Starobinsky Model in Rainbow Gravity

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- Two fundamental theories in physics
  - General Relativity
  - Quantum Physics
Quantum Effects in Gravity Theory

• Two fundamental theories in physics
  - General Relativity
  - Quantum Physics

• There are several candidates for Quantum Gravity theory
  - Loop Quantum Gravity
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  - String Theory
  - Gravity’s Rainbow
Introduction

1. Gravity’s Rainbow
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Gravity’s Rainbow (Magueijo, Smolin, 2004)

- A metric of curved spacetime is energy-dependent e.g. the flat rainbow Friedmann-Robertson-Walker (FRW) metric

\[ ds^2 = -\frac{1}{\tilde{f}^2(\varepsilon/M)} dt^2 + a^2(t)\delta_{ij} dx^i dx^j , \]  

(1)

where \( \tilde{f}(\varepsilon/M) \) is the rainbow function, \( \varepsilon \) is the energy of probe particle, \( a(t) \) is the scale factor and \( M \) is the energy scale such that

\[ \lim_{\varepsilon/M \to 0} \tilde{f}(\varepsilon/M) \to 1 . \]  

(2)

This ensures that the original FRW metric still holds in low energy limit.

- The probe particle can be any particle dominating the universe at each time.
- Difference of energy \( \varepsilon \) leads to different background spacetime.
Starobinsky’s Model (Starobinsky, 1980)

- The action of Starobinsky’s Model:

\[
S[g_{\mu\nu}] = \frac{1}{2\kappa^2} \int d^4x \sqrt{-g} \left( R + \frac{R^2}{6M^2} \right),
\]

where \( \kappa^2 = 8\pi G \) and \( M \) is the mass scale of inflation.

- The best inflation model

- This model is naturally consistent with the Planck data rather than the power-law inflation and there is no matter field in this model. That is why we use this model as the testing model of gravity’s rainbow.

- The Starobinsky’s Model is equivalent to the Einstein gravity with inflaton field.
Starobinsky’s Model in Rainbow Gravity

1. Field Equation and Background Solution
2. The Spectra of Perturbations
3. Fitting 2015 Planck Data
Field Equation and Background Solution

• In this framework, we use the flat rainbow FRW metric in the form

\[ ds^2 = -\frac{1}{\tilde{f}^2 (\varepsilon/M)} dt^2 + a^2(t) \delta_{ij} dx^i dx^j, \quad (4) \]

• To study the evolution of the universe, the energy of probe particle \( \varepsilon \) depends on the cosmic time.

• We propose that the rainbow function is in the form

\[ \tilde{f}^2 = 1 + \left( \frac{H}{M} \right)^{2\lambda}, \quad (5) \]

where \( \lambda \) is the rainbow parameter. During inflation, we assume that \( H^2 \gg M^2 \), then

\[ \tilde{f} \approx \left( \frac{H}{M} \right)^{\lambda}. \quad (6) \]
Field Equation and Background Solution

Then the Friedmann equation associated with the action (3) the metric (4) reads

\[
\ddot{H} - \frac{\dot{H}^2}{2H} + \frac{1}{2\tilde{f}^2} M^2 H + \frac{11}{6} H \left( \frac{\dot{f}}{\tilde{f}} \right)^2 + \frac{1}{3} \left( \frac{\ddot{f}}{\tilde{f}} \right)^3 + \frac{10}{3} \frac{\dot{H}}{\tilde{f}} + \frac{\dot{H}}{H} \left( \frac{\dot{f}}{\tilde{f}} \right)^2 \\
+ H \frac{\dddot{f}}{\tilde{f}} + \frac{1}{3} \frac{\dddot{f}}{\tilde{f}^2} + \frac{1}{3} \frac{\dddot{f}}{H \tilde{f}} = -3H^2 \left( \frac{\dot{f}}{\tilde{f}} + \frac{\dot{H}}{H} \right) .
\] (7)

Substituting this form of \( \tilde{f} \), we now get

\[
\frac{1}{2(1 + \lambda)} \frac{M^2 H}{\tilde{f}^2} + 3H\dot{H} + \frac{1}{6} \left( 17\lambda - 3 \right) \frac{\dot{H}^2}{H} + \frac{2\lambda^2 \dot{H}^3}{3H^3} \\
+ (1 + \frac{\lambda}{3} \frac{\dot{H}}{H^2}) \ddot{H} = 0 .
\] (8)
By imposing the slow-roll conditions ($\epsilon_1 < 1$), we obtain
\[
\dot{H} \simeq -\frac{M^{2\lambda+2}}{6(1 + \lambda)} H^{-2\lambda}.
\] (9)

Then the solution of these equations read
\[
H \simeq H_i - \frac{M^2}{6(1 + \lambda)} \left( \frac{M}{H_i} \right)^{2\lambda} (t - t_i),
\] (10)
\[
a \simeq a_i \exp \left\{ H_i (t - t_i) - \frac{M^2}{12(1 + \lambda)} \left( \frac{M}{H_i} \right)^{2\lambda} (t - t_i)^2 \right\}.
\] (11)

where $a_i$ and $H_i$ are the quantities at the initial time of inflation $t_i$. 
Field Equation and Background Solution

We can compute $\epsilon_1$ by using the solution (10), it gives

$$
\epsilon_1 = -\frac{\dot{H}}{H^2} = \frac{1}{6(1 + \lambda)} \left( \frac{M}{H} \right)^{2+2\lambda}.
$$

(12)

We can further calculate the number of e-folds by using (10) and (20), we obtain

$$
N \equiv \int_{t_i}^{t_f} Hdt
$$

(13)

$$
\simeq 3(1 + \lambda) \left( \frac{H_i}{M} \right)^{2+2\lambda} \simeq \frac{1}{2\epsilon_1(t_i)} ,
$$

(14)

where $t_f$ is the time at the end of inflation.
The Spectra of Perturbations

By doing the scalar and tensor perturbations:

- The power spectra $\mathcal{P}_s$ and $\mathcal{P}_T$ are given by

$$
\mathcal{P}_s \approx \frac{1}{12\pi} \left( \frac{M}{m_{pl}} \right)^2 \frac{1}{(1 + \lambda)^2 \epsilon_1^2},
$$

$$
\mathcal{P}_T \approx \frac{4}{\pi} \left( \frac{M}{m_{pl}} \right)^2,
$$

where $\mathcal{P}_s$ denotes the scalar part of perturbation and $\mathcal{P}_T$ denotes the tensor part associated with the gravitational waves.

- The spectral indices $n_s$ and $n_T$ are given by

$$
n_s - 1 \approx -4\epsilon_1, \quad n_T \approx 0
$$

where $n_s$ and $n_T$ denote the scalar and tensor part respectively.
The Spectra of Perturbations

One can define the tensor-to-scalar ratio

\[ r \equiv \frac{P_T}{P_s} \simeq 48(\lambda + 1)^2 \epsilon_1^2. \tag{18} \]

Since we have derived the relation

\[ N_k = \frac{1}{2 \epsilon_1(t_k)} \tag{19} \]

where \( t_k \) is a time at the Hubble radius crossing. So \( P_s, n_s, r \) can be rewritten as

\[ P_s = \frac{1}{3\pi} \left( \frac{M}{m_{pl}} \right)^2 \frac{N_k^2}{(1 + \lambda)^2}, \tag{20} \]

\[ n_s - 1 = -\frac{2}{N_k}, \tag{21} \]

\[ r = \frac{12(1 + \lambda)^2}{N_k^2}. \tag{22} \]
Fitting 2015 Planck Data

$r - n_s$ plane: Fixed $N_k$, varied $\lambda$

• From the Planck 2015 data, the amplitude of curvature perturbation at the scale $k = 0.05 \text{Mpc}^{-1}$ is given by $\mathcal{P}_s = (2.219 \pm 0.103) \times 10^{-9}$. Since, from Eqn. (20)

$$\mathcal{P}_s = \frac{1}{3\pi} \left( \frac{M}{m_{pl}} \right)^2 \frac{N_k^2}{(1 + \lambda)^2}, \quad (23)$$

with $N_k = 70$ and $\lambda \simeq 3.0$, the mass scale $M$ can be constrained as

$$M \simeq 2 \times 10^{-6} (1 + \lambda) m_{pl}$$

$$\sim 1.70 \times 10^{14} \text{GeV} \quad (24)$$

where we have used the reduced Planck mass is obtained by $m_{pl} = 1.22 \times 10^{19} \text{GeV}$. 
Conclusions

• The predictions of the Starobinsky’s model in rainbow gravity are well consistent with the data such that

\[ 42 \lesssim N_k \lesssim 87 \quad \text{and} \quad \lambda \lesssim 6.0 \,, \tag{25} \]

to be within 2\(\sigma\) C.L. of Planck’15 contours.

• One can also constrain the mass scale \(M\) (or equivalent to the inflaton mass) as

\[
M \simeq 2 \times 10^{-6} (1 + \lambda) m_{\text{pl}} \\
\simeq 1.70 \times 10^{14} \text{ GeV} \,, \tag{26}
\]

where \(N_k = 70\) and \(\lambda \simeq 3.0\).
References


**References**


References


Deformed Special Relativity (Kimberly, Magueijo, 2004)

Deformed Einstein’s Postulates:

• The laws of physics are the same in all inertial frames.

• The invariance of speed of light in vacuum in all inertial frames.

• The invariance of upper energy scale in the universe, Planck energy $E_p$, in all inertial frames.

These postulates lead to the modified dispersion relation (MDR)

$$\varepsilon^2 \tilde{f}^2(\varepsilon) - p^2 \tilde{g}^2(\varepsilon) = m^2,$$

(27)

where $\varepsilon$ is an energy of a probe particle, the functions $\tilde{f}(\varepsilon)$ and $\tilde{g}(\varepsilon)$ are called the rainbow functions.
Why $\tilde{f} \simeq (H/M)^\lambda$ ?

In the Jordan frame:

$$S[g_{\mu \nu}] = \frac{1}{2\kappa^2} \int d^4x \sqrt{-g} (R + \frac{R^2}{6M^2})$$  \hspace{1cm} (28)

In the Einstein frame:

$$S_E[g_E^{\mu \nu}, \phi_E] = \int d^4x \sqrt{-g_E} \left[ \frac{1}{2\kappa^2} R_E - \frac{1}{2} g_E^{\mu \nu} \partial_\mu \phi_E \partial_\nu \phi_E - V(\phi_E) \right],$$  \hspace{1cm} (29)

These two frames are connected by the conformal transformation $(g_E)_{\mu \nu} = \Omega^2 g_{\mu \nu}$ with $\Omega^2 = F$. The scalar field $\phi_E$ can be expressed by

$$\kappa \phi_E \equiv \sqrt{\frac{3}{2}} \ln F,$$  \hspace{1cm} (30)

and its potential is given by

$$V(\phi_E) = \frac{FR - f}{2\kappa^2 F^2}.$$  \hspace{1cm} (31)
Why $\tilde{f} \simeq (H/M)^{\lambda}$?

Since

$$\varepsilon_E(t) \propto \rho_E,$$  \hspace{1cm} (32)

and during inflation we can assume that

$$\rho_{\phi_E} \simeq V(\phi_E).$$ \hspace{1cm} (33)

From the eq. (31), the potential $V(\phi_E)$ can be written in terms of the Hubble parameter. So $\varepsilon_E(t)$ should be also written in terms of the Hubble parameter. That is why we assume that

$$\tilde{f}^2 = 1 + \left(\frac{H}{M}\right)^{2\lambda},$$ \hspace{1cm} (34)

it is plausible to propose that the rainbow function is in the power-law form of the Hubble parameter.
Deformed Special Relativity

This MDR can be as considered as the action of non-linear map $U : \mathcal{P} \to \mathcal{P}$, 

$$|p|^2 = \tilde{\eta}^{\mu\nu} U_\mu(p) U_\nu(p) ,$$  \hspace{1cm} (35)

where $\tilde{\eta}^{\mu\nu}$ is the usual Minkowski metric and 

$$U_\mu(\varepsilon, p_i) = (U_0, U_i) = (\tilde{f}(\varepsilon)\varepsilon, \tilde{g}(\varepsilon)p_i) .$$  \hspace{1cm} (36)

To obtain the rainbow Minkowski metric, we demand that

$$U_\mu(p)U^\mu(x) = p_\mu x^\mu ,$$  \hspace{1cm} (37)

where $p_\mu$ and $x^\mu$ are physical momentums and coordinates respectively.
Deformed Special Relativity

To satisfy the relation (37), $U^\mu(x)$ should be in the form

$$U^\mu(x) = (U^0, U^i) = \left( \frac{t}{\tilde{f}(\varepsilon)}, \frac{x^i}{\tilde{g}(\varepsilon)} \right). \quad (38)$$

Then the line element involved with this map $U^\mu(x)$ is given by

$$ds^2 = \tilde{\eta}_{\mu\nu} U^\mu(dx) U^\nu(dx) = -\frac{dt^2}{\tilde{f}^2(\varepsilon)} + \frac{1}{\tilde{g}^2(\varepsilon)} \delta_{ij} dx^i dx^j, \quad (39)$$

which is invariant under the deformed Lorentz transformation. So the rainbow Minkowski metric reads

$$\eta_{\mu\nu}(\varepsilon) = \text{diag} \left( -\frac{1}{\tilde{f}^2(\varepsilon)}, \frac{1}{\tilde{g}^2(\varepsilon)} , \frac{1}{\tilde{g}^2(\varepsilon)} , \frac{1}{\tilde{g}^2(\varepsilon)} \right). \quad (40)$$
Gravity’s Rainbow (Magueijo, Smolin, 2004)

Magueijo & Smolin proposed that the metrics of curved spacetime can be parametrized by one parameter $\varepsilon$,

$$g(\varepsilon) = \bar{g}^{\mu\nu}(x) e_\mu(\varepsilon) \otimes e_\nu(\varepsilon) ,$$

where $\bar{g}^{\mu\nu}(x)$ is a usual metric tensor and $e_\mu(\varepsilon)$ are the energy dependent frame fields defined by

$$e_0(\varepsilon) = \frac{1}{\tilde{f}(\varepsilon)} \tilde{e}_0 , \quad e_i(\varepsilon) = \frac{1}{\tilde{g}(\varepsilon)} \tilde{e}_i .$$

Note that $\tilde{e}_\mu$ are the energy independent frame fields. Then the Einstein field eq. can be modified as

$$G_{\mu\nu}(\varepsilon) = 8\pi G(\varepsilon) T_{\mu\nu}(\varepsilon) + g_{\mu\nu}(\varepsilon) \Lambda(\varepsilon) .$$
Gravity’s Rainbow (Magueijo, Smolin, 2004)

Deformed Equivalence principle:

Any physical experiments of particles and fields with energies $\epsilon$ performed in the freely falling frame are the same as the similar experiments performed in the inertial frame in the rainbow Minkowski spacetime.

Correspondence principle:

When $\epsilon \ll E_p$, gravity’s rainbow becomes the ordinary general theory of relativity i.e.

$$\lim_{\epsilon/E_p \to 0} \tilde{f}(\epsilon) = \lim_{\epsilon/E_p \to 0} \tilde{g}(\epsilon) = 1.$$  \hspace{1cm} (44)
Conclusions

Rainbow Universe (Ling, 2007)

To study the evolution of rainbow universe, let us consider the flat rainbow FRW metric in the form

\[ ds^2 = -\frac{1}{\tilde{f}^2(\varepsilon)} dt^2 + a^2(t) \delta_{ij} dx^i dx^j , \tag{45} \]

where \( \tilde{g}(\varepsilon) = 1 \). By doing the calculations of this metric with the usual Einstein field eq., we can come up with the Friedmann equations

\[ H^2 = \frac{8\pi G}{3} \frac{\rho}{\tilde{f}^2} , \tag{46} \]

\[ \dot{H} = -\frac{4\pi G(\rho + P)}{\tilde{f}^2} - H \frac{\dot{\tilde{f}}}{\tilde{f}} , \tag{47} \]

where we have used \( T^{\mu}_\nu = \text{diag}(-\rho, P, P, P) \). This approach often used to solve the Big Bang singularity problem.
Field Equation of $f(R)$ Gravity

Since we are interested in the flat FRW metric:

$$ds^2 = -dt^2 + a^2(t) \left( dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2 \right)$$  \hspace{1cm} (48)$$

and the perfect fluid:

$$T^{\mu}_{\nu}^{(M)} = \text{diag}(-\rho_M, P_M, P_M, P_M)$$  \hspace{1cm} (49)$$

So we can come up with the Friedmann equations

$$3FH^2 = \frac{FR - f}{2} - 3H\dot{F} + \kappa^2 \rho_M$$  \hspace{1cm} (50)$$

$$-2F\dot{H} = \ddot{F} - H\dot{F} + \kappa^2 (\rho_M + P_M)$$  \hspace{1cm} (51)$$

where $H = \dot{a}/a$. 
Field Equation and Background Solution

The perfect fluid:

\[ T_{\mu}^{(M)} = \text{diag}(-\rho_M, P_M, P_M, P_M) \]  

(52)

where \( \rho_M \) and \( P_M \) are the energy density and pressure respectively.

Then the Friedmann equations with rainbow effects read

\[ 3(FH^2 + \dot{H}\tilde{f}) + \dot{F}\frac{\tilde{f}}{\tilde{f}^2} = \frac{FR - f(R)}{2\tilde{f}^2} + \frac{\kappa^2 \rho_M}{\tilde{f}^2}, \]  

(53)

\[ \ddot{F} - H\dot{F} + 2F\dot{H} + 2FH\frac{\tilde{f}}{\tilde{f}^2} = -\frac{\kappa^2}{\tilde{f}^2}(\rho_M + P_M). \]  

(54)

\[ \ddot{R} + 3H\dot{R} + \frac{4\tilde{f}\dot{R}}{3\tilde{f}} + \frac{M^2 R}{\tilde{f}^2} = 0. \]  

(55)
Conclusions

Cosmological Perturbations with Rainbow

The full perturbed metric around the flat FRW metric

\[ ds^2 = -\frac{1 + 2\alpha}{\tilde{f}^2(\varepsilon)} dt^2 - \frac{2a(t)(\partial_i\beta - S_i)}{\tilde{f}(\varepsilon)} dtdx^i + a^2(t)(\delta_{ij} + 2\psi\delta_{ij} + 2\partial_i\partial_j\gamma + 2\partial_jF_i + h_{ij})dx^i dx^j , \]  

(56)

where \(\alpha, \beta, \psi, \gamma\) are scalar perturbations, \(S_i, F_i\) are vector perturbations, and \(h_{ij}\) are tensor perturbations. We can set \(S_i = F_i = 0\) since the vector perturbations decay rapidly during inflation. Consider the gauge transformation:

\[ \alpha \rightarrow \hat{\alpha} = \alpha + \frac{\dot{\tilde{f}}}{\tilde{f}}\delta t - \dot{\delta}t, \quad \beta \rightarrow \hat{\beta} = \beta - \frac{\delta t}{af} + af\dot{\delta}x \]  

(57)

\[ \psi \rightarrow \hat{\psi} = \psi - H\delta t, \quad \gamma \rightarrow \hat{\gamma} = \gamma - \delta x . \]  

(58)
Conclusions

Cosmological Perturbations with Rainbow

The gauge invariant quantities can be defined by

\[
\Phi = \alpha - \tilde{f} \frac{d}{dt} \left[ a^2 \tilde{f} \left( \dot{\gamma} + \frac{\beta}{a \tilde{f}} \right) \right],
\]

\[
\Psi = -\psi + a^2 \tilde{f}^2 H \left( \dot{\gamma} + \frac{\beta}{a \tilde{f}} \right),
\]

\[
R = \psi - \frac{H \delta F}{\tilde{F}},
\]

where \( R \) is the curvature perturbation. Choosing the Longitudinal gauge \( \beta = 0 \) and \( \gamma = 0 \) such that \( \Phi = \alpha \) and \( \Psi = -\psi \). So the line element (56) reduces to

\[
ds^2 = -\frac{1 + 2\Phi}{\tilde{f}^2(t)} dt^2 + a^2(t)(1 - 2\Psi)\delta_{ij} dx^i dx^j.
\]
 Scalar Perturbations with Rainbow

Introduce a quantity $A$ through

$$A \equiv 3(H\Phi + \dot{\Psi}).$$

(63)

Then the field equation associated with the metric (62) gives the following equations.

$$-rac{\nabla^2 \Psi}{a^2} + \tilde{f}^2 HA = -\frac{1}{2F} \left[ 3\tilde{f}^2 \left( H^2 + \dot{H} + \frac{\dot{\tilde{f}}}{\tilde{f}} \right) \delta F + \frac{\nabla^2 \delta F}{a^2} - 3\tilde{f}^2 H \delta \dot{F} 
+ 3\tilde{f}^2 H \dot{F} \Phi + \tilde{f}^2 \dot{F} A + \kappa^2 \delta \rho_M \right],$$

(64)

$$H\Phi + \dot{\Psi} = -\frac{1}{2F} (H\delta F + \dot{F} \Phi - \delta \dot{F}),$$

(65)
Scalar Perturbations with Rainbow

and

\[ \ddot{A} + \left( 2H + \frac{\dot{f}}{\tilde{f}} \right) A + 3\dot{H}\Phi + \frac{\nabla^2 \Phi}{a^2 \tilde{f}^2} + \frac{3H\dot{\Phi}}{\tilde{f}} = \frac{1}{2F} \left[ 3\delta \ddot{F} \right. \]

\[ + 3 \left( H + \frac{\dot{f}}{\tilde{f}} \right) \delta \dot{F} - 6H^2 \delta F - \frac{\nabla^2 \delta F}{a^2 \tilde{f}^2} - 3\dot{F}\dot{\Phi} - \dot{F}A - 3 \left( H + \frac{\dot{f}}{\tilde{f}} \right) \dot{\Phi} \]

\[ - 6\ddot{\Phi} + \frac{\kappa^2}{\tilde{f}^2} (3\delta P_M + \delta \rho_M) \right] , \]  

where the energy-momentum tensor has been chosen to be in the perfect fluid form.
Scalar Perturbations with Rainbow

In our model, we set $\delta \rho_M = 0$ and $\delta P_M = 0$ and also choose the condition $\delta F = 0$ such that $R = \psi = -\Psi$. Under this gauge choice, the equation (65) will become

$$\Phi = \frac{\dot{R}}{H + \dot{F}/2F}.$$  

(67)

Plugging the equation (67) into (64), we now obtain

$$A = -\frac{1}{H + \dot{F}/2F} \left[ \frac{\nabla^2 R}{a^2 \tilde{f}^2} + \frac{3HF\dot{R}}{2F(H + \dot{F}/2F)} \right].$$  

(68)
Scalar Perturbations with Rainbow

Using the background equation in the equation (66), it gives

\[
\dot{A} + \left( 2H + \frac{\dot{F}}{2F} \right) A + \frac{\dot{f}A}{\tilde{f}} + \frac{3\dot{F}\Phi}{2F} + \left[ \frac{3\ddot{F} + 6H\dot{F}}{2F} + \frac{\nabla^2}{a^2\tilde{f}^2} \right] \Phi
\]

\[
+ \frac{3\dot{F}\Phi\dot{\tilde{f}}}{2F\tilde{f}} = 0. \tag{69}
\]

Substituting the equations (67) and (68) into the equation (69), then the simple equation of $\mathcal{R}$ written in the Fourier space is given by

\[
\ddot{\mathcal{R}} + \frac{1}{a^3 Q_s} \frac{d}{dt} (a^3 Q_s) \dot{\mathcal{R}} + \frac{\dot{f}}{\tilde{f}} \ddot{\mathcal{R}} + \frac{k^2}{a^2\tilde{f}^2} \mathcal{R} = 0 , \tag{70}
\]

here $k$ denotes a comoving wavenumber.
Scalar Perturbations with Rainbow

Introduce the new variable $Q_s$ through

$$Q_s \equiv \frac{3 \dot{F}^2}{2 \kappa^2 F (H + \dot{F}/2F)^2}.$$ (71)

Also we can define the new parameters $z_s = a \sqrt{Q_s}$ and $u = z_s \mathcal{R}$, so the equation (70) reduces to

$$u'' + \left( k^2 - \frac{z_s''}{z_s} \right) u = 0,$$ (72)

where a prime is a derivative with respect to the conformal time

$$\eta = \int (a\tilde{\tau})^{-1} dt.$$
Scalar Perturbations with Rainbow

Let us define the Hubble flow parameters:

\[ \epsilon_1 \equiv -\frac{\dot{H}}{H^2}, \quad \epsilon_3 \equiv \frac{\dot{F}}{2HF}, \quad \epsilon_4 \equiv \frac{\dot{E}}{2HE}, \]  

(73)

where the variable \( E \) is defined by

\[ E \equiv \frac{3\dot{F}^2}{2\kappa^2}. \]  

(74)

From the definitions (73), \( Q_s \) can be re-expressed as

\[ Q_s = \frac{E}{FH^2(1 + \epsilon_3)^2} = \frac{6F\epsilon_3^2}{\kappa^2(1 + \epsilon_3)^2}. \]  

(75)
Scalar Perturbations with Rainbow

Since $\epsilon_i$ ($i = 1, 3, 4$) are assumed to be constant values ($\dot{\epsilon}_i \simeq 0$) during inflation, then

$$\eta = -\frac{1}{(1 - (1 + \lambda)\epsilon_1)\tilde{f}aH}. \quad (76)$$

Under the assumption of $\epsilon_i$, the term $z_s''/z_s$ in the equation (72) can be estimated as

$$\frac{z_s''}{z_s} = \left(1 + \epsilon_1 - \epsilon_3 + \epsilon_4\right)\left(1 - (\lambda + 1)\epsilon_1\right)^2 \frac{\nu^2_R - 1/4}{\eta^2}, \quad (77)$$

with

$$\nu^2_R = \frac{1}{4} + \frac{(1 + \epsilon_1 - \epsilon_3 + \epsilon_4)(2 - \lambda\epsilon_1 - \epsilon_3 + \epsilon_4)}{(1 - (\lambda + 1)\epsilon_1)^2}. \quad (78)$$
Scalar Perturbations with Rainbow

So the approximate solution to the equation (72) can be written in terms of a linear combination of the Hankel functions as

$$u = \frac{\sqrt{\pi|\eta|}}{2} e^{i(1+2\nu_R)\pi/4} \left[ c_1 H^{(1)}_{\nu_R}(k|\eta|) + c_2 H^{(2)}_{\nu_R}(k|\eta|) \right] ,$$  \hspace{1cm} (79)

where $c_1$, $c_2$ are integration constants. In the asymptotic past $k\eta \to -\infty$, the solution (79) will become $e^{-ik\eta}/\sqrt{2k}$ which leads to $c_1 = 1$ and $c_2 = 0$. Then we have

$$u = \frac{\sqrt{\pi|\eta|}}{2} e^{i(1+2\nu_R)\pi/4} H^{(1)}_{\nu_R}(k|\eta|) .$$  \hspace{1cm} (80)

Let us define the power spectrum as

$$\mathcal{P}_s \equiv \frac{4\pi k^3}{(2\pi)^3} |\mathcal{R}|^2 .$$  \hspace{1cm} (81)
Conclusions

Scalar Perturbations with Rainbow

Using the solution (80) and the relation $u = z_s R$ in the above definition, then it gives

$$\mathcal{P}_s = \frac{1}{Q_s} \left[ (1 - (1 + \lambda) \epsilon_1) \frac{\Gamma(\nu_R) H}{2 \pi \Gamma(3/2)} \left( \frac{H}{M} \right)^\lambda \right]^2 \left( \frac{k|\eta|}{2} \right)^{3 - 2\nu_R}. \quad (82)$$

where we have used that the limit as $k|\eta| \to 0$ then

$H^{(1)}_{\nu_R}(k|\eta|) \to -(i/\pi) \Gamma(\nu_R)(k|\eta|/2)^{-\nu_R}$, and $\mathcal{P}_s$ should be evaluated at $k = aH$ because the curvature perturbation $R$ is fixed after the Hubble radius crossing. Now, the spectral index $n_s$ can be defined by

$$n_s - 1 = \left. \frac{d \ln \mathcal{P}_s}{d \ln k} \right|_{k=aH} = 3 - 2\nu_R, \quad (83)$$

where $\nu_s$ is already expressed out in (103).
Scalar Perturbations with Rainbow

We can now further approximate the equation (83) by imposing that $|\epsilon_i| \ll 1$ for all $i$ during inflation, then $n_s$ reduces to

$$n_s - 1 \simeq -2(\lambda + 2)\epsilon_1 + 2\epsilon_3 - 2\epsilon_4 .$$

(84)

Subsequently, the power spectrum of the curvature perturbation $\mathcal{P}_s$ (82) can be rewritten as

$$\mathcal{P}_s \approx \frac{1}{Q_s} \left( \frac{H}{2\pi} \right)^2 \left( \frac{H}{M} \right)^{2\lambda} .$$

(85)
Tensor Perturbations with Rainbow

Let $h_{ij}$ be the tensor perturbations generally written in the form

$$h_{ij} = h_+ e^+_{ij} + h_\times e^\times_{ij},$$

(86)

where $e^+_{ij}$ and $e^\times_{ij}$ denote the polarization tensors which correspond to the two polarization states of $h_{ij}$. Let $\vec{k}$ be a wave vector along the $z$-axis, then $e^+_{xx} = -e^+_{yy} = 1$ and $e^\times_{xy} = e^\times_{yx} = 1$. Consider the tensor perturbation, the perturbed metric (56) reduces to

$$ds^2 = -\frac{dt^2}{\tilde{f}^2(t)} + a^2(t)h_\times dx dy + a^2(t) \left[(1 + h_+)dx^2 + (1 - h_+)dy^2 + dz^2\right].$$
Tensor Perturbations with Rainbow

Applying this metric to the field equation, then we can show that the Fourier components $h_\chi$ satisfy the following equation

$$\ddot{h}_\chi + \frac{(a^3 F)'}{a^3 F} \dot{h}_\chi + \frac{\dot{f}}{f} \dot{h}_\chi + \frac{k^2}{a^2 \tilde{f}^2} h_\chi = 0,$$

(87)

where $\chi$ denotes $+$ and $\times$. Introduce a new variable $z_t = a\sqrt{F}$ and $u_\chi = z_t h_\chi / \sqrt{16\pi G}$, then equation (87) will become

$$u''_\chi + \left( k^2 - \frac{z''_t}{z_t} \right) u_\chi = 0.$$

(88)

Assuming that $\dot{\epsilon}_i = 0$ during inflation, we obtain

$$\frac{z''_t}{z_t} = \frac{\nu_t^2 - 1/4}{\eta^2}, \quad \nu_t^2 = \frac{1}{4} + \frac{(1 + \epsilon_3)(2 - (1 + \lambda)\epsilon_1 + \epsilon_3)}{(1 - (1 + \lambda)\epsilon_1)^2}.$$

(89)
Tensor Perturbations with Rainbow

Again the solution to the equation (88) is approximately written in terms of linear combination of the Hankel functions, then the power spectrum $P_T$ after the Hubble radius crossing is

$$P_T = 4 \times \frac{16\pi G}{a^2 F} \frac{4\pi k^3}{(2\pi)^3} |u_\chi|^2$$

$$= \frac{16}{\pi} \left( \frac{H}{m_{pl}} \right)^2 \frac{1}{F} \left[ (1 - (1 + \lambda)\epsilon_1) \frac{\Gamma(\nu_t)}{\Gamma(3/2)} \left( \frac{H}{M} \right)^\lambda \right]^2 \left( \frac{k|\eta|}{2} \right)^{3-2\nu_t}.$$

During inflation, we demand that the Hubble flow parameters is very small ($|\epsilon_i| \ll 1$), so $\nu_t$ can be evaluated as

$$\nu_t \simeq \frac{3}{2} + (1 + \lambda)\epsilon_1 + \epsilon_3. \quad (90)$$
Tensor Perturbations with Rainbow

The spectral index of tensor perturbations $n_T$ that is given by

$$n_T = \left. \frac{d \ln P_T}{d \ln k} \right|_{k = aH} = 3 - 2\nu_t \simeq -2(1 + \lambda)\epsilon_1 - 2\epsilon_3 . \quad (91)$$

Using the conditions $|\epsilon_i| \ll 1$ again, we now get

$$P_T \simeq \frac{16}{\pi} \left( \frac{H}{m_{pl}} \right)^2 \frac{1}{F} \left( \frac{H}{M} \right)^{2\lambda} . \quad (92)$$

Furthermore, the last important parameter is the tensor-to-scalar ratio $r$ which is defined by

$$r \equiv \frac{P_T}{P_s} \simeq 48\epsilon_3^2 \quad (93)$$

where we have used the formulas of $P_s$ and $P_T$. 
Perturbations with Rainbow based on Starobinsky’s Model

For the Starobinsky’s model, we obtain

\[ \epsilon_3 \simeq -(1 + \lambda)\epsilon_1 , \]  

(94)

and

\[ \epsilon_4 = -(2\lambda + 1)\epsilon_1 . \]  

(95)

Then \( n_s, P_s, \) and \( r \) can be re-expressed as

\[ n_s - 1 \simeq -4\epsilon_1 , \]  

(96)

\[ P_s \simeq \frac{1}{12\pi} \left( \frac{M}{m_{pl}} \right)^2 \frac{1}{(1 + \lambda)^2\epsilon_1^2} , \]  

(97)

and

\[ r \simeq 48(\lambda + 1)^2\epsilon_1^2 . \]  

(98)
Perturbations with Rainbow based on $R^n$ Model

For $R^n$ model, $\epsilon_1$ is given by

$$\epsilon_1 = \frac{2 - n}{(n - 1)(2n - 1)(1 + \lambda)}.$$  \hspace{1cm} (99)

Also we obtain the relations among the Hubble flow parameters which are

$$\epsilon_3 \simeq -(n - 1)(\lambda + 1)\epsilon_1 = \frac{n - 2}{2n - 1},$$  \hspace{1cm} (100)

and

$$\epsilon_4 \simeq -(2n(\lambda + 1) - 2\lambda - 1)\epsilon_1 = -\frac{(2n(\lambda + 1) - 2\lambda - 1)(2 - n)}{(n - 1)(2n - 1)(1 + \lambda)}. \hspace{1cm} (101)$$
Perturbations with Rainbow based on $R^n$ Model

So we will get

$$n_s - 1 = n_T = \frac{2(n-2)^2}{2n^2 - 2n - 1},$$  \hspace{1cm} (102)

where we have substituted the expressions of $\epsilon_1$, $\epsilon_3$, and $\epsilon_4$ into

$$\nu^2_R = \frac{1}{4} + \frac{(1 + \epsilon_1 - \epsilon_3 + \epsilon_4)(2 - \lambda \epsilon_1 - \epsilon_3 + \epsilon_4)}{(1 - (\lambda + 1)\epsilon_1)^2},$$ \hspace{1cm} (103)

to obtain the exact form of $n_R$ i.e.

$$n_s = 3 - 2\nu_R.$$ \hspace{1cm} (104)

For $n_T$, the calculations are similar to the case of $n_R$. 