44th SLAC summer institute
Lecture I:
The Standard Model defined

Keith Ellis
IPPP, Durham

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- Feynman rules for perturbative QCD follow from Lagrangian

$$\mathcal{L} = -\frac{1}{4} F^A_{\alpha\beta} F^{\alpha\beta}_A + \sum_{\text{flavours}} \bar{q}_a (i\not{D} - m)_{ab} q_b + \mathcal{L}_{\text{gauge-fixing}}$$

$F^A_{\alpha\beta}$ is field strength tensor for spin-1 gluon field $\mathcal{A}^A_\alpha$,

$$F^A_{\alpha\beta} = \partial_\alpha \mathcal{A}^A_\beta - \partial_\beta \mathcal{A}^A_\alpha - g f^{ABC} \mathcal{A}^B_\alpha \mathcal{A}^C_\beta$$

Capital indices A, B, C run over 8 colour degrees of freedom of the gluon field. Third 'non-Abelian' term distinguishes QCD from QED, giving rise to triplet and quartic gluon self-interactions and ultimately to asymptotic freedom.

- QCD coupling strength is $\alpha_s \equiv g^2/4\pi$. Numbers f^{ABC} $(A, B, C = 1, ..., 8)$ are structure constants of the SU(3) colour group. Quark fields q_a $(a = 1, 2, 3)$ are in triplet colour representation. D is covariant derivative:

$$\begin{aligned}(D_\alpha)_{ab} &= \partial_\alpha \delta_{ab} + ig \left(t^C \mathcal{A}^C_\alpha\right)_{ab} \\ (D_\alpha)_{AB} &= \partial_\alpha \delta_{AB} + ig (T^C \mathcal{A}^C_\alpha)_{AB}\end{aligned}$$

- t and T are matrices in the fundamental and adjoint representations of SU(3), respectively:

$$t^A = \frac{1}{2}\lambda^A, \quad [t^A, t^B] = if^{ABC}t^C, \quad [T^A, T^B] = if^{ABC}T^C$$

where $(T^A)_{BC} = -if^{ABC}$. We use the metric $g^{\alpha\beta} = \text{diag}(1,-1,-1,-1)$ and set $\hbar = c = 1$. $\not{D}$ is symbolic notation for $\gamma^\alpha D_\alpha$. Normalisation of the t matrices is

$$\text{Tr}\, t^A t^B = T_R\, \delta^{AB}, \quad T_R = \frac{1}{2}.$$

- Colour matrices obey the relations:

$$\sum_A t^A_{ab} t^A_{bc} = C_F\, \delta_{ac}\,, \quad C_F = \frac{N^2-1}{2N}$$

$$\text{Tr}\, T^C T^D = \sum_{A,B} f^{ABC} f^{ABD} = C_A\, \delta^{CD}\,, \quad C_A = N$$

Thus $C_F = \frac{4}{3}$ and $C_A = 3$ for SU(3).

- QCD Lagrangian is invariant under local gauge transformations. That is, one can redefine quark fields independently at every point in space-time,

$$q_a(x) \to q'_a(x) = \exp(it \cdot \theta(x))_{ab} q_b(x) \equiv \Omega(x)_{ab} q_b(x)$$

without changing physical content.

- Covariant derivative is so called because it transforms in same way as field itself:

$$D_\alpha q(x) \to D'_\alpha q'(x) \equiv \Omega(x) D_\alpha q(x) \ .$$

(omitting the colour labels of quark fields from now on). Use this to derive transformation property of gluon field $\mathcal{A}$

$$\begin{aligned} D'_\alpha q'(x) &= (\partial_\alpha + igt \cdot \mathcal{A}'_\alpha)\Omega(x) q(x) \\ &\equiv (\partial_\alpha \Omega(x)) q(x) + \Omega(x) \partial_\alpha q(x) + igt \cdot \mathcal{A}'_\alpha \Omega(x) q(x) \end{aligned}$$

where $t \cdot \mathcal{A}_\alpha \equiv \sum_A t^A \mathcal{A}^A_\alpha$. Hence

$$t \cdot \mathcal{A}'_\alpha = \Omega(x) t \cdot \mathcal{A}_\alpha \Omega^{-1}(x) + \frac{i}{g} (\partial_\alpha \Omega(x)) \Omega^{-1}(x) \ .$$

- Transformation property of gluon field strength $F_{\alpha\beta}$ is

$$t \cdot F_{\alpha\beta}(x) \to t \cdot F'_{\alpha\beta}(x) = \Omega(x) F_{\alpha\beta}(x) \Omega^{-1}(x) \; .$$

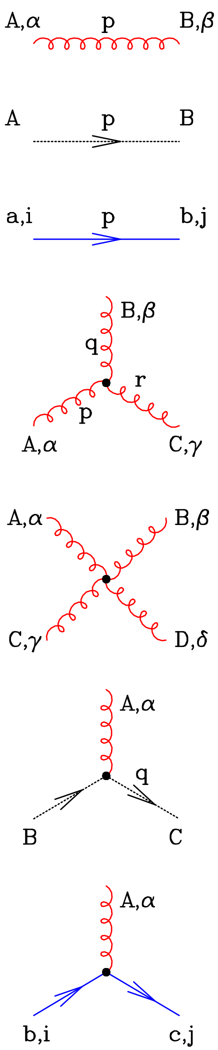
Contrast this with gauge-invariance of QED field strength. QCD field strength is not gauge invariant because of self-interaction of gluons. Carriers of the colour force are themselves coloured, unlike the electrically neutral photon.

- Note there is no gauge-invariant way of including a gluon mass. A term such as

$$m^2 \mathcal{A}^\alpha \mathcal{A}_\alpha$$

is not gauge invariant. This is similar to QED result for mass of the photon. On the other hand quark mass term is gauge invariant, under SU(3) gauge transformations.

- Use free piece of QCD Lagrangian to obtain inverse quark and gluon propagators.
- Quark propagator in momentum space obtained by setting $\partial^\alpha = -ip^\alpha$ for an incoming field.
- The $i\varepsilon$ prescription for pole of propagator is determined by causality, as in QED.



$$\delta^{AB}\left[-g^{\alpha\beta}+(1-\lambda)\frac{p^\alpha p^\beta}{p^2+i\epsilon}\right]\frac{i}{p^2+i\epsilon}$$

$$\delta^{AB}\frac{i}{(p^2+i\epsilon)}$$

$$\delta^{ab}\frac{i}{(\not{p}-m+i\epsilon)_{ji}}$$

$$-g\ f^{ABC}[(p-q)^\gamma g^{\alpha\beta}+(q-r)^\alpha g^{\beta\gamma}+(r-p)^\beta g^{\gamma\alpha}]$$
(all momenta incoming)

$$-ig^2\ f^{XAC}f^{XBD}\ [g^{\alpha\beta}g^{\gamma\delta}-g^{\alpha\delta}g^{\beta\gamma}]$$
$$-ig^2\ f^{XAD}f^{XBC}\ [g^{\alpha\beta}g^{\gamma\delta}-g^{\alpha\gamma}g^{\beta\delta}]$$
$$-ig^2\ f^{XAB}f^{XCD}\ [g^{\alpha\gamma}g^{\beta\delta}-g^{\alpha\delta}g^{\beta\gamma}]$$

$$g\ f^{ABC}q^\alpha$$

$$-ig\ (t^A)_{cb}\ (\gamma^\alpha)_{ji}$$

- Gluon propagator impossible to define without a choice of gauge. The choice

$$\mathcal{L}_{\text{gauge-fixing}} = -\frac{1}{2\,\lambda}\left(\partial^\alpha \mathcal{A}_\alpha^A\right)^2$$

defines covariant gauges with gauge parameter λ. Inverse gluon propagator is then

$$\Gamma^{(2)}_{\{AB,\ \alpha\beta\}}(p) = i\delta_{AB}\ \left[p^2 g_{\alpha\beta} - (1-\frac{1}{\lambda})p_\alpha p_\beta\right].$$

(Without gauge-fixing term this function would have no inverse.) Resulting propagator is in the table. $\lambda = 1$ (0) is Feynman (Landau) gauge.

- Gauge fixing explicitly breaks gauge invariance. However, in the end physical results will be independent of gauge. For convenience, we usually use Feynman gauge.
- In non-Abelian theories like QCD, covariant gauge-fixing term must be supplemented by a ghost term which we do not discuss here. Ghost field, shown by dashed lines in the above table, cancels unphysical degrees of freedom of gluon which would otherwise propagate in covariant gauges.

- Propagators determined from $-S$, interactions from S.
- Consider a theory which contains only a complex scalar field ϕ and an action which contains only bilinear terms, $S = \phi^* (K + K') \phi$.
- MOE: both K and K' are included in the free Lagrangian, $S_0 = \phi^* (K + K') \phi$. Using the above rule the propagator Δ for the ϕ field is given by

$$\Delta = \frac{-1}{K + K'}.$$

- JOE: K is regarded as the free Lagrangian, $S_0 = \phi^* K \phi$, and K' as the interaction Lagrangian, $S_I = \phi^* K' \phi$. Now S_I is included to all orders in perturbation theory by inserting the interaction term an infinite number of times:

$$\begin{aligned}\Delta &= \frac{-1}{K} + \left(\frac{-1}{K}\right) K' \left(\frac{-1}{K}\right) + \left(\frac{-1}{K}\right) K' \left(\frac{-1}{K}\right) K' \left(\frac{-1}{K}\right) + \cdots \\ &= \frac{-1}{K + K'}\end{aligned}$$

- An alternative choice of gauge fixing is provided by the *axial gauges* which are fixed in terms of another vector which we denote by b

$$\mathcal{L}_{\text{gauge-fixing}} = -\frac{1}{2\,\lambda}\left(b^\alpha \mathcal{A}^A_\alpha\right)^2 ,$$

The advantage of the axial class of gauge is that ghost fields are not required. However one pays for this simplicity because the gluon propagator is more complicated. The inverse propagator is

$$\Gamma^{(2)}_{\{AB,\ \alpha\beta\}}(p) = i\delta_{AB}\left[p^2 g_{\alpha\beta} - p_\alpha p_\beta + \frac{1}{\lambda} b_\alpha b_\beta\right].$$

The inverse of this matrix gives the gauge boson propagator,

$$\Delta^{(2)}_{\{BC,\ \beta\gamma\}}(p) = \delta_{BC}\,\frac{i}{p^2}\left[-g_{\beta\gamma} + \frac{b_\beta p_\gamma + p_\beta b_\gamma}{b\cdot p} - \frac{(b^2+\lambda p^2)p_\beta p_\gamma}{(b\cdot p)^2}\right].$$

Notice the new singularities at $b \cdot p = 0$.

- What are the properties of these gauges which make them interesting? Let us specialize to the case $\lambda = 0, b^2 = 0$, (light-cone gauge).

$$\Delta^{(2)}_{\{BC,\beta\gamma\}}(p) = \delta_{BC} \frac{i}{p^2} d_{\beta\gamma}(p,b)$$

where

$$d_{\beta\gamma} = -g_{\beta\gamma} + \frac{b_\beta p_\gamma + p_\beta b_\gamma}{b \cdot p} \ .$$

In the limit $p^2 \to 0$ we find that

$$b^\beta d_{\beta\gamma}(p,b) = 0, \ p^\beta d_{\beta\gamma}(p,b) = 0 \ .$$

Only two physical polarization states, orthogonal to b and p, propagate. For this reason these classes of gauges are called physical gauges. In the $p^2 \to 0$ limit we may decompose the numerator of the propagator into a sum over two polarizations:

$$d_{\alpha\beta} = \sum_i \varepsilon^{(i)}_\alpha(p) \varepsilon^{(i)}_\beta(p) \ .$$

In addition to the constraint $\varepsilon^{(i)}_\beta(p) p^\beta = 0$, which is always true, in an axial gauge we have the further constraint $\varepsilon^{(i)}_\beta(p) b^\beta = 0$.

$$\mathcal{L}_{\rm classical} = -\frac{1}{4}\sum_i W^{i\;\mu\nu} W^i_{\mu\nu} - \frac{1}{4} B^{\mu\nu} B_{\mu\nu} \;,$$

- $W^i_{\mu\nu}$ and $B_{\mu\nu}$ are the field strength tensors of the U(1) gauge field B and the SU(2) gauge fields W^i, (g_W is SU(2) gauge coupling.)

$$\begin{aligned} W^i_{\mu\nu} &= \partial_\mu W^i_\nu - \partial_\nu W^i_\mu - g_W \epsilon^{ijk} W^j_\mu W^k_\nu \\ B_{\mu\nu} &= \partial_\mu B_\nu - \partial_\nu B_\mu \;, \end{aligned}$$

- The Lagrangian evidently describes four massless vector bosons forming a singlet (B) and a triplet (W^1, W^2, W^3) under weak isospin.
- The coupling of the gauge fields to fermionic matter fields is implemented using the covariant derivative, which is

$$D^\mu = \delta_{ij}\partial^\mu + i g_W (T\cdot W^\mu)_{ij} + iY\delta_{ij} g'_W B^\mu$$

where g'_W is the U(1) gauge coupling.

- The matrices T are a representation of the SU(2) *weak isospin* algebra and the U(1) charge Y is called the *weak hypercharge*. In order to specify the coupling to matter we therefore have to choose the SU(2) representation, T, and the U(1) gauge charge, Y, for the matter fields.

$$[T^i, T^j] = i\epsilon^{ijk}T^k, \ \epsilon^{123} = 1 \ .$$

- Defining $W^{\pm}_{\mu} = (W^1_{\mu} \mp iW^2_{\mu})/\sqrt{2}$ and $T^{\pm} = T^1 \pm iT^2$ we have

$$W_{\mu} \cdot T = W^3_{\mu} T_3 + \frac{1}{\sqrt{2}} W^+_{\mu} T^+ + \frac{1}{\sqrt{2}} W^-_{\mu} T^-$$

where the matrices $T^{\pm}$and T^3 satisfy the relations

$$[T^+, T^-] = 2T^3, \quad [T^3, T^{\pm}] = \pm T^{\pm} \ .$$

T^+ and T^- are the weak isospin raising and lowering operators. For example, in the doublet representation of SU(2) we have

$$T^3 = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}, \ T^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \ T^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} .$$

- Inserting a mass term for the W and B fields breaks gauge invariance.

- Inserting a mass term for the W and B fields violates gauge invariance, but adopt a practical approach and add one anyway

$$\mathcal{L}_{\rm mass} = \frac{1}{2}\left[M^2 \sum_i W^i_\mu W^{i\,\mu} + M_0^2 B^\mu B_\mu + 2M_{03}^2 W^3_\mu B^\mu\right]$$

- including the mass term, the terms bilinear in the boson fields become,

$$\begin{aligned}\mathcal{L}_{\rm gauge} &= -\frac{1}{2} W^+_{\mu\nu} W^{-\,\mu\nu} + M^2 W^+_\mu W^{-\,\mu} \\ &- \frac{1}{4}\left[W^3_{\mu\nu} W^{3\,\mu\nu} + B_{\mu\nu} B^{\mu\nu}\right. \\ &- \frac{1}{2}\left[M^2 W^3_\mu (W^3)^\mu + M_0^2 B_\mu B^\mu + 2M_{03}^2 W^3_\mu B^\mu\right]\end{aligned}$$

- First line defines 2 electrically charged spin one bosons with mass $M = M_W$.
- The mass matrix for the electrically neutral fields is

$$\frac{1}{2}(W^3_\mu, B_\mu)\mathcal{M}\begin{pmatrix} W^{3\,\mu} \\ B^\mu \end{pmatrix}, \mathcal{M} = \begin{pmatrix} M^2 & M_{03}^2 \\ M_{03}^2 & M_0^2 \end{pmatrix}$$

- Matrix is not arbitrary, should have zero eigenvalue corresponding to the zero photon mass, $\det\mathcal{M} = 0 \quad\Rightarrow\quad M M_0 = M_{03}^2$

- We therefore redefine the electrically neutral fields by introducing rotated fields A_μ and Z_μ which propagate independently, $c_W = \cos\theta_W, s_W = \sin\theta_W$

$$\begin{pmatrix} W^3_\mu \\ B_\mu \end{pmatrix} = \begin{pmatrix} c_W & s_W \\ -s_W & c_W \end{pmatrix} \begin{pmatrix} Z_\mu \\ A_\mu \end{pmatrix},$$

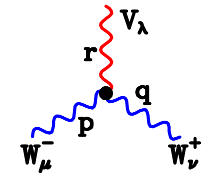
- Since the massless field must correspond to the massless eigenvalue, we have that

$$\begin{aligned} M_A^2 = (s_W, c_W)\mathcal{M}\begin{pmatrix} s_W \\ c_W \end{pmatrix} &= M^2 s_W^2 + 2MM_0 s_W c_W + M_0^2 c_W \\ &= M^2 (s_W + \frac{M_0}{M} c_W)^2 = 0, \rightarrow \frac{M_0}{M} = -\frac{s_W}{c_W} \end{aligned}$$

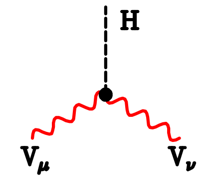
- Correspondingly the mass of the Z is,

$$\begin{aligned} M_Z^2 = (c_W, -s_W)\mathcal{M}\begin{pmatrix} c_W \\ -s_W \end{pmatrix} &= M^2 c_W^2 - 2MM_0 s_W c_W + M_0^2 s_W \\ &= M^2 (c_W - \frac{M_0}{M} s_W)^2 \\ &= \frac{M^2}{c_W^2} \end{aligned}$$

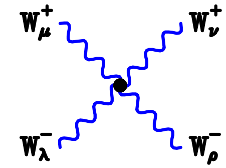
- 3 and 4 point vertices determined by the non-abelian term in the field strength.



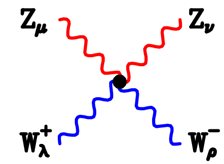
$+ig_V\ [(p-q)_\lambda g_{\mu\nu}+(q-r)_\mu g_{\nu\lambda}+(r-p)_\nu g_{\lambda\mu}]$
(all momenta incoming,
$g_A=e,\ g_Z=g_W\ \cos\theta_W$)



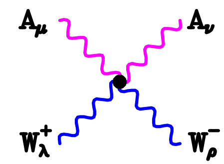
$+ig_{VH}M_W g_{\mu\nu}$
($g_{WH}=g_W,\ g_{ZH}=g_W/\cos^2\theta_W$)



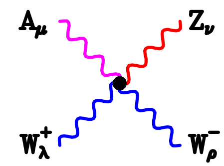
$+ig_W^2[2g_{\mu\nu}g_{\lambda\rho}-g_{\mu\lambda}g_{\nu\rho}-g_{\mu\rho}g_{\nu\lambda}]$



$-ig_W^2\cos^2\theta_W[2g_{\mu\nu}g_{\lambda\rho}-g_{\mu\lambda}g_{\nu\rho}-g_{\mu\rho}g_{\nu\lambda}]$



$-ie^2[2g_{\mu\nu}g_{\lambda\rho}-g_{\mu\lambda}g_{\nu\rho}-g_{\mu\rho}g_{\nu\lambda}]$



$-ieg_W\cos\theta_W[2g_{\mu\nu}g_{\lambda\rho}-g_{\mu\lambda}g_{\nu\rho}-g_{\mu\rho}g_{\nu\lambda}]$

$$\mathcal{L} = \bar{\psi}_R i(\partial\!\!\!/ + ig'_W Y_R B\!\!\!/)\psi_R + \bar{\psi}_L i(\partial\!\!\!/ + ig_W T \cdot W\!\!\!\!/ + ig'_W Y_L B\!\!\!/)\psi_L \ .$$

- The U(1) charges of the left- and right-handed fermions, Y_L and Y_R, are chosen to satisfy the relation $Q = T^3 + Y$,

$$\psi_L = \gamma_L \begin{pmatrix} \nu_e \\ e^- \end{pmatrix}, \gamma_L \begin{pmatrix} \nu_\mu \\ \mu^- \end{pmatrix}, \gamma_L \begin{pmatrix} \nu_\tau \\ \tau^- \end{pmatrix}$$

- The right-handed fields are all SU(2) singlets:

$$\psi_R = \gamma_R e^-, \ \gamma_R \mu^-, \ \gamma_R \tau^- \ .$$

Fermion			T_L^3	Y_L	T_R^3	Y_R	Q_f
u	c	t	$+\frac{1}{2}$	$+\frac{1}{6}$	0	$+\frac{2}{3}$	$+\frac{2}{3}$
d	s	b	$-\frac{1}{2}$	$+\frac{1}{6}$	0	$-\frac{1}{3}$	$-\frac{1}{3}$
ν_e	ν_μ	ν_τ	$+\frac{1}{2}$	$-\frac{1}{2}$	-	-	0
e^-	μ^-	τ^-	$-\frac{1}{2}$	$-\frac{1}{2}$	0	-1	-1

The interaction Lagrangian can be expressed in terms of physical fields by substituting for B and W^3

$$\begin{aligned}
\mathcal{L} &= \sum_f \bar\psi_f \left(i\partial\!\!\!/ - m_f - g_W \frac{m_f H}{2M_W} \right) \psi_f - e \sum_f Q_f \bar\psi_f \gamma_\mu \psi_f \, A^\mu \\
&- \frac{g_W}{\cos\theta_W} \sum_f \bar\psi_f \gamma^\mu (R_f \gamma_R + L_f \gamma_L) \psi_f Z_\mu \\
&- \frac{g_W}{\sqrt{2}} \sum_f \bar\psi_f (T^+ W^+_\mu \gamma^\mu \gamma_L + T^- W^-_\mu \gamma^\mu \gamma_L) \psi_f
\end{aligned}$$

The couplings of the fermions to the Z boson are, ($\gamma_{R/L} = \frac{1}{2}(1 \pm \gamma_5)$)

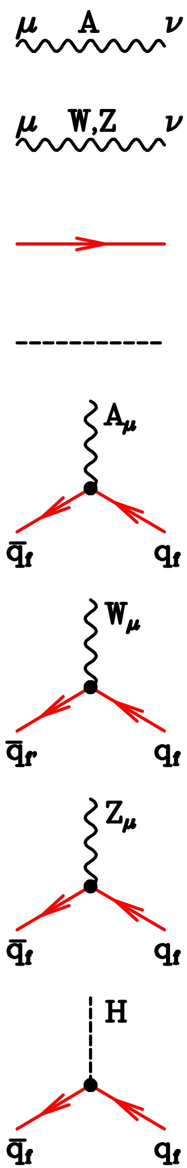
$$R_f = -Q_f \sin^2\theta_W , \quad L_f = T_f^3 - Q_f \sin^2\theta_W \ ,$$

where Q_f is the charge of the fermion in units of the positron electric charge e. The values of e and the weak SU(2) charge g_W are related by

$$e = g_W \sin\theta_W = g'_W \cos\theta_W \ .$$

Fermion rules

- The propagators are shown in the Unitary gauge.
- This gauge eliminates fields that do not correspond to physical particles.
- In this gauge the propagators have worse ultra-violet behaviour.
- The Weinberg angle fixes the coupling to the Z boson.
- Measurements of the Weinberg angle fix the ratio of the Z and W masses



$$[-g^{\mu\nu}+\frac{q^\mu q^\nu}{q^2}]\frac{i}{q^2}$$

$$[-g^{\mu\nu}+\frac{q^\mu q^\nu}{M^2}]\frac{i}{(q^2-M^2)}$$

$$\frac{i}{\not{q}-m}$$

$$\frac{i}{(q^2-M_H^2)}$$

$$-ieQ_f\gamma_\mu$$

$$\frac{-ig_W}{2\sqrt{2}}\gamma^\mu(1-\gamma_5)(T^+)_{f'f} \qquad g_W=\frac{e}{\sin\theta_W}$$

$$\frac{-ig_W}{2\cos\theta_W}\gamma^\mu(V_f-A_f\gamma_5)=\frac{-ig_W}{\cos\theta_W}\gamma^\mu(R_f\gamma_R+L_f\gamma_L$$

$(V_f=T_f^3-2Q_f\sin^2\theta_W,\ A_f=T_f^3),\ T_f^3=\pm1/2$

$(R_f=-Q_f\sin^2\theta_W,\ L_f=T_f^3-Q_f\sin^2\theta_W)$

$$\frac{-ig_W m_f}{2M_W}$$

- Returning to QCD we examine the concept of a running coupling.
- Consider dimensionless physical observable R which depends on a single large energy scale, $Q \gg m$ where m is any mass. Then we can set $m \to 0$ (assuming this limit exists), and dimensional analysis suggests that R should be independent of Q.
- This is not true in quantum field theory. Calculation of R as a perturbation series in the coupling $\alpha_{\rm S} = g^2/4\pi$ requires renormalization to remove ultraviolet divergences. This introduces a second mass scale μ — point at which subtractions which remove divergences are performed. Then R depends on the ratio Q/μ and is not constant. The renormalized coupling $\alpha_{\rm S}$ also depends on μ.
- But μ is arbitrary! Therefore, if we hold bare coupling fixed, R cannot depend on μ. Since R is dimensionless, it can only depend on Q^2/μ^2 and the renormalized coupling $\alpha_{\rm S}$. Hence

$$\mu^2 \frac{d}{d\mu^2} R\left(\frac{Q^2}{\mu^2}, \alpha_{\rm S}\right) \equiv \left[\mu^2 \frac{\partial}{\partial \mu^2} + \mu^2 \frac{\partial \alpha_{\rm S}}{\partial \mu^2} \frac{\partial}{\partial \alpha_{\rm S}}\right] R = 0 \; .$$

- Introducing

$$\tau = \ln\left(\frac{Q^2}{\mu^2}\right), \quad \beta(\alpha_S) = \mu^2 \frac{\partial \alpha_S}{\partial \mu^2},$$

we have

$$\left[-\frac{\partial}{\partial \tau} + \beta(\alpha_S)\frac{\partial}{\partial \alpha_S}\right] R = 0.$$

This renormalization group equation is solved by defining running coupling $\alpha_S(Q)$:

$$\tau = \int_{\alpha_S}^{\alpha_S(Q)} \frac{dx}{\beta(x)}, \quad \alpha_S(\mu) \equiv \alpha_S .$$

Then

$$\frac{\partial \alpha_S(Q)}{\partial \tau} = \beta(\alpha_S(Q)), \quad \frac{\partial \alpha_S(Q)}{\partial \alpha_S} = \frac{\beta(\alpha_S(Q))}{\beta(\alpha_S)} .$$

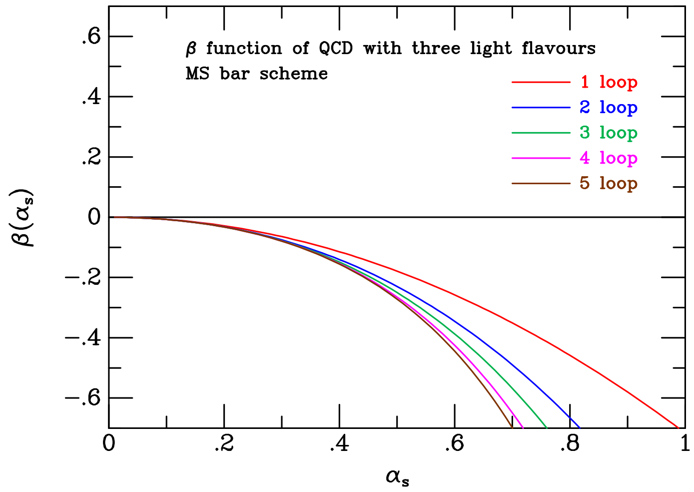
and hence $R(Q^2/\mu^2, \alpha_S) = R(1, \alpha_S(Q))$. Thus all scale dependence in R comes from running of $\alpha_S(Q)$.
- We shall see QCD is asymptotically free: $\alpha_S(Q) \to 0$ as $Q \to \infty$. Thus for large Q we can safely use perturbation theory. Then knowledge of $R(1, \alpha_S)$ to fixed order allows us to predict variation of R with Q.

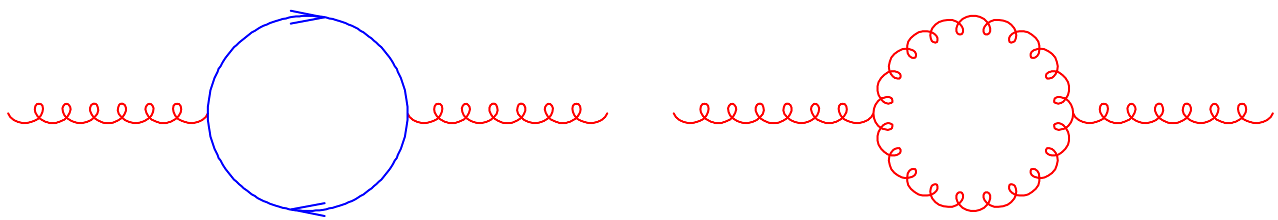
- Running of the QCD coupling $\alpha_{\rm S}$ is determined by the β function, which has the expansion

$$\beta(\alpha_{\rm S}) = -b\alpha_{\rm S}^2(1 + b'\alpha_{\rm S}) + \mathcal{O}(\alpha_{\rm S}^4)$$
$$b = \frac{(11C_A - 2N_f)}{12\pi}$$
$$b' = \frac{(17C_A^2 - 5C_A N_f - 3C_F N_f)}{2\pi(11C_A - 2N_f)}$$

where N_f is number of "active" light flavours. Terms up to $\mathcal{O}(\alpha_{\rm S}^7)$ are now known.

- if $\frac{d\alpha_{\rm S}}{d\tau} = -b\alpha_{\rm S}^2(1 + b'\alpha_{\rm S})$ and $\alpha_{\rm S} \to \bar{\alpha}_S(1 + c\bar{\alpha}_S)$, it follows that $\frac{d\bar{\alpha}_S}{d\tau} = -b\bar{\alpha}_S^2(1 + b'\bar{\alpha}_S) + O(\bar{\alpha}_S^4)$
- first two coefficients b, b' are thus invariant under scheme change.





(a) (b)

- Roughly speaking, quark loop diagram (a) contributes negative N_f term in b, while gluon loop (b) gives positive C_A contribution, which makes β function negative overall.
- QED β function is

$$\beta_{QED}(\alpha) = \frac{1}{3\pi}\alpha^2 + \dots$$

Thus b coefficients in QED and QCD have opposite signs.

- From earlier slides,

$$\frac{\partial \alpha_S(Q)}{\partial \tau} = -b\alpha_S^2(Q)\left[1 + b'\alpha_S(Q)\right] + \mathcal{O}(\alpha_S^4).$$

Neglecting b' and higher coefficients gives

$$\alpha_S(Q) = \frac{\alpha_S(\mu)}{1 + \alpha_S(\mu)b\tau}\ , \quad \tau = \ln\left(\frac{Q^2}{\mu^2}\right).$$

- As Q becomes large, $\alpha_S(Q)$ decreases to zero: this is asymptotic freedom. Notice that sign of b is crucial. In QED, $b < 0$ and coupling *increases* at large Q.
- Including next coefficient b' gives implicit equation for $\alpha_S(Q)$:

$$b\tau = \frac{1}{\alpha_S(Q)} - \frac{1}{\alpha_S(\mu)} + b' \ln\left(\frac{\alpha_S(Q)}{1+b'\alpha_S(Q)}\right) - b' \ln\left(\frac{\alpha_S(\mu)}{1+b'\alpha_S(\mu)}\right)$$

- What type of terms does the solution of the renormalization group equation take into account in the physical quantity R?
  Assume that R has perturbative expansion

$$R = \alpha_S + \mathcal{O}(\alpha_S^2)$$

Solution $R(1, \alpha_S(Q))$ can be re-expressed in terms of $\alpha_S(\mu)$:

$$\begin{aligned} R(1, \alpha_S(Q)) &= \alpha_S(\mu) \sum_{j=0}^{\infty} (-1)^j (\alpha_S(\mu) b\tau)^j \\ &= \alpha_S(\mu) \left[1 - \alpha_S(\mu) b\tau + \alpha_S^2(\mu)(b\tau)^2 + \dots\right] \end{aligned}$$

Thus there are logarithms of Q^2/μ^2 which are automatically resummed by using the running coupling. Neglected terms have fewer logarithms per power of α_S.

Lambda parameter

- Perturbative QCD tells us how $\alpha_S(Q)$ varies with Q, but its absolute value has to be obtained from experiment. Nowadays we usually choose as the fundamental parameter the value of the coupling at $Q = M_Z$, which is simply a convenient reference scale large enough to be in the perturbative domain.
- Also useful to express $\alpha_S(Q)$ directly in terms of a dimensionful parameter (constant of integration) Λ:

$$\ln\frac{Q^2}{\Lambda^2} = -\int_{\alpha_S(Q)}^{\infty}\frac{dx}{\beta(x)} = \int_{\alpha_S(Q)}^{\infty}\frac{dx}{bx^2(1+b'x+\ldots)}.$$

  Then (if perturbation theory were the whole story) $\alpha_S(Q)\to\infty$ as $Q\to\Lambda$. More generally, Λ sets the scale at which $\alpha_S(Q)$ becomes large.
- In leading order (LO) keep only first β-function b:

$$\alpha_S(Q) = \frac{1}{b\ln(Q^2/\Lambda^2)} \quad \text{(LO)}.$$

- In next-to-leading order (NLO) include also b':

$$\frac{1}{\alpha_S(Q)} + b' \ln\left(\frac{b'\alpha_S(Q)}{1 + b'\alpha_S(Q)}\right) = b \ln\left(\frac{Q^2}{\Lambda^2}\right).$$

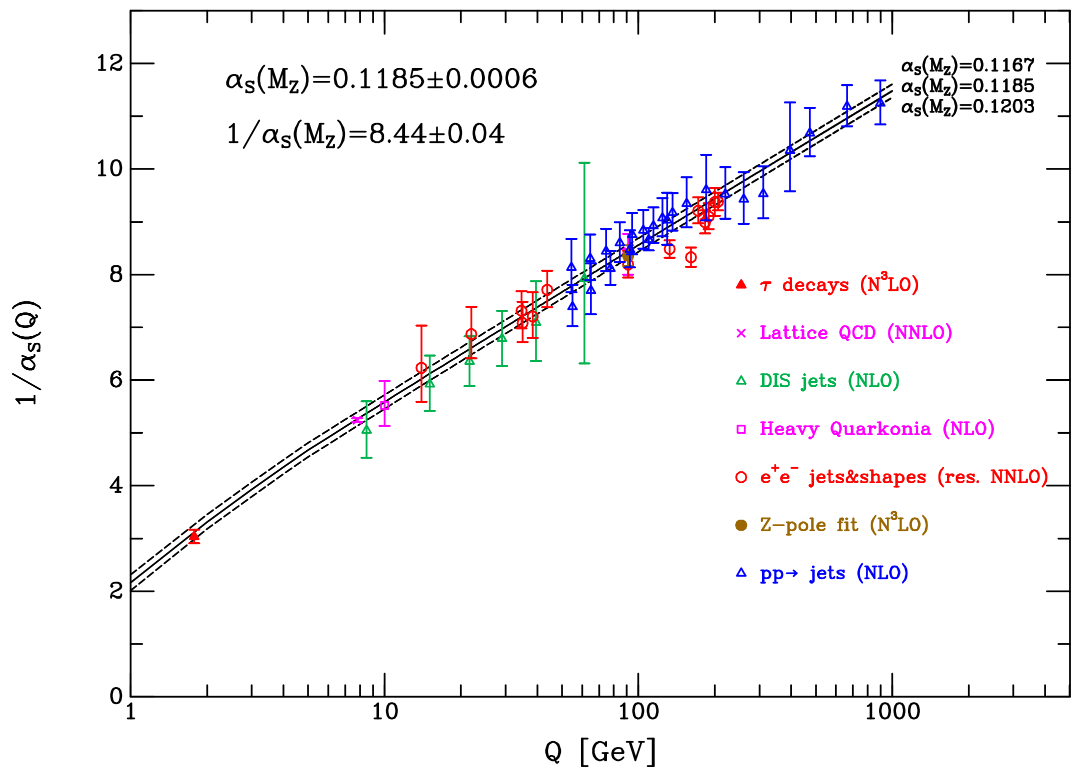
This can be solved numerically, or we can obtain an approximate solution to second order in $1/\log(Q^2/\Lambda^2)$:

$$\alpha_S(Q) = \frac{1}{b\ln(Q^2/\Lambda^2)}\left[1 - \frac{b'}{b}\frac{\ln\ln(Q^2/\Lambda^2)}{\ln(Q^2/\Lambda^2)}\right] \qquad \text{(NLO)}.$$

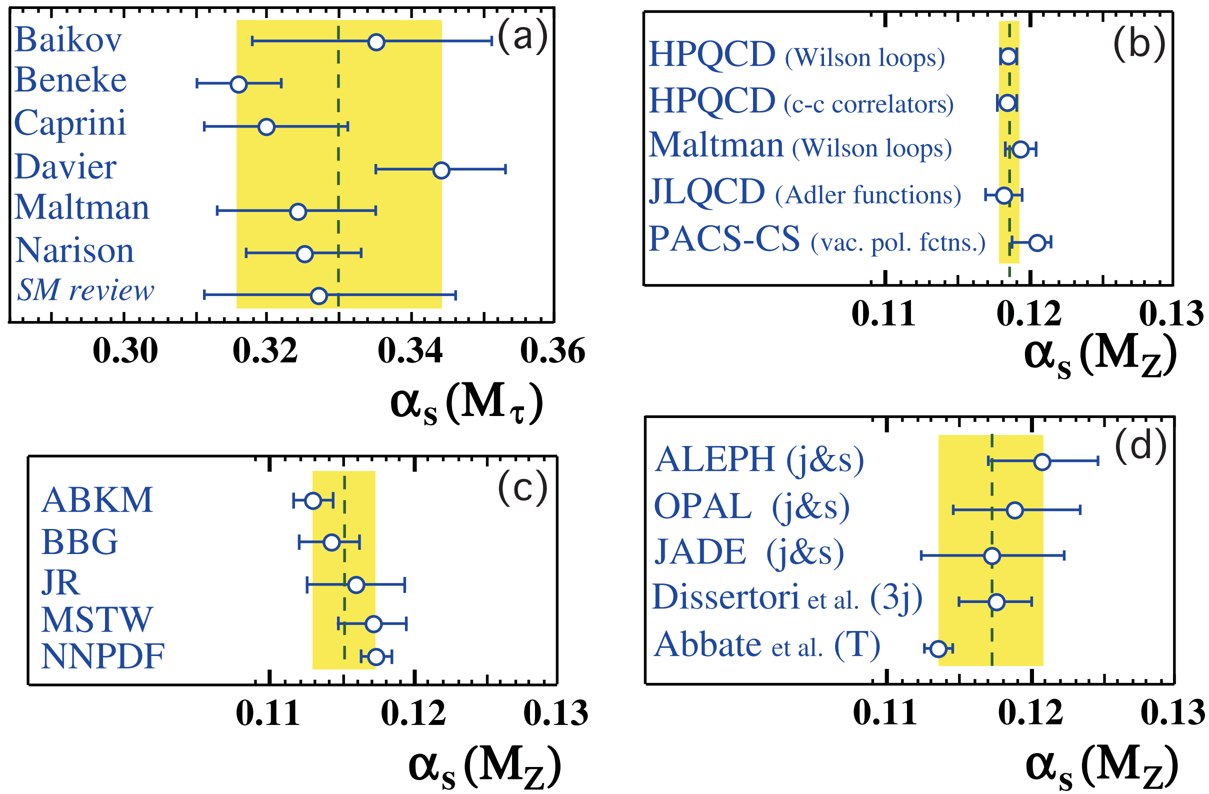
This is Particle Data Group (PDG) definition.

- Note that Λ depends on number of active flavours N_f. 'Active' means $m_q < Q$. Thus for $5 < Q < 175$ GeV we should use $N_f = 5$.

Measurement of α_s



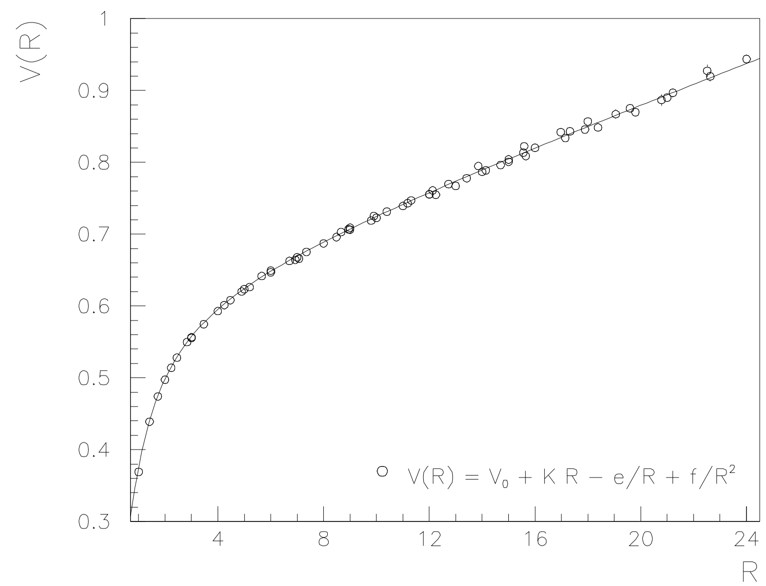
- Data from PDG September, 2013
- Evidence that $\alpha_s(Q)$ has a logarithmic fall-off with Q is persuasive.
- $1/\alpha_s$ as grows as $\ln(Q)$
- $1/\alpha_s(M_Z) = 8.44$, c.f QED: $1/\alpha(M_Z) = 128$.
- Radiative corrections, at least 15 times more important in QCD than QED.



$\alpha_S(M_Z) = 0.1187 \pm 0.0007$, arXiv:1210.0325, (2012)

Non-perturbative QCD

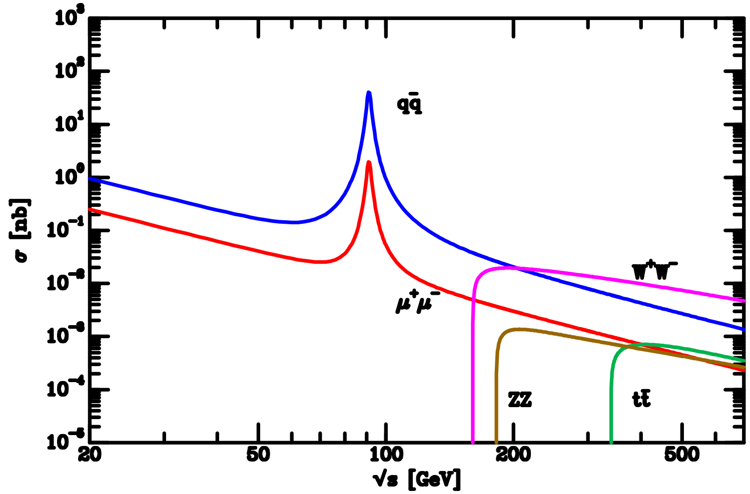
- Corresponding to asymptotic freedom at high momentum scales, we have infra-red slavery: $\alpha_S(Q)$ becomes large a low momenta, (long distances). Perturbation theory is not reliable for large α_S, so non-perturbative methods, (e.g. lattice) must be used.

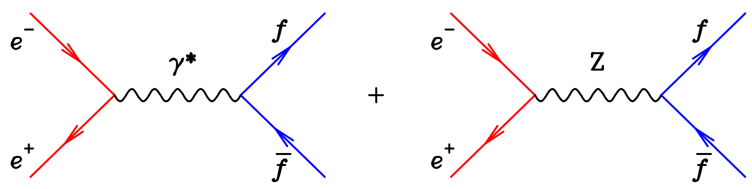


- Important low momentum scale phenomena
  - Confinement: partons (quarks and gluons) found only in colour singlet bound states, hadrons, size $\sim$ 1 fm. If we try top separate them it becomes energetically favourable to create extra partons from the vacuum.
  - Hadronization: partons produced in short distance interactions re-organize themselves to make the observed hadrons.

e^+e^- annihilation cross section

- $e^+e^- \to \mu^+\mu^-$ is a fundamental electroweak processes.
- Same type of process, $e^+e^- \to q\bar{q}$, will produce hadrons. Cross sections are roughly proportional.





- Since formation of hadrons is non-perturbative, how can PT give hadronic cross section? This can be understood by visualizing event in space-time:
- e^+ and e^- collide to form γ or Z^0 with virtual mass $Q = \sqrt{s}$. This fluctuates into $q\bar{q}$, $q\bar{q}g$,..., occupy space-time volume $\sim 1/Q$. At large Q, rate for this short-distance process given by PT.
- Subsequently, at much later time $\sim 1/\Lambda$, produced quarks and gluons form hadrons. This modifies outgoing state, but occurs too late to change original probability for event to happen.

- Well below Z^0, process $e^+e^- \to f\bar{f}$ is purely electromagnetic, with lowest-order (Born) cross section (neglecting quark masses)

$$\sigma_0 = \frac{4\pi\alpha^2}{3s}\, Q_f^2$$

Thus ($3 = N =$ number of possible $q\bar{q}$ colours)

$$R \equiv \frac{\sigma(e^+e^- \to \text{hadrons})}{\sigma(e^+e^- \to \mu^+\mu^-)} = \frac{\sum_q \sigma(e^+e^- \to q\bar{q})}{\sigma(e^+e^- \to \mu^+\mu^-)} = 3\sum_q Q_q^2\,.$$

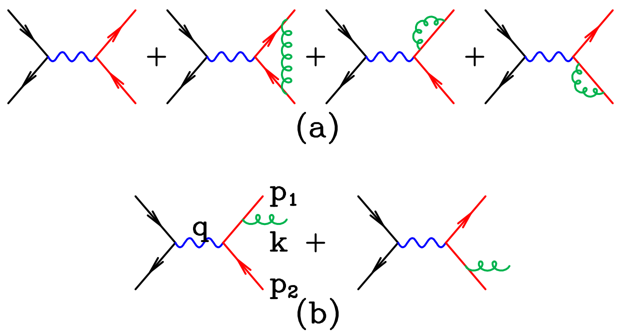
- On Z^0 pole, $\sqrt{s} = M_Z$, neglecting γ/Z interference

$$\sigma_0 = \frac{4\pi\alpha^2\kappa^2}{3\Gamma_Z^2}\,(A_e^2 + V_e^2)\,(A_f^2 + V_f^2)$$

where $\kappa = \sqrt{2}G_F M_Z^2/4\pi\alpha = 1/\sin^2(2\theta_W) \simeq 1.5$. Hence

$$R_Z = \frac{\Gamma(Z \to \text{hadrons})}{\Gamma(Z \to \mu^+\mu^-)} = \frac{\sum_q \Gamma(Z \to q\bar{q})}{\Gamma(Z \to \mu^+\mu^-)} = \frac{3\sum_q (A_q^2 + V_q^2)}{A_\mu^2 + V_\mu^2}$$

- Measured cross section is about 5% higher than σ_0, due to QCD corrections. For massless quarks, corrections to R and R_Z are equal. To $\mathcal{O}(\alpha_S)$ we have:



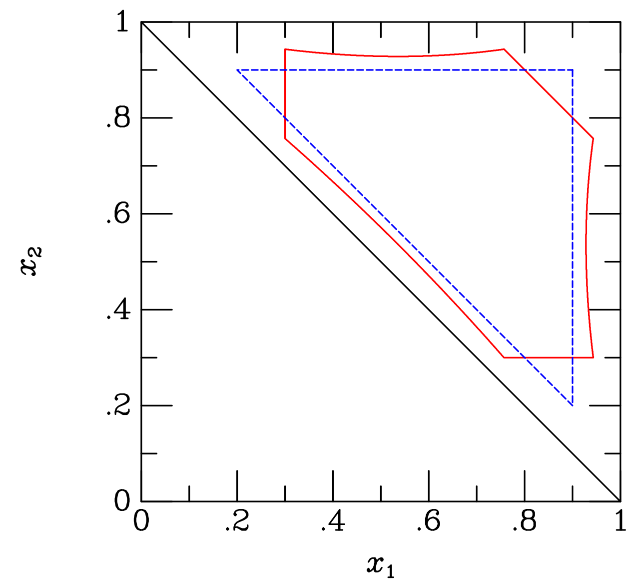
- Real emission diagrams (b):
- Write 3-body phase-space integration as $d\Phi_3 = [...] d\alpha\, d\beta\, d\gamma\, dx_1\, dx_2$
- α, β, γ are Euler angles of 3-parton plane

$$x_1 = 2p_1 \cdot q/q^2 = 2E_q/\sqrt{s}, \quad x_2 = 2p_2 \cdot q/q^2 = 2E_{\bar{q}}/\sqrt{s}.$$

- Applying Feynman rules and integrating over Euler angles:

$$\sigma^{q\bar{q}g} = 3\sigma_0 C_F \frac{\alpha_S}{2\pi} \int dx_1\, dx_2 \frac{x_1^2 + x_2^2}{(1-x_1)(1-x_2)}\ .$$

- Integration region: $0 \le x_1, x_2, x_3 \le 1$ where $x_3 = 2k \cdot q/q^2 = 2E_g/\sqrt{s} = 2 - x_1 - x_2$.



- Integral divergent at $x_{1,2} = 1$:

$$1 - x_1 = \frac{1}{2} x_2 x_3 (1 - \cos\theta_{qg}), \quad 1 - x_2 = \frac{1}{2} x_1 x_3 (1 - \cos\theta_{\bar{q}g})$$

- Divergences: collinear when $\theta_{qg} \to 0$ or $\theta_{\bar{q}g} \to 0$; soft when $E_g \to 0$, i.e. $x_3 \to 0$. Singularities are not physical – simply indicate breakdown of PT when energies and/or invariant masses approach QCD scale Λ.

- Collinear and/or soft regions do not in fact make important contribution to R. To see this, make integrals finite using dimensional regularization, $D = 4 + 2\epsilon$ with $\epsilon < 0$. Then

$$\sigma^{q\bar{q}g} = 2\sigma_0 \frac{\alpha_S}{\pi} H(\epsilon) \times \int \frac{dx_1 dx_2}{P(x_1, x_2)} \left[\frac{(1-\epsilon)(x_1^2 + x_2^2) + 2\epsilon(1-x_3)}{[(1-x_1)(1-x_2)]} - 2\epsilon \right]$$

where $$H(\epsilon) = \frac{3(1-\epsilon)(4\pi)^{2\epsilon}}{(3-2\epsilon)\Gamma(2-2\epsilon)} = 1 + \mathcal{O}(\epsilon)\ .$$

and $$P(x_1, x_2) = [(1-x_1)(1-x_2)(1-x_3)]^{\epsilon}$$

Hence

$$\sigma^{q\bar{q}g} = 2\sigma_0 \frac{\alpha_S}{\pi} H(\epsilon) \left[\frac{2}{\epsilon^2} + \frac{3}{\epsilon} + \frac{19}{2} - \pi^2 + \mathcal{O}(\epsilon) \right] .$$

- Soft and collinear singularities are regulated, appearing instead as poles at $D = 4$.
- Virtual gluon contributions (a): using dimensional regularization again

$$\sigma^{q\bar{q}} = 3\sigma_0 \left\{1 + \frac{2\alpha_S}{3\pi} H(\epsilon) \left[-\frac{2}{\epsilon^2} - \frac{3}{\epsilon} - 8 + \pi^2 + \mathcal{O}(\epsilon)\right]\right\} .$$

- Adding real and virtual contributions, poles cancel and result is finite as $\epsilon \to 0$:

$$R = 3\sum_q Q_q^2 \left\{1 + \frac{\alpha_S}{\pi} + \mathcal{O}(\alpha_S^2)\right\} .$$

- Thus R is an infrared safe quantity.

- Coupling α_S evaluated at renormalization scale μ. UV divergences in R cancel to $\mathcal{O}(\alpha_S)$, so coefficient of α_S independent of μ. At $\mathcal{O}(\alpha_S^2)$ and higher, UV divergences make coefficients renormalization scheme dependent:

$$
\begin{aligned}
R &= 3\,K_{QCD}\sum_q Q_q^2\,, \\
K_{QCD} &= 1+\frac{\alpha_S(\mu^2)}{\pi}+\sum_{n\geq 2} C_n\left(\frac{s}{\mu^2}\right)\left(\frac{\alpha_S(\mu^2)}{\pi}\right)^n
\end{aligned}
$$

- In $\overline{\text{MS}}$ scheme with scale $\mu=\sqrt{s}$,

$$C_2(1)=\frac{365}{24}-11\zeta(3)-[11-8\zeta(3)]\frac{N_f}{12}\simeq 1.986-0.115N_f$$

- Coefficient C_3 is also known.

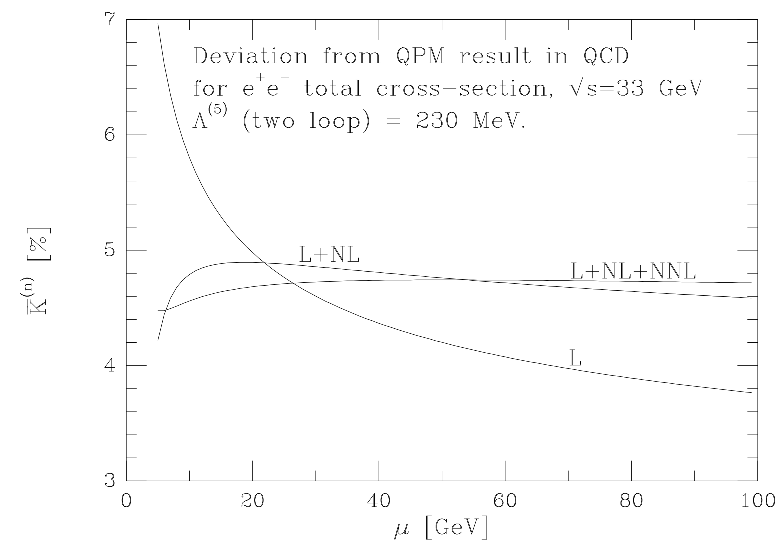
Scale dependence

- Scale dependence of $C_2,\ C_3\ \ldots$ fixed by requirement that, order-by-order, series should be independent of μ. For example

$$C_2\left(\frac{s}{\mu^2}\right) = C_2(1) - \frac{\beta_0}{4}\log\frac{s}{\mu^2}$$

where $\beta_0 = 4\pi b = 11 - 2N_f/3$.

- Scale and scheme dependence only cancels completely when series is computed to all orders. Scale change at $\mathcal{O}(\alpha_S^n)$ induces changes at $\mathcal{O}(\alpha_S^{n+1})$. The more terms are added, the more stable is prediction with respect to changes in μ.



- Residual scale dependence is an important source of uncertainty in QCD predictions. One can vary scale over some physically reasonable range, e.g. $\sqrt{s}/2 < \mu < 2\sqrt{s}$, to try to quantify this uncertainty. but there is no real substitute for a full higher-order calculation.

- Shape variables measure some aspect of shape of hadronic final state, e.g. whether it is pencil-like, planar, spherical etc.
- For $d\sigma/dX$ to be calculable in PT, shape variable X should be infrared safe, i.e. insensitive to emission of soft or collinear particles. In particular, X must be invariant under $\boldsymbol{p}_i \to \boldsymbol{p}_j + \boldsymbol{p}_k$ whenever $\boldsymbol{p}_j$ and $\boldsymbol{p}_k$ are parallel or one of them goes to zero.
- Examples are Thrust and C-parameter:

$$T = \max \frac{\sum_i |\boldsymbol{p}_i \cdot \boldsymbol{n}|}{\sum_i |\boldsymbol{p}_i|}$$

$$C = \frac{3}{2} \frac{\sum_{i,j} |\boldsymbol{p}_i|\,|\boldsymbol{p}_j|\,\sin^2\theta_{ij}}{(\sum_i |\boldsymbol{p}_i|)^2}$$

After maximization, unit vector $\boldsymbol{n}$ defines *thrust axis*.

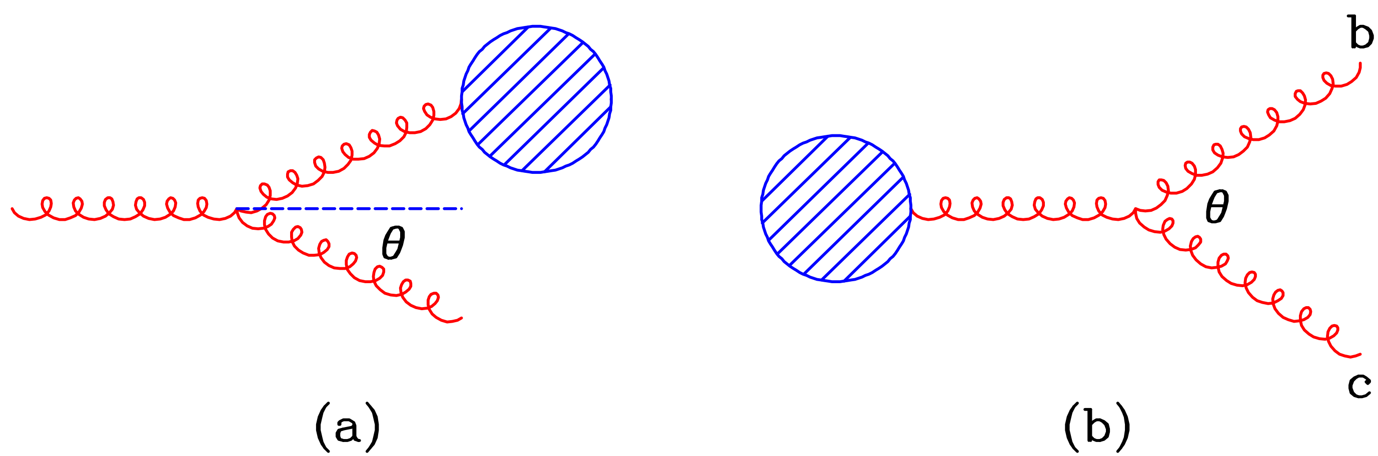
- In Born approximation final state is $q\bar{q}$ and $1 - T = C = 0$. Non-zero contribution at $\mathcal{O}(\alpha_S)$ comes from $e^+e^- \to q\bar{q}g$. Recall distribution of $x_i = 2E_i/\sqrt{s}$:

$$\frac{1}{\sigma}\frac{d^2\sigma}{dx_1 dx_2} = C_F \frac{\alpha_S}{2\pi} \frac{x_1^2 + x_2^2}{(1-x_1)(1-x_2)} \; .$$

- Distribution of shape variable X is obtained by integrating over x_1 and x_2 with constraint $\delta(X - f_X(x_1, x_2, x_3 = 2 - x_1 - x_2))$, i.e. along contour of constant X in (x_1, x_2)-plane.
- For thrust, $f_T = \max\{x_1, x_2, x_3\}$ and we find

$$\frac{1}{\sigma}\frac{d\sigma}{dT} = C_F \frac{\alpha_S}{2\pi}\left[\frac{2(3T^2 - 3T + 2)}{T(1-T)} \log\left(\frac{2T-1}{1-T}\right) - \frac{3(3T-2)(2-T)}{(1-T)}\right] .$$

- Even in high-energy, short-distance regime, long-distance aspects of QCD cannot be ignored. Soft or collinear gluon emission gives infrared divergences in PT. Light quarks ($m_q \ll \Lambda$) also lead to divergences in the limit $m_q \to 0$ (mass singularities).



- ☐ Spacelike branching: gluon splitting on incoming line (a)

$$p_b^2 = -2E_aE_c(1 - \cos\theta) \leq 0 \ .$$

Propagator factor $1/p_b^2$ diverges as $E_c \to 0$ (soft singularity) or $\theta \to 0$ (collinear or mass singularity).

If a and b are quarks, inverse propagator factor is

$$p_b^2 - m_q^2 = -2E_a E_c(1 - v_a \cos\theta) \leq 0 \ ,$$

Hence $E_c \to 0$ soft divergence remains; collinear enhancement becomes a divergence as $v_a \to 1$, i.e. when quark mass is negligible. If emitted parton c is a quark, vertex factor cancels $E_c \to 0$ divergence.

- Timelike branching: gluon splitting on outgoing line (b)

  $$p_a^2 = 2E_b E_c(1 - \cos\theta) \geq 0 \ .$$

  Diverges when either emitted gluon is soft (E_b or $E_c \to 0$) or when opening angle $\theta \to 0$. If b and/or c are quarks, collinear/mass singularity in $m_q \to 0$ limit. Again, soft quark divergences cancelled by vertex factor.
- Similar infrared divergences in loop diagrams, associated with soft and/or collinear configurations of virtual partons within region of integration of loop momenta.
- Infrared divergences indicate dependence on long-distance aspects of QCD not correctly described by PT. Divergent (or enhanced) propagators imply propagation of partons over long distances. When distance becomes comparable with hadron size ~ 1 fm, quasi-free partons of perturbative calculation are confined/hadronized non-perturbatively, and apparent divergences disappear.

- Can still use PT to perform calculations, provided we limit ourselves to two classes of observables:
  - Infrared safe quantities, i.e. those insensitive to soft or collinear branching. Infrared divergences in PT calculation either cancel between real and virtual contributions or are removed by kinematic factors. Such quantities are determined primarily by hard, short-distance physics; long-distance effects give power corrections, suppressed by inverse powers of a large momentum scale.
  - Factorizable quantities, i.e. those in which infrared sensitivity can be absorbed into an overall non-perturbative factor, to be determined experimentally.
- In either case, infrared divergences must be *regularized* during PT calculation, even though they cancel or factorize in the end.
  - Gluon mass regularization: introduce finite gluon mass, set to zero at end of calculation. However, as we saw, gluon mass breaks gauge invariance.
  - Dimensional regularization: analogous to that used for ultraviolet divergences, except we must *increase* dimension of space-time, $\epsilon = 2 - \frac{D}{2} < 0$. Divergences are replaced by powers of $1/\epsilon$.

Recap

- QCD is an SU(3) gauge theory of quarks (3 colours) and gluons (8 colours,self interacting)
- The Electroweak SM is based on $SU(2) \otimes U(1)$ gauge theory.
- Renormalization of dimensionless observables depending on a single large scale implies that the scale dependence enters through the running coupling.
- Asymptotic freedom implies that IR-safe quantities can be calculated in perturbation theory.
- $\alpha(M_Z) \simeq 0.118$ in five flavour $\overline{MS}$-renormalization scheme.
- Perturbative QCD has infrared singularities due to collinear or soft parton emission. We can calculate infra-red safe or factorizable quantities in perturbation theory.

- QCD and Collider Physics
  (Cambridge Monographs on Particle Physics, Nuclear Physics and Cosmology)
  by R. K. Ellis, W.J. Stirling and B.R. Webber
  for errata, see
  http://www.hep.phy.cam.ac.uk/theory/webber/QCDupdates.html