Starobinsky Model in Rainbow Gravity

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Outlines

1. Introductions
   - Gravity's Rainbow
   - Starobinsky's Model

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Quantun Effects in Gravity Theory

- Two fundamental theories in physics
  - General Relativity
  - Quantum Physics

Quantun Effects in Gravity Theory

- Two fundamental theories in physics
  - General Relativity
  - Quantum Physics
- There are several candidates for Quantum Gravity theory
  - Loop Quantum Gravity
  - Non-Commutative Geometry
  - String Theory
  - **Gravity's Rainbow**

Introduction

1. Gravity's Rainbow
2. Starobinsky's Model

Gravity's Rainbow (Magueijo, Smolin, 2004)

• A metric of curved spacetime is energy-dependent e.g. the flat rainbow Friedmann-Robertson-Walker (FRW) metric

$$ds^2 = -\frac{1}{\tilde{f}^2(\varepsilon/M)}dt^2 + a^2(t)\delta_{ij}dx^i dx^j \ , \tag{1}$$

where $\tilde{f}(\varepsilon/M)$ is the rainbow function, ε is the energy of probe particle, $a(t)$ is the scale factor and M is the energy scale such that

$$\lim_{\varepsilon/M\to 0} \tilde{f}(\varepsilon/M) \to 1 \ . \tag{2}$$

This ensures that the original FRW metric still holds in low energy limit.

• The probe particle can be any particle dominating the universe at each time.

• Difference of energy ε leads to different background spacetime.

Starobinsky's Model (Starobinsky, 1980)

- **The action of Starobinsky's Model:**

$$S[g_{\mu\nu}] = \frac{1}{2\kappa^2}\int d^4x\sqrt{-g}(R + \frac{R^2}{6M^2}), \tag{3}$$

where $\kappa^2 = 8\pi G$ and M is the mass scale of inflation.

- The best inflation model

- This model is naturally consistent with the Planck data rather than the power-law inflation and there is no matter field in this model. That is why we use this model as the testing model of gravity's rainbow.

- The Starobinsky's Model is equivalent to the Einstein gravity with inflaton field.

Starobinsky's Model in Rainbow Gravity

1. Field Equation and Background Solution
2. The Spectra of Perturbations
3. Fitting 2015 Planck Data

Field Equation and Background Solution

• In this framework, we use the flat rainbow FRW metric in the form

$$ds^2 = -\frac{1}{\tilde{f}^2(\varepsilon/M)}dt^2 + a^2(t)\delta_{ij}dx^i dx^j \ , \tag{4}$$

• To study the evolution of the universe, the energy of probe particle ε depends on the cosmic time.

• We propose that the rainbow function is in the form

$$\tilde{f}^2 = 1 + \left(\frac{H}{M}\right)^{2\lambda} \ , \tag{5}$$

where λ is the rainbow parameter. During inflation, we assume that $H^2 \gg M^2$, then

$$\tilde{f} \simeq \left(\frac{H}{M}\right)^{\lambda} \ . \tag{6}$$

Field Equation and Background Solution

Then the Friedmann equation associated with the action (3) the metric (4) reads

$$\ddot{H}-\frac{\dot{H}^2}{2H}+\frac{1}{2\tilde{f}^2}M^2H+\frac{11}{6}H\left(\frac{\dot{\tilde{f}}}{\tilde{f}}\right)^2+\frac{1}{3}\left(\frac{\dot{\tilde{f}}}{\tilde{f}}\right)^3+\frac{10}{3}\dot{H}\frac{\dot{\tilde{f}}}{\tilde{f}}+\frac{\dot{H}}{H}\left(\frac{\dot{\tilde{f}}}{\tilde{f}}\right)^2$$
$$+H\frac{\ddot{\tilde{f}}}{\tilde{f}}+\frac{1}{3}\frac{\dot{\tilde{f}}\ddot{\tilde{f}}}{\tilde{f}^2}+\frac{1}{3}\frac{\ddot{H}}{H}\frac{\dot{\tilde{f}}}{\tilde{f}}=-3H^2\left(\frac{\dot{\tilde{f}}}{\tilde{f}}+\frac{\dot{H}}{H}\right). \quad (7)$$

Substituting this form of $\tilde{f}$, we now get

$$\frac{1}{2(1+\lambda)}\frac{M^2H}{\tilde{f}^2}+3H\dot{H}+\frac{1}{6}(17\lambda-3)\frac{\dot{H}^2}{H}+\frac{2\lambda^2\dot{H}^3}{3H^3}$$
$$+(1+\frac{\lambda}{3}\frac{\dot{H}}{H^2})\ddot{H}=0\,. \quad (8)$$

Field Equation and Background Solution

By imposing the slow-roll conditions ($\epsilon_1 < 1$), we obtain

$$\dot{H} \simeq -\frac{M^{2\lambda+2}}{6(1+\lambda)} H^{-2\lambda}. \tag{9}$$

Then the solution of these equations read

$$H \simeq H_i - \frac{M^2}{6(1+\lambda)} \left(\frac{M}{H_i}\right)^{2\lambda} (t - t_i), \tag{10}$$

$$a \simeq a_i \exp\left\{ H_i(t - t_i) - \frac{M^2}{12(1+\lambda)} \left(\frac{M}{H_i}\right)^{2\lambda} (t - t_i)^2 \right\}. \tag{11}$$

where a_i and H_i are the quantities at the initial time of inflation t_i.

Field Equation and Background Solution

We can compute ϵ_1 by using the solution (10), it gives

$$\epsilon_1 = -\frac{\dot{H}}{H^2} = \frac{1}{6(1+\lambda)}\left(\frac{M}{H}\right)^{2+2\lambda}. \tag{12}$$

We can further calculate the number of e-folds by using (10) and (20), we obtain

$$N \equiv \int_{t_i}^{t_f} H dt \tag{13}$$

$$\simeq 3(1+\lambda)\left(\frac{H_i}{M}\right)^{2+2\lambda} \simeq \frac{1}{2\epsilon_1(t_i)}, \tag{14}$$

where t_f is the time at the end of inflation.

The Spectra of Perturbations

By doing the scalar and tensor perturbations:

- The power spectra $\mathcal{P}_s$ and $\mathcal{P}_T$ are given by

$$\mathcal{P}_s \simeq \frac{1}{12\pi}\left(\frac{M}{m_{pl}}\right)^2 \frac{1}{(1+\lambda)^2\epsilon_1^2}\,, \tag{15}$$

$$\mathcal{P}_T \simeq \frac{4}{\pi}\left(\frac{M}{m_{pl}}\right)^2\,, \tag{16}$$

where $\mathcal{P}_s$ denotes the scalar part of perturbation and $\mathcal{P}_T$ denotes the tensor part associated with the gravitational waves.

- The spectral indices n_s and n_T are given by

$$n_s - 1 \simeq -4\epsilon_1 \quad , \quad n_T \simeq 0 \tag{17}$$

where n_s and n_T denote the scalar and tensor part respectively.

The Spectra of Perturbations

One can define the tensor-to-scalar ratio

$$r \equiv \frac{\mathcal{P}_T}{\mathcal{P}_s} \simeq 48(\lambda+1)^2\epsilon_1^2 \ . \tag{18}$$

Since we have derived the relation

$$N_k = \frac{1}{2\epsilon_1(t_k)} \tag{19}$$

where t_k is a time at the Hubble radius crossing. So $\mathcal{P}_s$, n_s, r can be rewritten as

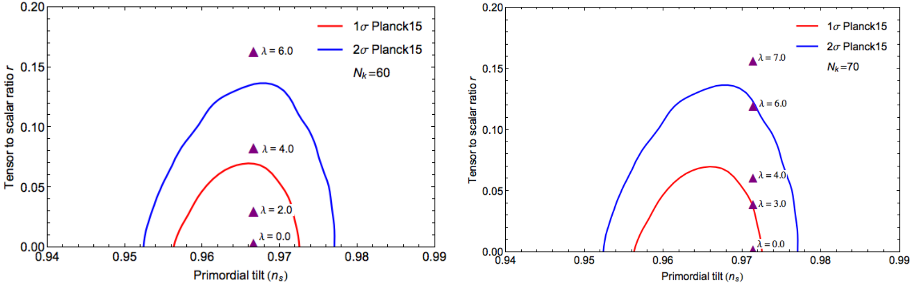
$$\mathcal{P}_s = \frac{1}{3\pi}\left(\frac{M}{m_{pl}}\right)^2 \frac{N_k^2}{(1+\lambda)^2} \ , \tag{20}$$

$$n_s - 1 = -\frac{2}{N_k} \ , \tag{21}$$

$$r = \frac{12(1+\lambda)^2}{N_k^2} \ . \tag{22}$$

Fitting 2015 Planck Data

$r - n_s$ plane: Fixed N_k, varied λ



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Fitting 2015 Planck Data

• From the Planck 2015 data, the amplitude of curvature perturbation at the scale $k = 0.05\,\mathrm{Mpc}^{-1}$ is given by $\mathcal{P}_s = (2.219 \pm 0.103) \times 10^{-9}$. Since, from Eqn. (20)

$$\mathcal{P}_s = \frac{1}{3\pi}\left(\frac{M}{m_{pl}}\right)^2 \frac{N_k^2}{(1+\lambda)^2}\,, \tag{23}$$

with $N_k = 70$ and $\lambda \simeq 3.0$, the mass scale M can be constrained as

$$\begin{aligned} M &\simeq 2 \times 10^{-6}(1+\lambda)\, m_{pl} \\ &\sim 1.70 \times 10^{14}\ \mathrm{GeV} \end{aligned} \tag{24}$$

where we have used the reduced Planck mass is obtained by $m_{pl} = 1.22 \times 10^{19}\ \mathrm{GeV}$.

Conclusions

• The predictions of the Starobinsky's model in rainbow gravity are well consistent with the data such that

$$42 \lesssim N_k \lesssim 87 \quad \text{and} \quad \lambda \lesssim 6.0 \,, \tag{25}$$

to be within 2σ C.L. of Planck'15 contours.

• One can also constrain the mass scale M (or equivalent to the inflaton mass) as

$$\begin{aligned} M &\simeq 2 \times 10^{-6}(1+\lambda)\, m_{pl} \\ &\sim 1.70 \times 10^{14}\ \text{GeV} \,, \end{aligned} \tag{26}$$

where $N_k = 70$ and $\lambda \simeq 3.0$.

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Deformed Special Relativity (Kimberly, Magueijo, 2004)

Deformed Einstein's Postulates:

• The laws of physics are the same in all inertial frames.

• The invariance of speed of light in vacuum in all inertial frames.

• The invariance of upper energy scale in the universe, Planck energy E_p, in all inertial frames.

These postulates lead to **the modified dispersion relation (MDR)**

$$\varepsilon^2 \tilde{f}^2(\varepsilon) - p^2 \tilde{g}^2(\varepsilon) = m^2 \ , \tag{27}$$

where ε is an energy of a probe particle, the functions $\tilde{f}(\varepsilon)$ and $\tilde{g}(\varepsilon)$ are called the rainbow functions.

Why $\tilde{f} \simeq (H/M)^\lambda$?

In the Jordan frame:

$$S[g_{\mu\nu}] = \frac{1}{2\kappa^2}\int d^4x\sqrt{-g}(R + \frac{R^2}{6M^2}) \tag{28}$$

In the Einstein frame:

$$S_E[g_E^{\mu\nu}, \phi_E] = \int d^4x\sqrt{-g_E}\left[\frac{1}{2\kappa^2}R_E - \frac{1}{2}g_E^{\mu\nu}\partial_\mu\phi_E\partial_\nu\phi_E - V(\phi_E)\right], \tag{29}$$

These two frames are connected by the conformal transformation $(g_E)_{\mu\nu} = \Omega^2 g_{\mu\nu}$ with $\Omega^2 = F$. The scalar field ϕ_E can be expressed by

$$\kappa\phi_E \equiv \sqrt{\frac{3}{2}}\ln F \ , \tag{30}$$

and its potential is given by

$$V(\phi_E) = \frac{FR - f}{2\kappa^2 F^2} \ . \tag{31}$$

Why $\tilde{f} \simeq (H/M)^\lambda$?

Since

$$\varepsilon_E(t) \propto \rho_E \ , \tag{32}$$

and during inflation we can assume that

$$\rho_{\phi_E} \approx V(\phi_E) \ . \tag{33}$$

From the eq. (31), the potential $V(\phi_E)$ can be written in terms of the Hubble parameter. So $\varepsilon_E(t)$ should be also written in terms of the Hubble parameter. That is why we assume that

$$\tilde{f}^2 = 1 + \left(\frac{H}{M}\right)^{2\lambda} \ , \tag{34}$$

it is plasible to propose that the rainbow function is in the power-law form of the Hubble parameter.

Deformed Special Relativity

This MDR can be as considered as the action of non-linear map $U : \mathcal{P} \rightarrow \mathcal{P}$,

$$|p|^2 = \bar{\eta}^{\mu\nu} U_\mu(p) U_\nu(p) \ , \tag{35}$$

where $\bar{\eta}^{\mu\nu}$ is the usual Minkowski metric and

$$U_\mu(\varepsilon, p_i) = (U_0, U_i) = (\tilde{f}(\varepsilon)\varepsilon, \tilde{g}(\varepsilon)p_i) \ . \tag{36}$$

To obtain the rainbow Minkowski metric, we demand that

$$U_\mu(p) U^\mu(x) = p_\mu x^\mu \ , \tag{37}$$

where p_μ and x^μ are physical momentums and coordinates respectively.

Deformed Special Relativity

To satisfy the relation (37), $U^{\mu}(x)$ should be in the form

$$U^{\mu}(x) = (U^0, U^i) = \left(\frac{t}{\tilde{f}(\varepsilon)}, \frac{x^i}{\tilde{g}(\varepsilon)}\right) \ . \tag{38}$$

Then the line element involved with this map $U^{\mu}(x)$ is given by

$$ds^2 = \bar{\eta}_{\mu\nu} U^{\mu}(dx) U^{\nu}(dx) = -\frac{dt^2}{\tilde{f}^2(\varepsilon)} + \frac{1}{\tilde{g}^2(\varepsilon)} \delta_{ij} dx^i dx^j \ , \tag{39}$$

which is invariant under the deformed Lorentz transformation. So the rainbow Minkowski metric reads

$$\eta_{\mu\nu}(\varepsilon) = \mathrm{diag}\left(-\frac{1}{\tilde{f}^2(\varepsilon)}, \frac{1}{\tilde{g}^2(\varepsilon)}, \frac{1}{\tilde{g}^2(\varepsilon)}, \frac{1}{\tilde{g}^2(\varepsilon)}\right) \ . \tag{40}$$

Gravity's Rainbow (Magueijo, Smolin, 2004)

Magueijo & Smolin proposed that the metrics of curved spacetime can be parametrized by one parameter ε,

$$g(\varepsilon) = \bar{g}^{\mu\nu}(x) e_\mu(\varepsilon) \otimes e_\nu(\varepsilon) \ , \tag{41}$$

where $\bar{g}^{\mu\nu}(x)$ is a usual metric tensor and $e_\mu(\varepsilon)$ are the energy dependent frame fields defined by

$$e_0(\varepsilon) = \frac{1}{\tilde{f}(\varepsilon)} \tilde{e}_0 \quad , \quad e_i(\varepsilon) = \frac{1}{\tilde{g}(\varepsilon)} \tilde{e}_i \ . \tag{42}$$

Note that $\tilde{e}_\mu$ are the energy independent frame fields. Then the Einstein field eq. can be modified as

$$G_{\mu\nu}(\varepsilon) = 8\pi G(\varepsilon) T_{\mu\nu}(\varepsilon) + g_{\mu\nu}(\varepsilon) \Lambda(\varepsilon) \ . \tag{43}$$

Gravity's Rainbow (Magueijo, Smolin, 2004)

Deformed Equivalence principle:

Any physical experiments of particles and fields with energies ε performed in the freely falling frame are the same as the similar experiments performed in the inertial frame in the rainbow Minkowski spacetime.

Correspondence principle:

When $\varepsilon \ll E_p$, gravity's rainbow becomes the ordinary general theory of relativity i.e.

$$\lim_{\varepsilon/E_p \to 0} \tilde{f}(\varepsilon) = \lim_{\varepsilon/E_p \to 0} \tilde{g}(\varepsilon) = 1 \ . \tag{44}$$

Rainbow Universe (Ling, 2007)

To study the evolution of rainbow universe, let us consider the flat rainbow FRW metric in the form

$$ds^2 = -\frac{1}{\tilde{f}^2(\varepsilon)}dt^2 + a^2(t)\delta_{ij}dx^i dx^j \ , \tag{45}$$

where $\tilde{g}(\varepsilon) = 1$. By doing the calculations of this metric with the usual Einstein field eq., we can come up with the Friedmann equations

$$H^2 = \frac{8\pi G}{3}\frac{\rho}{\tilde{f}^2} \ , \tag{46}$$

$$\dot{H} = -\frac{4\pi G(\rho + P)}{\tilde{f}^2} - H\frac{\dot{\tilde{f}}}{\tilde{f}} \ , \tag{47}$$

where we have used $T^{\mu}_{\nu} = \mathrm{diag}(-\rho, P, P, P)$. This approach often used to solve the Big Bang singularity problem.

Field Equation of $f(R)$ Gravity

Since we are interested in **the flat FRW metric**:

$$ds^2 = -dt^2 + a^2(t)\left(dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2\right) \tag{48}$$

and **the perfect fluid**:

$$T^{\mu(M)}_{\nu} = \mathrm{diag}(-\rho_M, P_M, P_M, P_M) \tag{49}$$

So we can come up with the Friedmann equations

$$3FH^2 = \frac{FR - f}{2} - 3H\dot{F} + \kappa^2 \rho_M \tag{50}$$

$$-2F\dot{H} = \ddot{F} - H\dot{F} + \kappa^2(\rho_M + P_M) \tag{51}$$

where $H = \dot{a}/a$.

Field Equation and Background Solution

The perfect fluid:

$$T^{\mu(M)}_{\nu} = \mathrm{diag}(-\rho_M, P_M, P_M, P_M) \tag{52}$$

where ρ_M and P_M are the energy density and pressure respectively.

Then the Friedmann equations with rainbow effects read

$$\begin{aligned} 3(FH^2 + H\dot{F}) + \dot{F}\frac{\dot{\tilde{f}}}{\tilde{f}} &= \frac{FR - f(R)}{2\tilde{f}^2} + \frac{\kappa^2 \rho_M}{\tilde{f}^2}, & (53) \\ \ddot{F} - H\dot{F} + 2F\dot{H} + 2FH\frac{\dot{\tilde{f}}}{\tilde{f}} &= -\frac{\kappa^2}{\tilde{f}^2}(\rho_M + P_M). & (54) \end{aligned}$$

$$\ddot{R} + 3H\dot{R} + \frac{4\dot{\tilde{f}}\dot{R}}{3\tilde{f}} + \frac{M^2 R}{\tilde{f}^2} = 0. \tag{55}$$

Cosmological Perturbations with Rainbow

The full perturbed metric around the flat FRW metric

$$ds^2 = -\frac{1+2\alpha}{\tilde{f}^2(\varepsilon)}dt^2 - \frac{2a(t)(\partial_i\beta - S_i)}{\tilde{f}(\varepsilon)}dtdx^i + a^2(t)(\delta_{ij} + 2\psi\delta_{ij} + 2\partial_i\partial_j\gamma + 2\partial_j F_i + h_{ij})dx^i dx^j \ , \quad (56)$$

where α, β, ψ, γ are scalar perturbations, S_i, F_i are vector perturbations, and h_{ij} are tensor perturbations. We can set $S_i = F_i = 0$ since the vector perturbations decay rapidly during inflation. Consider the gauge transformation:

$$\alpha \rightarrow \hat{\alpha} = \alpha + \frac{\dot{\tilde{f}}}{\tilde{f}}\delta t - \dot{\delta t}, \quad \beta \rightarrow \hat{\beta} = \beta - \frac{\delta t}{a\tilde{f}} + a\tilde{f}\dot{\delta x} \quad (57)$$

$$\psi \rightarrow \hat{\psi} = \psi - H\delta t, \quad \gamma \rightarrow \hat{\gamma} = \gamma - \delta x \ . \quad (58)$$

Cosmological Perturbations with Rainbow

The gauge invariant quantities can be defined by

$$\Phi = \alpha - \tilde{f}\frac{d}{dt}\left[a^2\tilde{f}\left(\dot{\gamma} + \frac{\beta}{a\tilde{f}}\right)\right], \tag{59}$$

$$\Psi = -\psi + a^2\tilde{f}^2 H\left(\dot{\gamma} + \frac{\beta}{a\tilde{f}}\right), \tag{60}$$

$$\mathcal{R} = \psi - \frac{H\delta F}{\dot{F}}, \tag{61}$$

where $\mathcal{R}$ is the curvature perturbation. Choosing the Longitudinal gauge $\beta = 0$ and $\gamma = 0$ such that $\Phi = \alpha$ and $\Psi = -\psi$. So the line element (56) reduces to

$$ds^2 = -\frac{1+2\Phi}{\tilde{f}^2(t)}dt^2 + a^2(t)(1-2\Psi)\delta_{ij}dx^i dx^j. \tag{62}$$

Scalar Perturbations with Rainbow

Introduce a quantity A through

$$A \equiv 3(H\Phi + \dot{\Psi}). \tag{63}$$

Then the field equation associated with the metric (62) gives the following equations.

$$\begin{aligned} -\frac{\nabla^2\Psi}{a^2} + \tilde{f}^2 HA &= -\frac{1}{2F}\left[3\tilde{f}^2\left(H^2 + \dot{H} + \frac{\dot{\tilde{f}}}{\tilde{f}}\right)\delta F + \frac{\nabla^2 \delta F}{a^2} - 3\tilde{f}^2 H \delta\dot{F} \right. \\ &+ \left. 3\tilde{f}^2 H\dot{F}\Phi + \tilde{f}^2\dot{F}A + \kappa^2\delta\rho_M\right], \end{aligned} \tag{64}$$

$$H\Phi + \dot{\Psi} = -\frac{1}{2F}(H\delta F + \dot{F}\Phi - \delta\dot{F}), \tag{65}$$

Scalar Perturbations with Rainbow

and

$$\begin{aligned}
\dot{A} + \left(2H + \frac{\dot{\tilde{f}}}{\tilde{f}}\right) A &+ 3\dot{H}\Phi + \frac{\nabla^2\Phi}{a^2\tilde{f}^2} + \frac{3H\Phi\dot{\tilde{f}}}{\tilde{f}} = \frac{1}{2F}\Big[3\delta\ddot{F} \\
+3\left(H + \frac{\dot{\tilde{f}}}{\tilde{f}}\right)\delta\dot{F} - 6H^2\delta F &- \frac{\nabla^2\delta F}{a^2\tilde{f}^2} - 3\dot{F}\dot{\Phi} - \dot{F}A - 3\left(H + \frac{\dot{\tilde{f}}}{\tilde{f}}\right)\dot{F}\Phi \\
&- 6\ddot{F}\Phi + \frac{\kappa^2}{\tilde{f}^2}(3\delta P_M + \delta\rho_M)\Big], \qquad (66)
\end{aligned}$$

where the energy-momentum tensor has been chosen to be in the perfect fluid form.

Scalar Perturbations with Rainbow

In our model, we set $\delta\rho_M = 0$ and $\delta P_M = 0$ and also choose the condition $\delta F = 0$ such that $\mathcal{R} = \psi = -\Psi$. Under this gauge choice, the equation (65) will become

$$\Phi = \frac{\dot{\mathcal{R}}}{H + \dot{F}/2F} \ . \tag{67}$$

Plugging the equation (67) into (64), we now obtain

$$A = -\frac{1}{H + \dot{F}/2F}\left[\frac{\nabla^2\mathcal{R}}{a^2\tilde{f}^2} + \frac{3H\dot{F}\dot{\mathcal{R}}}{2F(H + \dot{F}/2F)}\right] \ . \tag{68}$$

Scalar Perturbations with Rainbow

Using the background equation in the equation (66), it gives

$$\dot{A} + \left(2H + \frac{\dot{F}}{2F}\right) A + \frac{\dot{\tilde{f}} A}{\tilde{f}} + \frac{3\dot{F}\dot{\Phi}}{2F} + \left[\frac{3\ddot{F} + 6H\dot{F}}{2F} + \frac{\nabla^2}{a^2 \tilde{f}^2}\right] \Phi + \frac{3\dot{F}}{2F} \frac{\Phi \dot{\tilde{f}}}{\tilde{f}} = 0. \quad (69)$$

Substituting the equations (67) and (68) into the equation (69), then the simple equation of $\mathcal{R}$ written in the Fourier space is given by

$$\ddot{\mathcal{R}} + \frac{1}{a^3 Q_s} \frac{d}{dt}(a^3 Q_s)\dot{\mathcal{R}} + \frac{\dot{\tilde{f}}}{\tilde{f}} \dot{\mathcal{R}} + \frac{k^2}{a^2 \tilde{f}^2} \mathcal{R} = 0 \ , \quad (70)$$

here k denotes a comoving wavenumber.

Scalar Perturbations with Rainbow

Introduce the new variable Q_s through

$$Q_s \equiv \frac{3\dot{F}^2}{2\kappa^2 F(H + \dot{F}/2F)^2} \ . \tag{71}$$

Also we can define the new parameters $z_s = a\sqrt{Q_s}$ and $u = z_s\mathcal{R}$, so the equation (70) reduces to

$$u'' + \left(k^2 - \frac{z_s''}{z_s}\right) u = 0 \ , \tag{72}$$

where a prime is a derivative with respect to the conformal time $\eta = \int (a\tilde{f})^{-1} dt$.

Scalar Perturbations with Rainbow

Let us define the Hubble flow parameters:

$$\epsilon_1 \equiv -\frac{\dot{H}}{H^2}, \quad \epsilon_3 \equiv \frac{\dot{F}}{2HF}, \quad \epsilon_4 \equiv \frac{\dot{E}}{2HE} \ , \tag{73}$$

where the variable E is defined by

$$E \equiv \frac{3\dot{F}^2}{2\kappa^2} \ . \tag{74}$$

From the definitions (73), Q_s can be re-expressed as

$$Q_s = \frac{E}{FH^2(1+\epsilon_3)^2} = \frac{6F\epsilon_3^2}{\kappa^2(1+\epsilon_3)^2} \ . \tag{75}$$

Scalar Perturbations with Rainbow

Since ϵ_i ($i = 1, 3, 4$) are assumed to be constant values ($\dot{\epsilon}_i \simeq 0$) during inflation, then

$$\eta = -\frac{1}{(1-(1+\lambda)\epsilon_1)\tilde{f}aH} \ . \tag{76}$$

Under the assumption of ϵ_i, the term z_s''/z_s in the equation (72) can be estimated as

$$\frac{z_s''}{z_s} = \frac{\nu_{\mathcal{R}}^2 - 1/4}{\eta^2} \ , \tag{77}$$

with

$$\nu_{\mathcal{R}}^2 = \frac{1}{4} + \frac{(1+\epsilon_1-\epsilon_3+\epsilon_4)(2-\lambda\epsilon_1-\epsilon_3+\epsilon_4)}{(1-(\lambda+1)\epsilon_1)^2} \ . \tag{78}$$

Scalar Perturbations with Rainbow

So the approximate solution to the equation (72) can be written in terms of a linear combination of the Hankel functions as

$$u = \frac{\sqrt{\pi|\eta|}}{2} e^{i(1+2\nu_{\mathcal{R}})\pi/4} \left[c_1 \mathrm{H}^{(1)}_{\nu_{\mathcal{R}}}(k|\eta|) + c_2 \mathrm{H}^{(2)}_{\nu_{\mathcal{R}}}(k|\eta|) \right] , \tag{79}$$

where c_1, c_2 are integration constants. In the asymptotic past $k\eta \to -\infty$, the solution (79) will become $e^{-ik\eta}/\sqrt{2k}$ which leads to $c_1 = 1$ and $c_2 = 0$. Then we have

$$u = \frac{\sqrt{\pi|\eta|}}{2} e^{i(1+2\nu_{\mathcal{R}})\pi/4} \mathrm{H}^{(1)}_{\nu_{\mathcal{R}}}(k|\eta|) . \tag{80}$$

Let us define the power spectrum as

$$\mathcal{P}_s \equiv \frac{4\pi k^3}{(2\pi)^3} |\mathcal{R}|^2 . \tag{81}$$

Scalar Perturbations with Rainbow

Using the solution (80) and the relation $u = z_s\mathcal{R}$ in the above definition, then it gives

$$\mathcal{P}_s = \frac{1}{Q_s}\left[(1-(1+\lambda)\epsilon_1)\frac{\Gamma(\nu_{\mathcal{R}})H}{2\pi\Gamma(3/2)}\left(\frac{H}{M}\right)^{\lambda}\right]^2\left(\frac{k|\eta|}{2}\right)^{3-2\nu_{\mathcal{R}}} . \qquad (82)$$

where we have used that the limit as $k|\eta| \to 0$ then $\mathrm{H}^{(1)}_{\nu_{\mathcal{R}}}(k|\eta|) \to -(i/\pi)\Gamma(\nu_{\mathcal{R}})(k|\eta|/2)^{-\nu_{\mathcal{R}}}$, and $\mathcal{P}_s$ should be evaluated at $k = aH$ because the curvature perturbation $\mathcal{R}$ is fixed after the Hubble radius crossing. Now, the spectral index n_s can be defined by

$$n_s - 1 = \left.\frac{d\ln\mathcal{P}_s}{d\ln k}\right|_{k=aH} = 3 - 2\nu_{\mathcal{R}} \ , \qquad (83)$$

where ν_s is already expressed out in (103).

Scalar Perturbations with Rainbow

We can now further approximate the equation (83) by imposing that $|\epsilon_i| \ll 1$ for all i during inflation, then n_s reduces to

$$n_s - 1 \simeq -2(\lambda + 2)\epsilon_1 + 2\epsilon_3 - 2\epsilon_4 \ . \tag{84}$$

Subsequently, the power spectrum of the curvature perturbation $\mathcal{P}_s$ (82) can be rewritten as

$$\mathcal{P}_s \approx \frac{1}{Q_s}\left(\frac{H}{2\pi}\right)^2\left(\frac{H}{M}\right)^{2\lambda} \ . \tag{85}$$

Tensor Perturbations with Rainbow

Let h_{ij} be the tensor perturbations generally written in the form

$$h_{ij} = h_+ e_{ij}^+ + h_\times e_{ij}^\times \ , \tag{86}$$

where e_{ij}^+ and $e_{ij}^\times$ denote the polarization tensors which correspond to the two polarization states of h_{ij}. Let $\vec{k}$ be a wave vector along the z-axis, then $e_{xx}^+ = -e_{yy}^+ = 1$ and $e_{xy}^\times = e_{yx}^\times = 1$. Consider the tensor perturbation, the perturbed metric (56) reduces to

$$ds^2 = -\frac{dt^2}{\tilde{f}^2(t)} + a^2(t)h_\times dxdy + a^2(t)\left[(1+h_+)dx^2 + (1-h_+)dy^2 + dz^2\right].$$

Tensor Perturbations with Rainbow

Applying this metric to the field equation, then we can show that the Fourier components h_χ satisfy the following equation

$$\ddot{h}_\chi + \frac{(a^3 F)^\cdot}{a^3 F}\dot{h}_\chi + \frac{\dot{\tilde{f}}}{\tilde{f}}\dot{h}_\chi + \frac{k^2}{a^2 \tilde{f}^2} h_\chi = 0 \ , \tag{87}$$

where χ denotes $+$ and $\times$. Introduce a new variable $z_t = a\sqrt{F}$ and $u_\chi = z_t h_\chi / \sqrt{16\pi G}$, then equation (87) will become

$$u''_\chi + \left(k^2 - \frac{z''_t}{z_t}\right) u_\chi = 0 \ . \tag{88}$$

Assuming that $\dot{\epsilon}_i = 0$ during inflation, we obtain

$$\frac{z''_t}{z_t} = \frac{\nu_t^2 - 1/4}{\eta^2} \ , \qquad \nu_t^2 = \frac{1}{4} + \frac{(1+\epsilon_3)(2-(1+\lambda)\epsilon_1 + \epsilon_3)}{(1-(1+\lambda)\epsilon_1)^2} \ . \tag{89}$$

Tensor Perturbations with Rainbow

Again the solution to the equation (88) is approximately written in terms of linear combination of the Hankel functions, then the power spectrum P_T after the Hubble radius crossing is

$$\begin{aligned}\mathcal{P}_T &= 4 \times \frac{16\pi G}{a^2 F} \frac{4\pi k^3}{(2\pi)^3} |u_\chi|^2 \\ &= \frac{16}{\pi} \left(\frac{H}{m_{pl}}\right)^2 \frac{1}{F} \left[(1-(1+\lambda)\epsilon_1) \frac{\Gamma(\nu_t)}{\Gamma(3/2)} \left(\frac{H}{M}\right)^\lambda \right]^2 \left(\frac{k|\eta|}{2}\right)^{3-2\nu_t} .\end{aligned}$$

During inflation, we demand that the Hubble flow parameters is very small ($|\epsilon_i| \ll 1$), so ν_t can be evaluated as

$$\nu_t \simeq \frac{3}{2} + (1+\lambda)\epsilon_1 + \epsilon_3 \ . \tag{90}$$

Tensor Perturbations with Rainbow

The spectral index of tensor perturbations n_T that is given by

$$n_T = \left.\frac{d\ln\mathcal{P}_T}{d\ln k}\right|_{k=aH} = 3 - 2\nu_t \simeq -2(1+\lambda)\epsilon_1 - 2\epsilon_3 \ . \tag{91}$$

Using the conditions $|\epsilon_i| \ll 1$ again, we now get

$$\mathcal{P}_T \simeq \frac{16}{\pi}\left(\frac{H}{m_{pl}}\right)^2 \frac{1}{F}\left(\frac{H}{M}\right)^{2\lambda} \ . \tag{92}$$

Furthermore, the last important parameter is the tensor-to-scalar ratio r which is defined by

$$r \equiv \frac{\mathcal{P}_T}{\mathcal{P}_s} \simeq 48\epsilon_3^2 \tag{93}$$

where we have used the formulas of $\mathcal{P}_s$ and $\mathcal{P}_T$.

Perturbations with Rainbow based on Starobinsky's Model

For the Starobinsky's model, we obtain

$$\epsilon_3 \simeq -(1+\lambda)\epsilon_1 \ , \tag{94}$$

and

$$\epsilon_4 = -(2\lambda+1)\epsilon_1 \ . \tag{95}$$

Then n_s, $\mathcal{P}_s$, and r can be re-expressed as

$$n_s - 1 \simeq -4\epsilon_1 \ , \tag{96}$$

$$\mathcal{P}_s \simeq \frac{1}{12\pi}\left(\frac{M}{m_{pl}}\right)^2 \frac{1}{(1+\lambda)^2\epsilon_1^2} \ , \tag{97}$$

and

$$r \simeq 48(\lambda+1)^2\epsilon_1^2 \ . \tag{98}$$

Perturbations with Rainbow based on R^n Model

For R^n model, ϵ_1 is given by

$$\epsilon_1 = \frac{2-n}{(n-1)(2n-1)(1+\lambda)} \ . \tag{99}$$

Also we obtain the relations among the Hubble flow parameters which are

$$\epsilon_3 \simeq -(n-1)(\lambda+1)\epsilon_1 = \frac{n-2}{2n-1} \ , \tag{100}$$

and

$$\epsilon_4 \simeq -(2n(\lambda+1)-2\lambda-1)\epsilon_1 = -\frac{(2n(\lambda+1)-2\lambda-1)(2-n)}{(n-1)(2n-1)(1+\lambda)} \ . \tag{101}$$

Perturbations with Rainbow based on R^n Model

So we will get

$$n_s - 1 = n_T = -\frac{2(n-2)^2}{2n^2 - 2n - 1} \ , \tag{102}$$

where we have substituted the expressions of ϵ_1, ϵ_3, and ϵ_4 into

$$\nu_{\mathcal{R}}^2 = \frac{1}{4} + \frac{(1+\epsilon_1-\epsilon_3+\epsilon_4)(2-\lambda\epsilon_1-\epsilon_3+\epsilon_4)}{(1-(\lambda+1)\epsilon_1)^2} \ , \tag{103}$$

to obtain the exact form of $n_{\mathcal{R}}$ i.e.

$$n_s = 3 - 2\nu_{\mathcal{R}} \ . \tag{104}$$

For n_T, the calculations are similar to the case of $n_{\mathcal{R}}$.