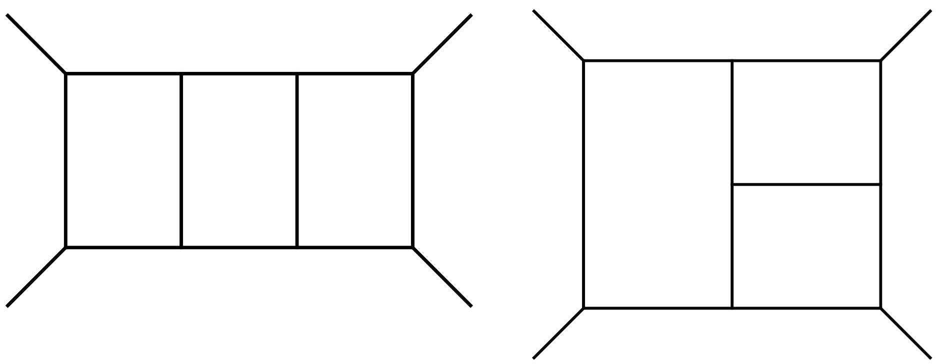
[Ossola, Papadopoulos, Pittau, Nucl. Phys. B763 (2007)]

- today: automated computations of one-loop amplitudes

NLO revolution

Examples of analytic progress: integrals with massless internal lines

- massless 2-2 scattering to 3 loops



[JMH, Smirnov, Smirnov] JHEP 1307 (2013) 128

$$s = (p_1 + p_2)^2 \qquad t = (p_2 + p_3)^2$$

$$x = t/s$$

[Di Vita, Mastrolia, Schubert, Yundin, 2014] JHEP 1409 (2014) 148

non-planar integrals: [JMH, A.V. Smirnov, V.A. Smirnov, 2013] JHEP 1403 (2014) 088

- all two-loop integrals for vector boson production pp to V V'

[JMH, Melnikov, Smirnov] JHEP 1405 (2014) 090

[Caola, JMH, Melnikov, Smirnov] JHEP 1409 (2014) 043

for pp to VV: [Gehrmann, von Manteuffel, Tancredi, Weihs] JHEP 1406 (2014) 032

[Gehrmann, Tancredi, Weihs] JHEP 1308 (2013) 070

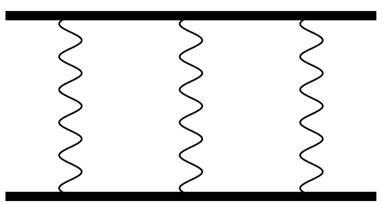
- new: planar two-loop 5-point integrals

[Gehrmann, JMH, Lo Presti, arXiv:1511.05409]

[Papadopoulos, Tommasini, Wever, arXiv:1511.09404]

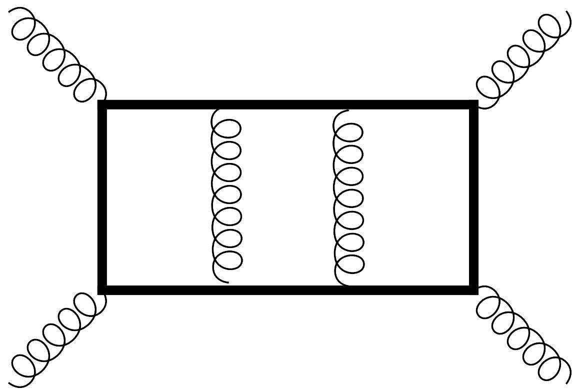
Examples of analytic progress: integrals with massive internal lines

- integrals for Bhabha scattering
  [JMH, V. Smirnov, JHEP 1311 (2013) 041]



scales:
s, t, m^2

- scattering amplitudes & cross sections in massive toy model in N=4 sYM



[JMH, S. Caron-Huot, JHEP 1406 (2014) 114]

s, t, m^2

3 loops and 3 scales!

- NLO QCD corrections to H to Z gamma

[Bonciani, Del Duca, Frellesvig, JMH, Moriello, JHEP 1508 (2015) 108]

[Gehrmann, Guns, Kara, JHEP 1509 (2015) 038]

- integrals needed for flavor physics [Huber, Kraenkl, JHEP 1504 (2015) 140]

- …

Loop integrands and integrals

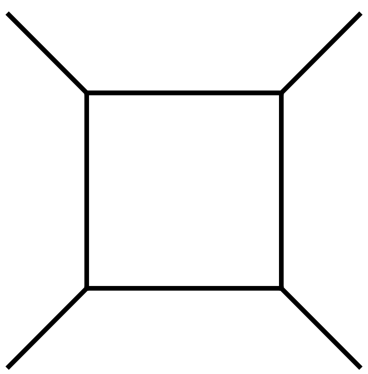
Typical steps in an amplitude calculation:

- determine and simplify loop integrand
- write it in a convenient basis of loop integrals
- carry out the loop integrations

Analyzing loop integrands: maximal cuts, leading singularities

- maximal cuts / leading singuarities

$D_1 = k^2 \quad D_2 = (k+p_1)^2 \quad D_3 = (k+p_1+p_2)^2 \quad D_4 = (k+p_1+p_2+p_3)^2$



$$= \int d^4k \delta(D_1)\delta(D_2)\delta(D_3)\delta(D_4) \sim \frac{1}{st}$$

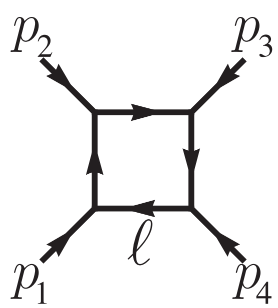
residues of integrand at poles: leading singularities

- observation: integrals with constant leading singularities have very nice properties

`d-log forms`

- sometimes, loop integrand can be rewritten in suggestive form

$$\mathcal{A}_4^{\ell=0} \times$$



$$= \mathcal{A}_4^{\ell=0} \times \int \frac{d^4\ell \;\; (p_1+p_2)^2 (p_1+p_3)^2}{\ell^2 (\ell+p_1)^2 (\ell+p_1+p_2)^2 (\ell-p_4)^2}$$

[Arkani-Hamed et al, 2012]
[Caron-Huot, talk at Trento, 2012]
[Lipstein and Mason, 2013-2014]

$$\frac{d^4\ell \;\; (p_1+p_2)^2 (p_1+p_3)^2}{\ell^2 (\ell+p_1)^2 (\ell+p_1+p_2)^2 (\ell-p_4)^2}$$
$$= d\log\left(\frac{\ell^2}{(\ell-\ell^*)^2}\right) d\log\left(\frac{(\ell+p_1)^2}{(\ell-\ell^*)^2}\right) d\log\left(\frac{(\ell+p_1+p_2)^2}{(\ell-\ell^*)^2}\right) d\log\left(\frac{(\ell-p_4)^2}{(\ell-\ell^*)^2}\right)$$

- makes leading singularities obvious

- both for planar and non-planar integrals

[Arkani-Hamed et al, 2014; Bern et al., 2015]

Leading singularities as guiding principle for an integral basis

- conjecture: integrals with constant leading singularities give rise to ‘pure’ functions

[Arkani-Hamed et al, 2012]

[Arkani-Hamed, Bourjaily, Cachzao, Trnka, 2010]

- pure functions are (rational linear combinations of) polylogarithmic functions of uniform weight

e.g. $$\mathrm{Li}_3(1 - x/y) + \frac{1}{2}\log^3(x) + \pi^2 \log(y)$$

- pure functions satisfy simple differential equations
- although first understood in N=4 sYM, this also applies for integrals needed for QCD

Differential equations (DE) technique

- idea: differentiate Feynman integral w.r.t. external variables, e.g. s, t, masses

Some general facts:

- a given Feynman integral f satisfies an n-th order DE
- equivalently described by a system of n first-order equations for $\vec{f}$

$$\partial_x \vec{f}(x,\epsilon) = A(x,\epsilon)\vec{f}(x,\epsilon)$$

Long and successful history:

[Kotikov, 1991] [Remiddi, 1997] [Gehrmann, Remiddi, 2000] [...]

New idea: use integrals with constant leading singularities as basis for DE system [JMH, 2013]

Canonical form of the differential equations

Example: one dimensionless variable x ; $D = 4 - 2\epsilon$

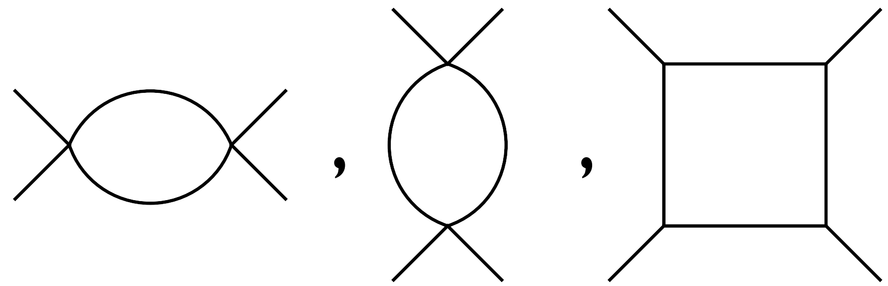
$$\partial_x \vec{f}(x;\epsilon) = \epsilon \sum_k \frac{A_k}{x - x_k} \vec{f}(x;\epsilon)$$

- the above equations decouple at D=4
- expansion to any order in ϵ is linear algebra

answer: multiple polylogarithms of uniform weight (‘transcendentality’)

- asymptotic behavior $\vec{f}(x;\epsilon) \sim (x - x_k)^{\epsilon A_k} \vec{f}_0(\epsilon)$
- natural extension to multi-variable case

Example: one-loop four-point integral

- basis $f = \{$  $\}$

$x = t/s$

$D = 4 - 2\epsilon$

differential equations

$$\partial_x f = \epsilon \left[\frac{a}{x} + \frac{b}{1+x} \right] f \qquad a = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ -2 & 0 & -1 \end{pmatrix} \qquad b = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 2 & 2 & 1 \end{pmatrix}$$

- (regular) singular points $s = 0\,, \quad t = 0\,, \quad u = -s - t = 0$
- asymptotic behavior governed by matrices a, b

- Solution: expand to any order in ϵ

$f = \sum_{k\geq 0} \epsilon^k f^{(k)}$ $\qquad f^{(k)}$ is k-fold iterated integral (uniform weight)

Multi-variable case and the alphabet

- Natural generalization to multi-variable case

$$d\vec{f}(\vec{x};\epsilon) = \epsilon\, d\left[\sum_k A_k \log \alpha_k(\vec{x})\right] \vec{f}(\vec{x};\epsilon)$$

A_k → constant matrices; $\alpha_k(\vec{x})$ → letters (alphabet)

- Examples of alphabets:

4-point on-shell: $\alpha = \{x, 1+x\}$

two-variable example (from 1-loop Bhabha scattering): $\alpha = \{x\,, 1 \pm x\,, y\,, 1 \pm y\,, x+y\,, 1+xy\}$

[J.M.H., Smirnov]

``hexagon functions`` in N=4 SYM:

$$\alpha = \{x, y, z, 1-x, 1-y, 1-z, \\ 1-xy, 1-xz, 1-yz, 1-xyz\}$$

[Goncharov, Spradlin, Vergu, Volovich] [Caron-Huot, He]
[Dixon, Drummond, J.M.H.] [Dixon et al.]

- Matrices and letters determine solution
- Immediate to solve in terms of iterated integrals

Beyond iterated integrals

- One such class are elliptic functions, needed e.g. in top quark physics [Czakon and Mitov, 2010]

- proposal for generalization to the elliptic case [JMH, 2014] (and beyond): $A(x,\epsilon)$ linear in ϵ e.g. sunrise integral:

$$A(x,\epsilon) = \frac{1}{x}\begin{pmatrix} -2\epsilon & 0 & 0 \\ 0 & 1 & -\frac{1}{4}-\frac{\epsilon}{2} \\ 0 & 0 & 1-\epsilon \end{pmatrix} + \frac{1}{1+x}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 3 & 3+9\epsilon & -1-2\epsilon \end{pmatrix} + \frac{1}{1/9+x}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -3 & 1+3\epsilon & -1-2\epsilon \end{pmatrix} .$$

- outlook: connect this to work on elliptic polylogarithms [Brown et al, Vanhove et al, Weinzierl et al.]

Analytic results for scattering amplitudes

- There has been enormous progress in analytically computing Feynman integrals

- result for Feynman integrals are as compact and simple as possible

- What about the amplitudes?

- Does the simplicity remain (if one looks hard enough)?

All-plus five-point two-loop amplitude

divergence structure

$$A_5^{(2)} = A_5^{(1)} \left[-\sum_{i=1}^{5} \frac{1}{\epsilon^2} \left(\frac{\mu^2}{-v_i} \right)^{\epsilon} \right] + R\, F_5^{(2)} + \mathcal{O}(\epsilon)\,,$$

finite part

$$F_5^{(2)} = \frac{5\pi^2}{12} F_5^{(1)} + \sum_{i=0}^{4} \sigma^i \left\{ \frac{v_5 \mathrm{tr}\left[(1-\gamma_5)\not{p}_4 \not{p}_5 \not{p}_1 \not{p}_2\right]}{(v_2+v_3-v_5)} I_{23,5} + \frac{1}{6} \frac{\mathrm{tr}\left[(1-\gamma_5)\not{p}_4 \not{p}_5 \not{p}_1 \not{p}_2\right]^2}{v_1 v_4} + \frac{10}{3} v_1 v_2 + \frac{2}{3} v_1 v_3 \right\}.$$

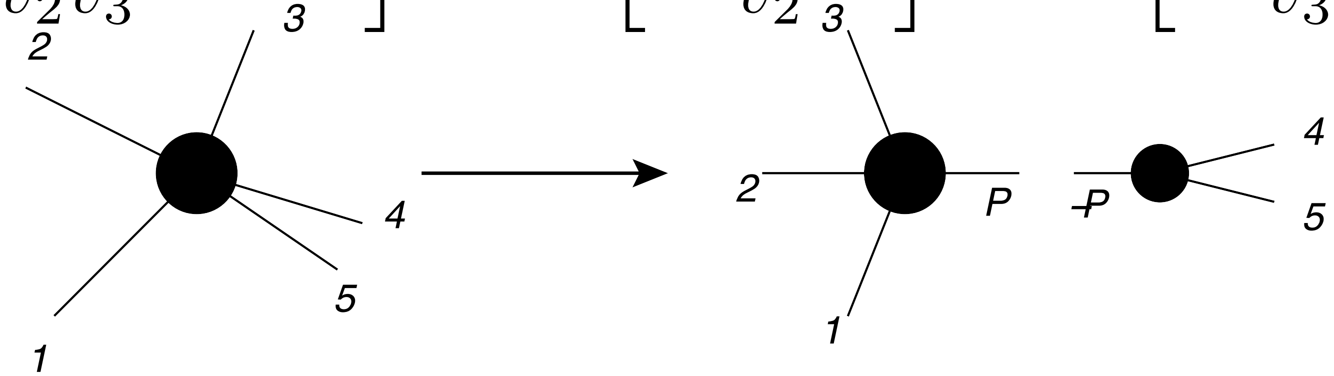
$v_1 = 2p_1 \cdot p_2$ σ : cyclic shift

[Gehrmann, JMH, Lo Presti, 2015]

only dilogarithms needed

$$I_{23,5} = \zeta_2 + \mathrm{Li}_2\left[\frac{(v_5-v_2)(v_5-v_3)}{v_2 v_3}\right] - \mathrm{Li}_2\left[\frac{v_5-v_3}{v_2}\right] - \mathrm{Li}_2\left[\frac{v_5-v_2}{v_3}\right].$$

collinear check:



Conclusions

- N=4 sYM - inspired methods are useful in QCD
- Feynman integrals are no longer the bottleneck of NNLO calculations

we are seeing the beginning of a NNLO revolution!

Outlook

- classification of special functions relevant to two-loop scattering amplitudes
- many further developments in N=4 sYM