Collective Instabilities and Collisional Effects for a 2D Model of a Beam in a Storage Ring

G. Turchetti, C. Benedetti*
Dipartimento di Fisica, Università di Bologna. INFN Sezione di Bologna

Abstract
We consider a collisional 2D model for a beam in a ring. In the smooth focusing approximation the relaxation time scales according to Landau's theory, but the p.d.f of momentum jumps has a power law decaying queue. A new hybrid regime is found for the equipartitioning due to the interplay between collisional and collective effects. The moments equations of a small perturbation to the KV distribution are analytically determined and the stability conditions follow from Floquet's theory.

1 Introduction

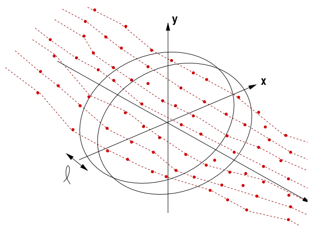
Our model consists in replacing the point charges of a coasting beam (or trains of long bunches) in a ring with parallel filaments, assuming strong longitudinal coherence. Assuming axial symmetry for a beam of radius R, denoting by n the particles per unit volume, we have that the number of particles per unit length is $N_p = n\pi R^2$. Denoting by $\ell = n^{-1/3}$ the specific length, the number of filaments is

$$N = N_p\ell = N_p^{2/3}\, R^{2/3}\pi^{1/3}\,.$$

For a typical beam with $N_p = 10^{11}$ particles/m and $R = 5$ mm we have $N = 10^6$. In figure 1 a sketch of our filaments is shown. Denoting by (x_i, y_i) the coordinate of a filament in the transverse plane and r_{ij} the distance between the filaments i and j, the Hamiltonian reads

$$H_{\,\mathrm{tot}} = \frac{1}{2}\sum_{i=1}^{N}\left(p_{xi}^2 + p_{yi}^2 + \omega_{x0}^2\, x_i^2 + \omega_{y0}^2\, y_i^2\right) - \frac{\xi}{N}\sum_{i<j}\log r_{ij} \qquad\qquad p_{xi} = \frac{dx_i}{ds}\,. \quad (1)$$

The parameters on which H depends, bare phase advances $\omega_{0\,x}$, $\omega_{0\,y}$ and perveance ξ, are N independent. Indeed we vary N keeping the charge per unit length $Q = N\,q = N_p\,e$ and the mass per unit length $M = N\,m = N_p\,m_p$ fixed, having denoted by e, m_p and q, m respectively the charge and mass of a particle and of a filament of unit length.



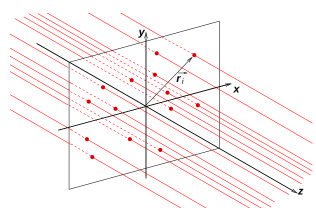


Figure 1. Real beam (left), the parallel filaments (right-)

2 Relaxation and equipartition

We have compared the results obtained by integrating Hamilton's equations for (1) with kinetic Landau's theory, whose 2D version we have developed. The relaxation from any initial distribution ρ_0 like KV to the Maxwell-Boltzmann distribution follows an exponential law $\rho = \rho_0\, e^{-\alpha s} + \rho_{MB}\,(1 - e^{-\alpha s})$ and

*email: turchett@bo.infn.it, Tel:+39 051-2091095, Fax: +39-051-247244

using an asymptotic approximation for the 2D Coulomb cross section it can be proved that the relaxation time scales as N. By varying N from 10^3 to 10^4 we have found that $\tau = c\,N$ with an uncertainty $\Delta c/c$ comparable with the statistical error $N^{-1/2}$ [1]. From Landau's theory we have

$$\tau = c\,\frac{N}{\xi^{3/2}}\,\frac{\epsilon_x\epsilon_y}{(\langle\, x^2\,\rangle\langle\, y^2\,\rangle)^{1/4}}$$

where we have assumed a proportionality between the cutoff of the Coulomb potential $R_{\rm \ cut}$ and the Debye radius $R_{\rm \ cut} \ = R\,[\,k_B\,T/(2\xi)\,]^{1/2}$. This introduces a unique calibration constant c, fixed by a single simulation. An excellent agreement is found between the theoretical value of τ and the simulations, see [2]. In figure 2 we show a comparison between the simulations and Landau's theory once the calibration constant has been fixed.
When an unstable resonance like the Montague integer one $\nu_x = \nu_y$ is present, a dynamic equipartition occurs with a time scale of order 1. In presence of a resonance which does not cause a complete dynamic equipartition, collisions inject particles into the resonance and an equipartition faster with respect to the pure collisional case due to an interplay with dynamical effects is observed (see [3]).

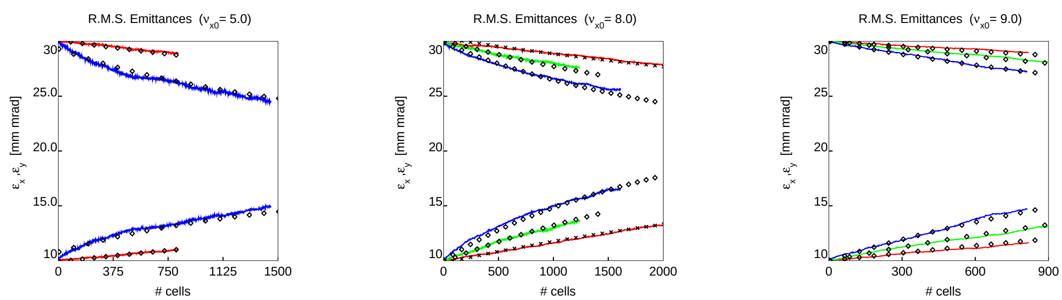


Figure 2. Comparison between a simulation (curves) and Landau's theory (diamonds). The parameters are: N=2048, $\epsilon_x(0)/\epsilon_y(0)$=30/10 mm mrad ν_{x0}=5 (left), ν_{x0}=8 (center), ν_{x0}=9 (right), ν_{y0}=6.21. The colors refer to different values of the perveance defined by assigning the value of the depressed vertical tunes: ν_y/ν_{y0}=0.8 (red), ν_y/ν_{y0}=0.7 (green), ν_y/ν_{y0}=0.6 (blue).

3 The momentum jumps

Landau's theory assumes that the collisions are binary, soft and frequent. The analysis of the time series for the momentum jumps obtained by a very accurate integration of the equations of motion in order to resolve the hard collisions shows significant deviations with respect to the previous hypothesis. The momentum jumps are obtained after subtracting the mean field motion in the interval Δs, see [4]

$$\Delta\mathbf{p}_k = \mathbf{p}(s_k) - \mathbf{p}(s_{k-1}) + \omega^2\mathbf{r}(s_{k-1})\Delta s\,.$$

The relevant feature is that in the p.d.f. of the momentum jumps a slowly decaying queue is present due to the rare hard collisions. The distribution can be fitted by a Student distribution $\Sigma(3)$ (see figure 3)

$$\rho_{\rm \ Stud}\ (\Delta p,\beta) = \frac{1}{\pi}\int_0^\infty \cos(u\Delta p)e^{-u\beta}(1+u)^\beta du \simeq \frac{2\beta}{\pi}\,\frac{1}{(\Delta p)^4} \qquad \text{for} \quad \Delta p\to\infty\,.$$

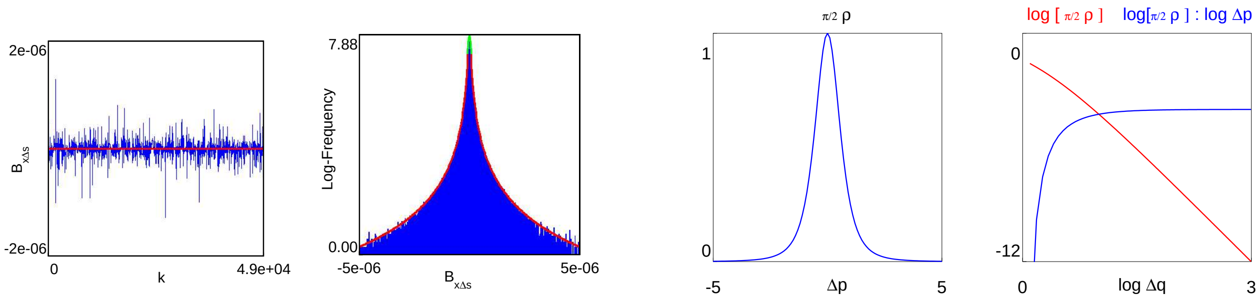


Figure 3. Horizontal component of momentum jumps Δp_x: time series of the jumps (left), histogram of the momentum jumps and fit with the Student's distribution (center left). Plot of $\rho_{\text{ Stud}} \times \pi/2$ for β=1 (center right). Plot of $\log[\rho_{\text{ Stud}} \times\pi/2]$ versus $\log \Delta q$ (right).

The p.d.f. decays algebraically with the fourth power and has a finite variance. As a consequence the central theorem applies. However approximating the process with a Wiener noise is rather crude, since the contribution of the hard collisions is lost. The behaviour for $\Delta p \to 0$ is a Gaussian whose variance σ can be analytically estimated according to $\sigma^2 = \frac{1}{2}\rho_s(\mathbf{r})\xi^2 N^{-1} \log N$. Another relevant parameter is the decorrelation time $(\Delta t)_{\text{ dec}} \propto \ell/v_{\text{ rel}} \propto (N\rho_s(\mathbf{r}))^{-1/2}\, v_{\text{rel}}^{-1}$. We propose to approximate the effect of collisions with a stochastic process whose p.d.f. is the observed one. The mean field dynamics is described by a PIC code, which solves the Poisson-Vlasov, with the desired accuracy provided that the number of pseudoparticles is large enough. The time step $(\Delta t)_{\text{ PIC}}$ can be chosen much larger than the Δt used in the microscopic simulations. Letting $n = (\Delta t)_{\text{ PIC}} /(\Delta t)_{\text{ dec}}$, the momentum change to be inserted in the PIC simulations is $(\Delta \mathbf{p})_{\text{ PIC}} = (\Delta \mathbf{p}_1 + \ldots + \Delta \mathbf{p}_n)(\Delta t)_{\text{ dec}} /\Delta t$, where $\Delta \mathbf{p}_k$ are chosen randomly according to the Student's distribution. In order to preserve the kinetic energy, the momentum is renormalized according to $\mathbf{p}_i' = C(\mathbf{p}_i + \Delta \mathbf{p}_i)$ where $C^2 = \sum_{i=1}^n (\mathbf{p} + \Delta \mathbf{p}_i)^2 / \sum_{i=1}^n \mathbf{p}^2$. With this choice the relaxation and equipartition processes observed in the microscopic model are well reproduced. An alternative is to write the Fokker-Planck-Poisson-Vlasov equation including a Student's noise, imposing an Einstein-like relation between drift and diffusion coefficients in order to preserve the second order moments. The integro-differential equations for the p.d.f, typical of Levy flights, render this approach quite involved.

4 The mean fields limit

A consequence of the scaling law for the relaxation time $\tau \propto N$, is that in the limit $N \to \infty$ the mean field theory is recovered. This limit has been recently proved in a completely rigorous framework [5]. In the periodic focusing case no stationary limit distribution exists, nor any rigorous result proving that the mean field theory is recovered as $N \to \infty$ is available at present. However there is strong numerical evidence that this is the case. The collisional model for N large ($\geq 10^4$) and a PIC code with a large number of pseudoparticles ($\geq 10^6$) give the same results within 1% for hundreds of betatron periods. In this case since the hard collisions need not to be resolved the time step requirements can be relaxed and the "collisional code" can be used to explore the collective effects just as a PIC code. Other consistency checks come from the analysis of collective instabilities. To this end an analytic approach based on the equations of moments for the linearized Poisson-Vlasov equations has been developed. The equations for the moments of order k couple only to the lower order moments and read

$$\frac{d\mu}{ds} = A(s)\mu(s) \qquad\qquad A(s+L) = A(s)$$

where μ is the moments vector. The matrix $A(s)$ is determined analytically and the eigenvalues of the monodromy matrix $M = X(L)$, where $X(s)$ is the fundamental matrix, determine the stability condition. The agreement in the emittance growth rate between predictions of the moments theory and PIC simulations is good (see [3]).

References

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