44th SLAC summer institute
Lecture II:
The QCD parton model: Partons and Vector Bosons

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Parton branching - kinematics
Massless Dirac equation
Branching probabilities
DGLAP equation
Evolution of Parton distributions
Sudakov form factor
Hadron-hadron processes
Factorization of the cross section
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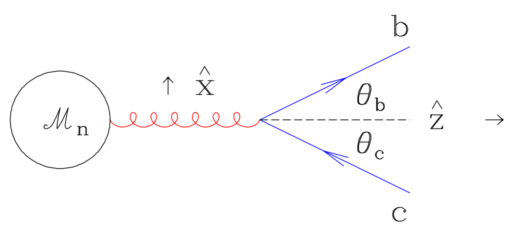

$$\begin{aligned} p_a &= (E_a + \frac{p_a^2}{4E_a}, 0, 0, E_a - \frac{p_a^2}{4E_a}) \\ p_b &= (E_b, +E_b \sin\theta_b, 0, +E_b \cos\theta_b) \\ p_c &= (E_c, -E_c \sin\theta_c, 0, +E_c \cos\theta_c) \end{aligned}$$

- the kinematics and notation for the branching of parton a into $b + c$. We assume that

$$p_b^2,\, p_c^2 \ll p_a^2 \equiv t$$

- a is an outgoing parton, which is called timelike branching since $t > 0$.
- The opening angle is $\theta = \theta_b + \theta_c$. Defining the energy fraction as

$$z = E_b/E_a = 1 - E_c/E_a \ ,$$

we have for small angles, $t = 2E_bE_c(1 - \cos\theta) = z(1-z)E_a^2\theta^2$

- using transverse momentum conservation, $(E_b\theta_b = E_c\theta_c)$,

$$\theta = \frac{1}{E_a}\sqrt{\frac{t}{z(1-z)}} = \frac{\theta_b}{1-z} = \frac{\theta_c}{z} \ .$$

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Massless Dirac equation

- The fermions involved in high energy processes can often be taken to be massless.
- We choose an explicit representation for the gamma matrices. The Bjorken and Drell representation is,

$$\gamma^0 = \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & -\mathbf{1} \end{pmatrix}, \gamma^i = \begin{pmatrix} \mathbf{0} & \sigma^i \\ -\sigma^i & \mathbf{0} \end{pmatrix}, \gamma^5 = \begin{pmatrix} \mathbf{0} & \mathbf{1} \\ \mathbf{1} & \mathbf{0} \end{pmatrix},$$

  The Weyl representation is more suitable at high energy

$$\gamma^0 = \begin{pmatrix} \mathbf{0} & \mathbf{1} \\ \mathbf{1} & \mathbf{0} \end{pmatrix}, \gamma^i = \begin{pmatrix} \mathbf{0} & -\sigma^i \\ \sigma^i & \mathbf{0} \end{pmatrix}, \gamma^5 = \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & -\mathbf{1} \end{pmatrix},$$

  In the Weyl representation upper and lower components have different helicities.
- Both representations satisfy the same commutation relations (West coast metric!)

$$\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}, \quad \gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$$

- in the Weyl representation $\gamma^0\gamma^i = \begin{pmatrix} \sigma^i & \mathbf{0} \\ \mathbf{0} & -\sigma^i \end{pmatrix}$. σ are the Pauli matrices.

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- In the Weyl representation

$$p_\mu \gamma^\mu = \sqrt{p^+ p_-} \begin{pmatrix} 0 & 0 & \sqrt{\frac{p^+}{p^-}} & e^{-i\varphi} \\ 0 & 0 & e^{+i\varphi} & \sqrt{\frac{p^-}{p^+}} \\ \sqrt{\frac{p^-}{p^+}} & -e^{-i\varphi} & 0 & 0 \\ -e^{+i\varphi} & \sqrt{\frac{p^+}{p^-}} & 0 & 0 \end{pmatrix}$$

$$e^{\pm i\varphi_p} \equiv \frac{p^1 \pm ip^2}{\sqrt{(p^1)^2 + (p^2)^2}} = \frac{p^1 \pm ip^2}{\sqrt{p^+ p^-}}, \qquad p^\pm = p^0 \pm p^3.$$

- The massless spinors solns of Dirac eqn, $\not{p} u_+(p) = \not{p} u_-(p) = 0$ are

$$u_+(p) = \begin{bmatrix} \sqrt{p^+} \\ \sqrt{p^-} e^{i\varphi_p} \\ 0 \\ 0 \end{bmatrix}, \quad u_-(p) = \begin{bmatrix} 0 \\ 0 \\ \sqrt{p^-} e^{-i\varphi_p} \\ -\sqrt{p^+} \end{bmatrix},$$

- In this representation the Dirac conjugate spinors are

$$\overline{u}_+(p) \equiv u_+^\dagger(p)\gamma^0 = \left[0, 0, \sqrt{p^+}, \sqrt{p^-} e^{-i\varphi_p}\right], \quad \overline{u}_-(p) = \left[\sqrt{p^-} e^{i\varphi_p}, -\sqrt{p^+}, 0, 0\right]$$

Normalization $u_\pm^\dagger u_\pm = 2p^0$

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- Consider the case where

$$
\begin{aligned}
p_a &= (E_a + \frac{p_a^2}{4E_a}, 0, 0, E_a - \frac{p_a^2}{4E_a}) \\
p_b &\sim (E_b, +E_b\theta_b, 0, +E_b) \\
p_c &\sim (E_c, -E_c\theta_c, 0, +E_c)
\end{aligned}
$$

- Thus for example, ($\theta_b = (1-z)\theta, \theta_c = z\theta$)

$$
u_+^\dagger(p) = \sqrt{2E_b}\left[1, \frac{\theta_b}{2}, 0, 0\right], \quad u_+(p_c) \equiv v_-(p_c) = \sqrt{2E_c}\begin{bmatrix} 1 \\ -\frac{\theta_c}{2} \\ 0 \\ 0 \end{bmatrix}
$$

Hence for polarization vectors $\varepsilon_{in} = (0,1,0,0), \varepsilon_{out} = (0,0,1,0)$

$$
g\bar{u}_+^b\, \gamma^0\gamma^1\, v_-^c = g\sqrt{4E_bE_c}\left(1, \frac{\theta_b}{2}\right)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ -\frac{\theta_c}{2} \end{pmatrix} = -g\sqrt{E_bE_c}(\theta_b - \theta_c)
$$

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$$-g\bar{u}_+^b \gamma_\mu \varepsilon_a^{\text{in}\,\mu} v_-^c = g\sqrt{E_b E_c}(\theta_b - \theta_c) = g\sqrt{z(1-z)}(1-2z)E_a\theta \approx g(1-2z)\sqrt{t},$$

$$-g\bar{u}_+^b \gamma_\mu \varepsilon_a^{\text{out}\,\mu} v_-^c = ig\sqrt{E_b E_c}(\theta_b + \theta_c) = ig\sqrt{z(1-z)}E_a\theta \approx ig\sqrt{t}.$$

- The squared branching probabilities both vanish in the forward direction
- the matrix element relation for the branching is

$$|\mathcal{M}_{n+1}|^2 \sim \frac{g^2}{t} T_R F(z;\varepsilon_a,\lambda_b,\lambda_c)|\mathcal{M}_n|^2$$

where the colour factor is now Tr$(t^A t^A)/8 = T_R = 1/2$. The non-vanishing functions $F(z;\varepsilon_a,\lambda_b,\lambda_c)$ for quark and antiquark helicities λ_b and λ_c are

ε_a	λ_b	λ_c	$F(z;\varepsilon_a,\lambda_b,\lambda_c)$
in	$\pm$	$\mp$	$(1-2z)^2$
out	$\pm$	$\mp$	1

Summing over the polarizations we get

$$2\Big[(1-2z)^2 + 1\Big] = 4(z^2 + (1-z)^2).$$

This is the branching probability for gluon into a quark, $P_{qg} = T_R(z^2 + (1-z)^2)$

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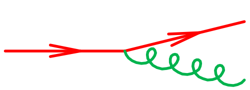
$$d\sigma_{n+1} = d\sigma_n \, \frac{dt}{t} \, dz \, \frac{d\phi}{2\pi} \, \frac{\alpha_S}{2\pi} \, CF, \quad \int \frac{d\phi}{2\pi} CF = \hat{P}_{ba}(z)$$

where $\hat{P}_{ba}(z)$ is the appropriate splitting function, (C=colour factor,F=polarization dependent splitting function)

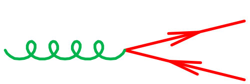
$$d\sigma_{n+1} = d\sigma_n \, \frac{dt}{t} \, dz \, \frac{\alpha_S}{2\pi} \, \hat{P}_{ba}(z) \; .$$

- Including all the color factors we find the results for the unregulated branching probabilities.

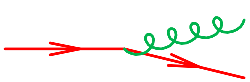
$$\hat{P}_{qq}(z) = C_F \left[\frac{1+z^2}{(1-z)}\right] ,$$



$$\hat{P}_{qg}(z) = T_R \left[z^2 + (1-z)^2\right], \quad T_R = \frac{1}{2} ,$$



$$\hat{P}_{gq}(z) = C_F \left[\frac{1+(1-z)^2}{z}\right] ,$$



$$\hat{P}_{gg}(z) = C_A \left[\frac{z}{(1-z)} + \frac{1-z}{z} + z\,(1-z)\right]$$

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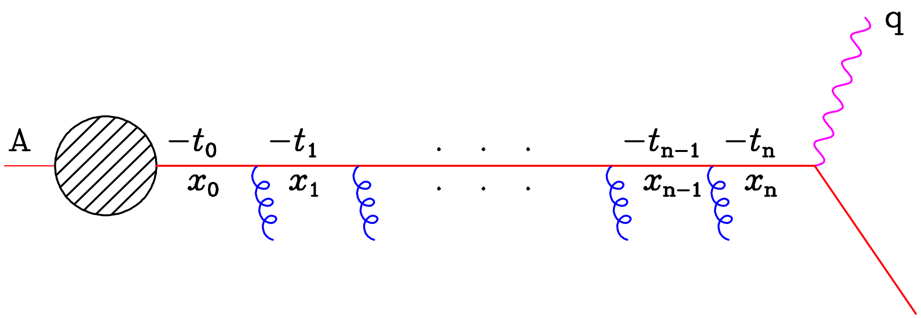
- Consider enhancement of higher-order contributions due to multiple small-angle parton emission, for example in deep inelastic scattering (DIS)



- Incoming quark from target hadron, initially with low virtual mass-squared $-t_0$ and carrying a fraction x_0 of hadron's momentum, moves to more virtual masses and lower momentum fractions by successive small-angle emissions, and is finally struck by photon of virtual mass-squared $q^2 = -Q^2$.
- Cross section will depend on Q^2 and on momentum fraction distribution of partons seen by virtual photon at this scale, $D(x, Q^2)$.

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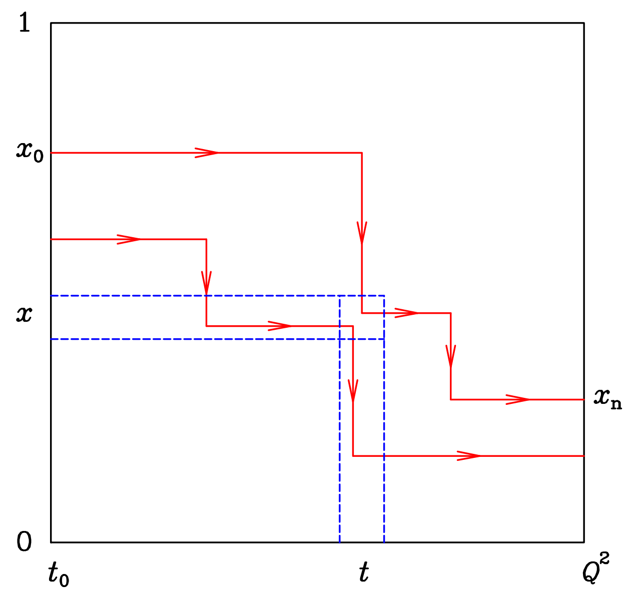
- To derive evolution equation for Q^2-dependence of $D(x, Q^2)$, first introduce pictorial representation of evolution, also useful later for Monte Carlo simulation.



- Represent sequence of branchings by path in (t, x)-space. Each branching is a step downwards in x, at a value of t equal to (minus) the virtual mass-squared after the branching.
- At $t = t_0$, paths have distribution of starting points $D(x_0, t_0)$ characteristic of target hadron at that scale. Then distribution $D(x, t)$ of partons at scale t is just the x-distribution of paths at that scale.

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Change in parton distribution

- Consider change in the parton distribution $D(x,t)$ when t is increased to $t+\delta t$. This is number of paths arriving in element $(\delta t, \delta x)$ minus number leaving that element, divided by δx.
- Number arriving is branching probability times parton density integrated over all higher momenta $x' = x/z$,

$$\begin{aligned}\delta D_{\text{in}}(x,t) &= \frac{\delta t}{t}\int_x^1 dx'\, dz \frac{\alpha_{\text{S}}}{2\pi}\hat{P}(z)D(x',t)\,\delta(x-zx') \\ &= \frac{\delta t}{t}\int_0^1 \frac{dz}{z}\frac{\alpha_{\text{S}}}{2\pi}\hat{P}(z)D(x/z,t)\end{aligned}$$

- For the number leaving element, must integrate over lower momenta $x' = zx$:

$$\begin{aligned}\delta D_{\text{out}}(x,t) &= \frac{\delta t}{t}D(x,t)\int_0^x dx'\, dz \frac{\alpha_{\text{S}}}{2\pi}\hat{P}(z)\,\delta(x'-zx) \\ &= \frac{\delta t}{t}D(x,t)\int_0^1 dz \frac{\alpha_{\text{S}}}{2\pi}\hat{P}(z)\end{aligned}$$

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Change in parton distribution

- Change in population of element is

$$\begin{aligned} \delta D(x,t) &= \delta D_{\text{in}} - \delta D_{\text{out}} \\ &= \frac{\delta t}{t} \int_0^1 dz \, \frac{\alpha_{\text{S}}}{2\pi} \hat{P}(z) \left[\frac{1}{z} D(x/z,t) - D(x,t) \right] . \end{aligned}$$

- Introduce plus-prescription with definition

$$\int_0^1 dx \; f(x) \, g(x)_+ = \int_0^1 dx \; [f(x) - f(1)] \, g(x) \; .$$

- Using this we can define regularized splitting function

$$P(z) = \hat{P}(z)_+$$

- Plus-prescription, like the Dirac-delta function, is only defined under integral sign.
- Plus-prescription includes some of the effects of virtual diagrams.

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We obtain the Dokshitzer-Gribov-Lipatov-Altarelli-Parisi (DGLAP) evolution equation:

$$t\frac{\partial}{\partial t}D(x,t) = \int_x^1 \frac{dz}{z}\frac{\alpha_\mathrm{S}}{2\pi}P(z)D(x/z,t)\ .$$

- Here $D(x,t)$ represents parton momentum fraction distribution inside incoming hadron probed at scale t.
- In timelike branching, it represents instead hadron momentum fraction distribution produced by an outgoing parton. Boundary conditions and direction of evolution are different, but evolution equation remains the same.
- For several different types of partons, must take into account different processes by which parton of type i can enter or leave the element $(\delta t, \delta x)$. This leads to coupled DGLAP evolution equations of form

$$t\frac{\partial}{\partial t}D_i(x,t) = \sum_j \int_x^1 \frac{dz}{z}\frac{\alpha_\mathrm{S}}{2\pi}P_{ij}(z)D_j(x/z,t)\ .$$

- Quark ($i=q$) can enter element via either $q \to qg$ or $g \to q\bar{q}$, but can only leave via $q \to qg$. Thus plus-prescription applies only to $q \to qg$ part, giving

$$P_{qg}(z) = \hat{P}_{qg}(z) = T_R\left[z^2+(1-z)^2\right], \quad P_{qq}(z) = \hat{P}_{qq}(z)_+ = C_F\left(\frac{1+z^2}{1-z}\right)_+$$

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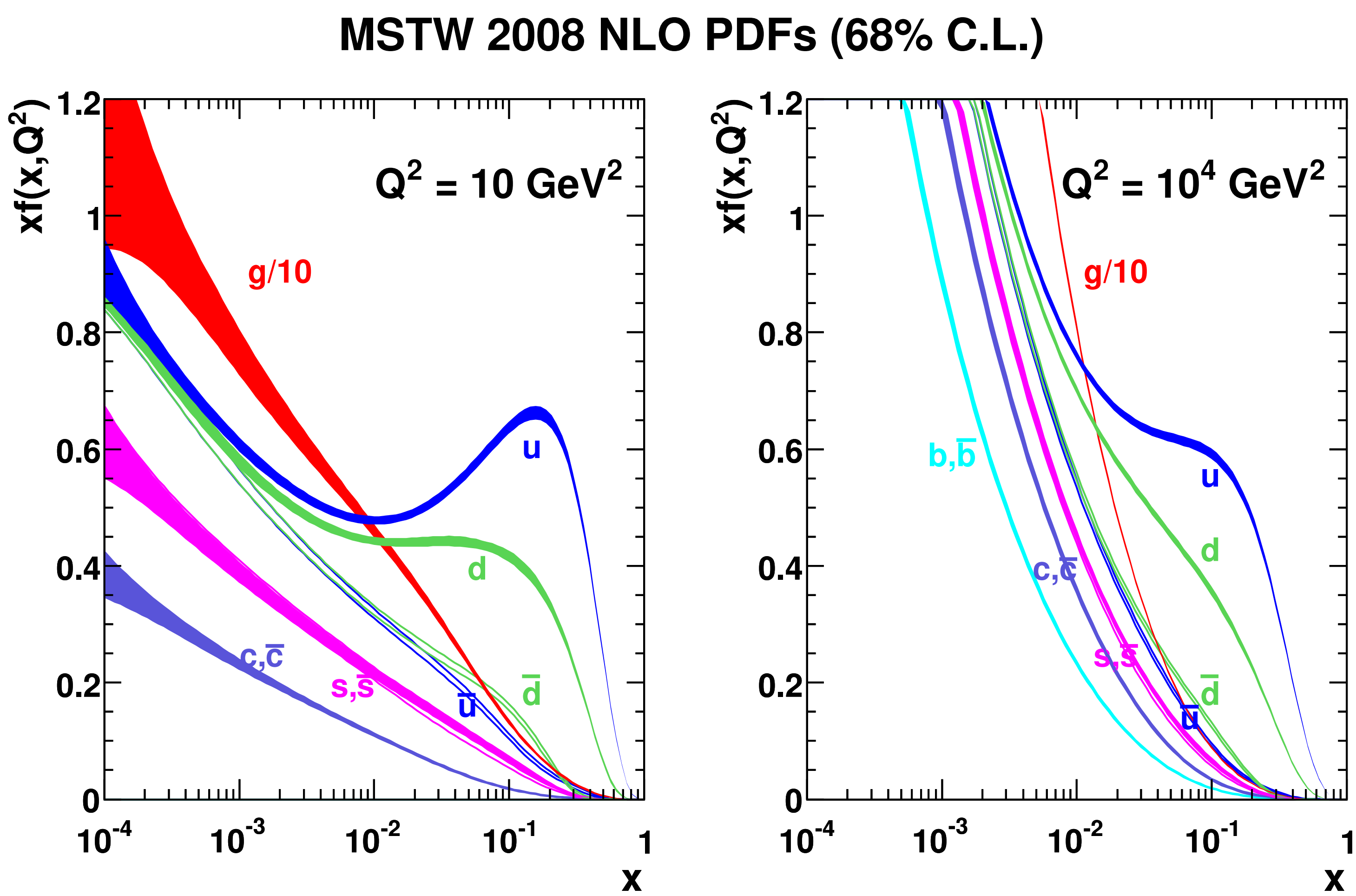

- Scale dependent parton distributions determined from experiment.
- Their behaviour with Q^2 (large x:shrinkage, small x:growth), determined by DGLAP eqn.
- N^{2}LO terms (and partial N^{3}LO terms) in DGLAP equation now known.

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Sudakov form factor

- DGLAP equations convenient for evolution of parton distributions. To study structure of final states, slightly different form is useful. Consider again simplified treatment with only one type of branching. Introduce Sudakov form factor:

$$\Delta(t) \equiv \exp\left[-\int_{t_0}^{t} \frac{dt'}{t'} \int dz\, \frac{\alpha_S}{2\pi} \hat{P}(z)\right] ,$$

- the DGLAP equation derived previously can be written as,

$$\frac{tD(x,t)}{dt} = \int_0^1 dz\, \frac{\alpha_S}{2\pi} \hat{P}(z) \left[\frac{1}{z} D(x/z,t) - D(x,t)\right] .$$

- This can be written in terms of the Sudakov form factor as

$$\begin{aligned} t\frac{\partial}{\partial t} D(x,t) &= \int \frac{dz}{z} \frac{\alpha_S}{2\pi} \hat{P}(z) D(x/z,t) + \frac{D(x,t)}{\Delta(t)} t \frac{\partial}{\partial t} \Delta(t) , \\ t\frac{\partial}{\partial t}\left(\frac{D}{\Delta}\right) &= \frac{1}{\Delta} \int \frac{dz}{z} \frac{\alpha_S}{2\pi} \hat{P}(z) D(x/z,t) . \end{aligned}$$

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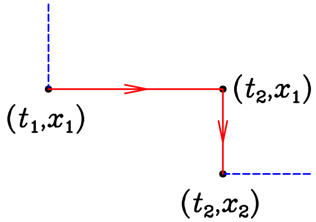
$$D(x,t) = \Delta(t) D(x,t_0) + \int_{t_0}^{t} \frac{dt'}{t'} \frac{\Delta(t)}{\Delta(t')} \frac{\alpha_S}{2\pi} \int \frac{dz}{z} P(z) D(x/z,t') \,.$$

- the first term on the right-hand side is the contribution from paths that do not branch between scales t_0 and t.
- Thus the Sudakov form factor $\Delta(t)$ is simply the probability of evolving from t_0 to t without branching.
- The second term is the contribution from all paths which have their last branching at scale t'.
- The basic problem that the Monte Carlo branching algorithm has to solve is as follows: given the virtual mass scale and momentum fraction (t_1, x_1) after some step of the evolution, or as initial conditions, generate the values (t_2, x_2) after the next step.



- t_2 and x_2 can be generated with the right distributions with two random numbers by solving the following relations,

$$\frac{\Delta(t_2)}{\Delta(t_1)} = \mathcal{R}$$

$$\int_{\epsilon}^{x_2/x_1} dz \frac{\alpha_S}{2\pi} P(z) = \mathcal{R}' \int_{\epsilon}^{1-\epsilon} dz \frac{\alpha_S}{2\pi} P(z)$$

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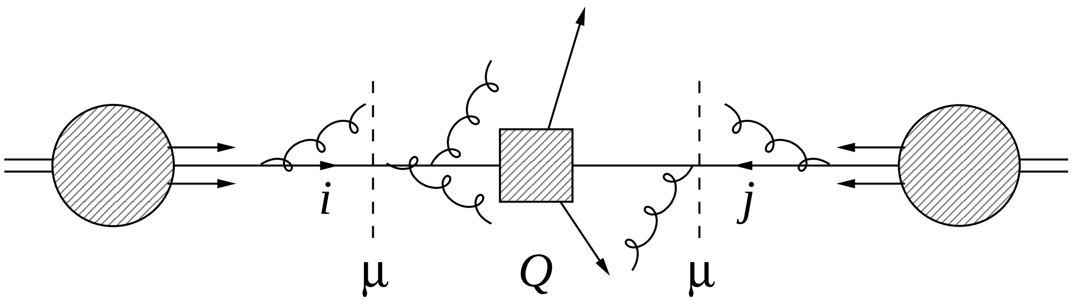
- In hard hadron-hadron scattering, constituent partons from each incoming hadron interact at short distance (large momentum transfer Q^2).



- For hadron momenta P_1, P_2 ($S = 2P_1 \cdot P_2$), form of cross section is

$$\sigma(S) = \sum_{i,j} \int dx_1 dx_2 D_i(x_1, \mu^2) D_j(x_2, \mu^2) \hat{\sigma}_{ij}(\hat{s} = x_1 x_2 S, \alpha_S(\mu^2), Q^2/\mu^2)$$

where μ^2 is factorization scale and $\hat{\sigma}_{ij}$ is subprocess cross section for parton types i, j.

- Notice that factorization scale is in principle arbitrary: affects only what we call part of subprocess or part of initial-state evolution (parton shower).
- Unlike e^+e^- or ep, we may have interaction between spectator partons, leading to soft underlying event and/or multiple hard scattering.

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- Why does the factorization property hold and when it should fail?
- For a heuristic argument, consider the vector boson production, the simplest hard process involving two hadrons

$$H_1(P_1) + H_2(P_2) \to V + X.$$

- Do the partons in hadron H_1, through the influence of their colour fields, change the distribution of partons in hadron H_2 before the vector boson is produced? Soft gluons which are emitted long before the collision are potentially troublesome.
- A simple model from classical electrodynamics. The vector potential due to an electromagnetic current density J is given by

$$A^\mu(t,\vec{x}) = \int dt'\,d\vec{x}'\,\frac{J^\mu(t',\vec{x}')}{|\vec{x}-\vec{x}'|}\,\delta(t'+|\vec{x}-\vec{x}'|-t)\;,$$

where the delta function provides the retarded behaviour required by causality.

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- Consider a particle with charge e travelling in the positive z direction with constant velocity β. The non-zero components of the current density are

$$\begin{aligned} J^t(t', \vec{x}') &= e\delta(\vec{x}' - \vec{r}(t'))\,, \\ J^z(t', \vec{x}') &= e\beta\delta(\vec{x}' - \vec{r}(t')), \quad \vec{r}(t') = \beta t' \hat{z}, \end{aligned}$$

$\hat{z}$ is a unit vector in the z direction. At an observation point (the supposed position of hadron H_2) described by coordinates x, y and z, the vector potential (either performing the integrations using the current density given above, or by Lorentz transformation of the scalar potential in the rest frame of the particle) is

$$\begin{aligned} A^t(t, \vec{x}) &= \frac{e\gamma}{\sqrt{[x^2 + y^2 + \gamma^2(\beta t - z)^2]}} \\ A^x(t, \vec{x}) &= 0 \\ A^y(t, \vec{x}) &= 0 \\ A^z(t, \vec{x}) &= \frac{e\gamma\beta}{\sqrt{[x^2 + y^2 + \gamma^2(\beta t - z)^2]}}, \end{aligned}$$

where $\gamma^2 = 1/(1 - \beta^2)$. Target hadron H_2 is at rest near the origin, so that $\gamma \approx s/m^2$.

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- Note that for large γ and fixed non-zero $(\beta t - z)$ some components of the potential tend to a constant independent of γ, suggesting that there will be non-zero fields which are not in coincidence with the arrival of the particle, even at high energy.
- However at large γ the potential is a pure gauge piece, $A^\mu = \partial^\mu \chi$ where χ is a scalar function
- Covariant formulation using the vector potential A has large fields which have no effect.
- For example, the electric field along the z direction is

$$E^z(t, \vec{x}) = F^{tz} \equiv \frac{\partial A^z}{\partial t} + \frac{\partial A^t}{\partial z} = \frac{e\gamma(\beta t - z)}{[x^2 + y^2 + \gamma^2(\beta t - z)^2]^{\frac{3}{2}}}.$$

The leading terms in γ cancel and the field strengths are of order $1/\gamma^2$ and hence of order m^4/s^2. The model suggests the force experienced by a charge in the hadron H_2, at any fixed time before the arrival of the quark, decreases as m^4/s^2.

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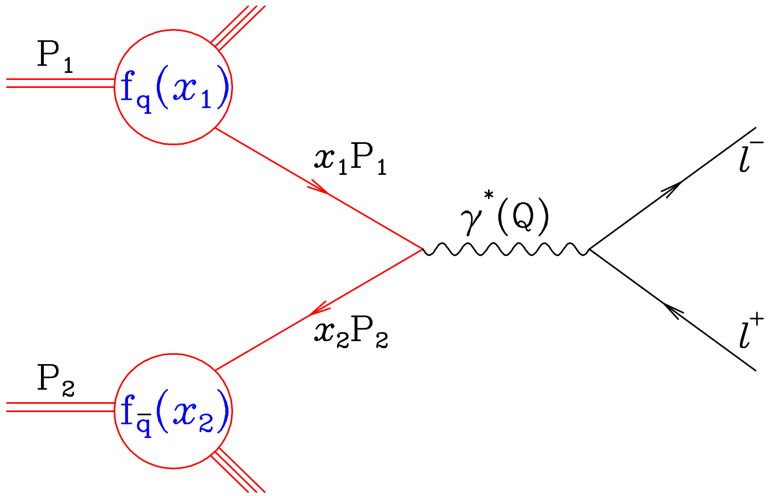

- Mechanism for Lepton pair production, W-production, Z-production, Vector-boson pairs, ...
- Collectively known as the Drell-Yan process.
- Colour average $1/N$.

$$\frac{d\hat{\sigma}}{dQ^2} = \frac{\sigma_0}{N} Q_q^2 \, \delta(\hat{s} - Q^2), \qquad \sigma_0 = \frac{4\pi\alpha^2}{3Q^2}, \quad \text{cf } e^+e^- \text{ annihilation.}$$

- In the CM frame of the two hadrons, the momenta of the incoming partons are

$$p_1 = \frac{\sqrt{s}}{2}(x_1, 0, 0, x_1), \quad p_2 = \frac{\sqrt{s}}{2}(x_2, 0, 0, -x_2) \; .$$

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The square of the $q\bar{q}$ collision energy $\hat{s}$ is related to the overall hadron-hadron collision energy by $\hat{s} = (p_1 + p_2)^2 = x_1 x_2 s$. The parton-model cross section for this process is:

$$\frac{d\sigma}{dM^2} = \int_0^1 dx_1 dx_2 \sum_q \{f_q(x_1) f_{\bar{q}}(x_2) + (q \leftrightarrow \bar{q})\} \frac{d\hat{\sigma}}{dM^2}(q\bar{q} \to l^+ l^-)$$

$$= \frac{\sigma_0}{Ns} \int_0^1 \frac{dx_1}{x_1} \frac{dx_2}{x_2} \delta(1-z) \left[\sum_q Q_q^2 \{f_q(x_1) f_{\bar{q}}(x_2) + (q \leftrightarrow \bar{q})\}\right] .$$

- For later convenience we have introduced the variable $z = \frac{Q^2}{\hat{s}} = \frac{Q^2}{x_1 x_2 s}$.
- The sum here is over quarks only and the $\bar{q}q$ contributions are indicated explicitly.

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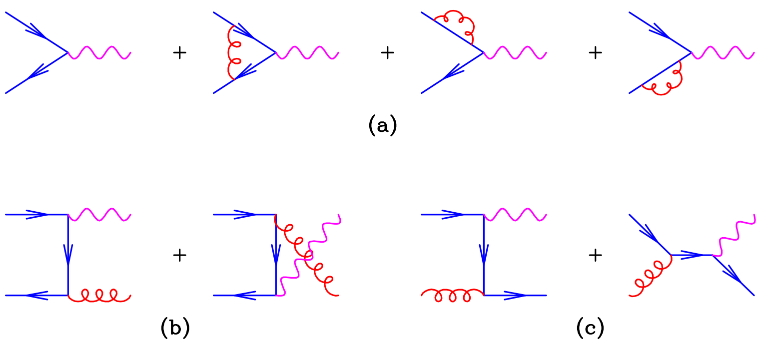
Lepton pair production at next-to-leading order



- The contribution of the real diagrams (in four dimensions) is

$$|M|^2 \sim g^2 C_F \left[\frac{u}{t} + \frac{t}{u} + \frac{2Q^2 s}{ut}\right] = g^2 C_F \left[\left(\frac{1+z^2}{1-z}\right)\left(\frac{-s}{t} + \frac{-s}{u}\right) - 2\right]$$

  where $z = Q^2/s, s+t+u = Q^2$.
- Note that the real diagrams contain collinear singularities, $u \to 0, t \to 0$ and soft singularities, $z \to 1$.
- The coefficient of the divergence is the unregulated branching probability $\hat{P}_{qq}(z)$.
- Ignore for simplicity the diagrams with incoming gluons.

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- Control the divergences by continuing the dimensionality of space-time, $d = 4 - 2\epsilon$, (technically this is dimensional reduction).

$$PS = \frac{c_\Gamma}{8\pi}\left(\frac{1}{Q^2}\right)^\epsilon z^\epsilon (1-z)^{1-2\epsilon}\int_0^1 dy\,(y(1-y))^{-\epsilon}$$

where

$$s = \frac{Q^2}{z},\quad t = -\frac{Q^2}{z}(1-z)(1-y)\quad u = -\frac{Q^2}{z}(1-z)y,\quad y = \frac{1}{2}(1+\cos\theta).$$

- Performing the phase space integration, the total contribution of the real diagrams is

$$\begin{aligned}\sigma_R \;&=\; \frac{\alpha_S}{2\pi}C_F\left(\frac{\mu^2}{Q^2}\right)^\epsilon c_\Gamma\left[\left(\frac{2}{\epsilon^2}+\frac{3}{\epsilon}-\frac{\pi^2}{3}\right)\delta(1-z)-\frac{2}{\epsilon}P_{qq}(z)\right.\\ &-\left.\;2(1-z)+4(1+z^2)\left[\frac{\ln(1-z)}{1-z}\right]_+ - 2\frac{1+z^2}{(1-z)}\ln z\right]\end{aligned}$$

with $c_\Gamma = (4\pi)^\epsilon/\Gamma(1-\epsilon)$.

- The contribution of the virtual diagrams is (neglecting terms of order ϵ)

$$\sigma_V = \delta(1-z)\left[1+\frac{\alpha_S}{2\pi}C_F\left(\frac{\mu^2}{Q^2}\right)^\epsilon c'_\Gamma\left(-\frac{2}{\epsilon^2}-\frac{3}{\epsilon}-6+\pi^2\right)\right]$$

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- Adding it up we get in dim-reduction

$$\begin{aligned}\sigma_{R+V} &= \frac{\alpha_S}{2\pi} C_F \left(\frac{\mu^2}{Q^2}\right)^{\epsilon} c_\Gamma \left[\left(\frac{2\pi^2}{3} - 6\right)\delta(1-z) - \frac{2}{\epsilon} P_{qq}(z) - 2(1-z)\right. \\ &+ \left. 4(1+z^2)\left[\frac{\ln(1-z)}{1-z}\right]_+ - 2\frac{1+z^2}{(1-z)}\ln z\right]\end{aligned}$$

- The divergences, proportional to the branching probability , are universal.
- We will factorize them into the parton distributions. We perform the mass factorization by subtracting the counterterm, (The finite terms are necessary to get us to the $\overline{MS}$-scheme).

$$2\frac{\alpha_S}{2\pi} C_F \left[\frac{-c_\Gamma}{\epsilon} P_{qq}(z) - (1-z) + \delta(1-z)\right]$$

$$\begin{aligned}\hat{\sigma} &= \frac{\alpha_S}{2\pi} C_F \left[\left(\frac{2\pi^2}{3} - 8\right)\delta(1-z) + 4(1+z^2)\left[\frac{\ln(1-z)}{1-z}\right]_+ - 2\frac{1+z^2}{(1-z)}\ln z\right. \\ &+ \left. 2P_{qq}(z)\ln\frac{Q^2}{\mu^2}\right]\end{aligned}$$

- Similar correction for incoming gluons.

Application to W, Z production

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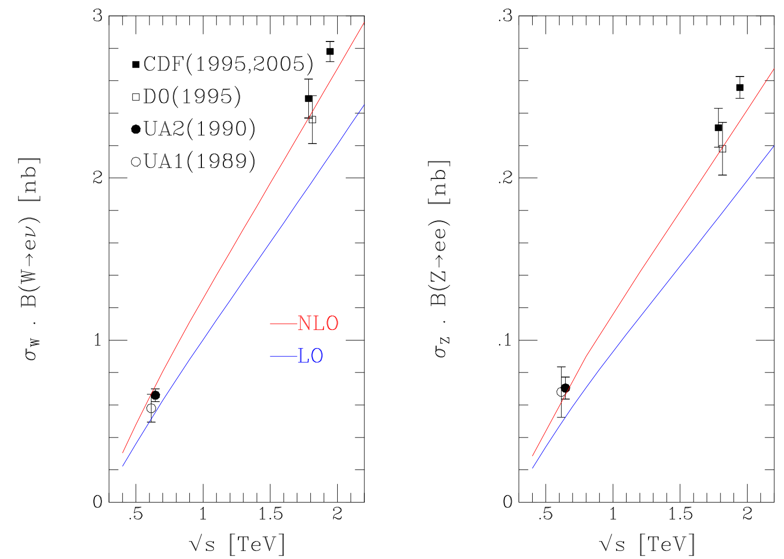
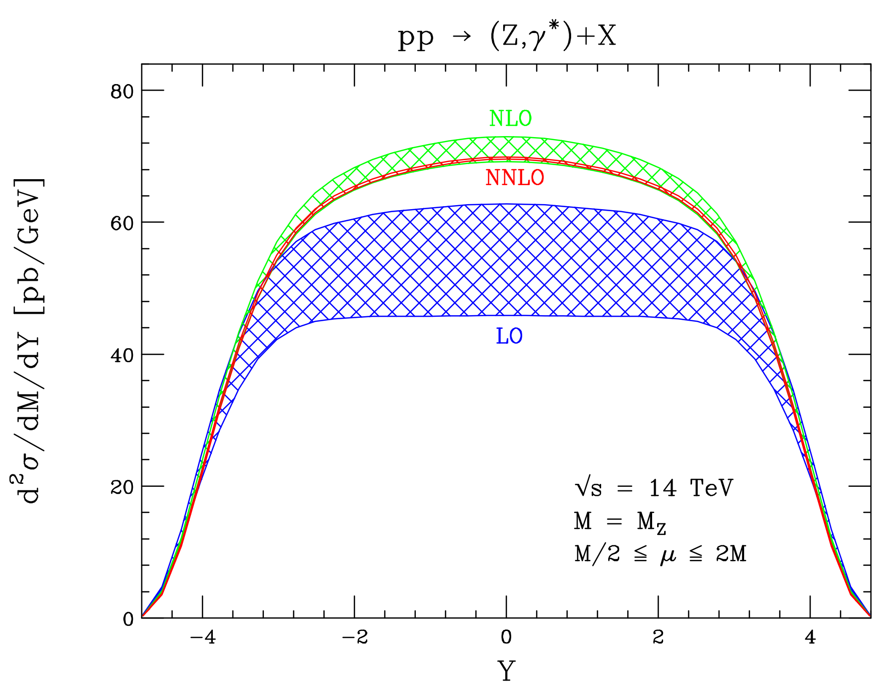
- Agreement with NLO theory is good.
- LO curves lie about 25% too low.
- NNLO results are also known and lead to a further modest (4%) increase at the Tevatron.
- NLO corrections for Z and W production at $\sqrt{s} = 13$ TeV remain a 22% effect.
- NNLO corrections are small at the LHC





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- We would like to go beyond the results for the total cross section to get results for distributions.
- We have two separate divergent integrals which must be combined before numerical integration

$$\sigma_{NLO} = \int_{m+1} d\sigma^R + \int_m d\sigma^V$$

- Note that the jet definition can be arbitratrily complicated.

$$d\sigma^R = PS_{m+1} |\mathcal{M}_{m+1}|^2 F^J_{m+1}(p_1, \ldots p_{m+1})$$

  We need to combine the two pieces, which reside in phase-spaces of different dimensionality, without knowledge of F^J.
- Divergences regularized in $d = 4 - 2\epsilon$ dimensions.
- Two solutions: phase space slicing and subtraction.
- Illustrate with a simple one-dimensional example.

$$|\mathcal{M}_{m+1}|^2 \equiv \frac{1}{x}\mathcal{M}(x), \quad |\mathcal{M}_m|^2 \equiv \frac{1}{\epsilon}\nu + k$$

  x is the energy of an emitted gluon.

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- Divergences regularized in $d = 4 - 2\epsilon$ dimensions. Two solutions: phase space slicing and subtraction.
- Thus the full cross section in d dimensions is

$$\sigma = \int_0^1 \frac{dx}{x^{1+\epsilon}} \mathcal{M}(x) F_1^J(x) + (\frac{1}{\epsilon}\nu + k) F_0^J$$

- Infrared safety: $F_1^J(0) = F_0^J$, KLN cancellation theorem, $\mathcal{M}(0) = \nu$
- Exact identity

$$\begin{aligned}\sigma &= \int_0^1 \frac{dx}{x^{1+\epsilon}} \left[\mathcal{M}(x) F_1^J(x) - \mathcal{M}(0) F_1^J(0)\right] + \int_0^1 \frac{dx}{x^{1+\epsilon}} \nu F_0^J + (\frac{1}{\epsilon}\nu + k) F_0^J \\ &= \int_0^1 \frac{dx}{x} \left[\mathcal{M}(x) F_1^J(x) - \mathcal{M}(0) F_1^J(0)\right] + k F_0^J\end{aligned}$$

- In practice we have to introduce a cutoff to protect from numerical overflow.

$$\sigma = \int_\delta^1 \frac{dx}{x} \left[\mathcal{M}(x) F_1^J(x) - \mathcal{M}(0) F_1^J(0)\right] + k F_0^J$$

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Thus the full cross section in d dimensions is

$$
\begin{aligned}
\sigma &= \int_0^1 \frac{dx}{x^{1+\epsilon}} \mathcal{M}(x) F_1^J(x) + (\frac{1}{\epsilon}\nu + k) F_0^J \\
&= \int_\delta^1 \frac{dx}{x^{1+\epsilon}} \mathcal{M}(x) F_1^J(x) + \int_0^\delta \frac{dx}{x^{1+\epsilon}} \mathcal{M}(x) F_1^J(x) + (\frac{1}{\epsilon}\nu + k) F_0^J \\
&\approx \int_\delta^1 \frac{dx}{x} \mathcal{M}(x) F_1^J(x) + \mathcal{M}(0) F_1^J(0) \int_0^\delta \frac{dx}{x^{1+\epsilon}} + (\frac{1}{\epsilon}\nu + k) F_0^J \\
&= \int_\delta^1 \frac{dx}{x} \mathcal{M}(x) F_1^J(x) + \ln(\delta)\nu F_0^J + k F_0^J
\end{aligned}
$$

- δ must be chosen small enough that the power corrections of order δ can be neglected.
- Important to establish that the final result is independent of the slicing parameter δ.
- large numerical cancellations at $\delta \to 0$.

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Subtraction method in detail

- Direct integration is good for the total cross section, but for differential distributions, (to which we want to apply cuts), we need a Monte Carlo method.
- We use a general subtraction procedure at NLO.
- at NLO the cross section for two initial partons a and b and for m outgoing partons, is given by

$$\sigma_{ab} = \sigma_{ab}^{LO} + \sigma_{ab}^{NLO}$$

where

$$\begin{aligned}\sigma_{ab}^{LO} &= \int_m d\sigma_{ab}^{B} \\ \sigma_{ab}^{NLO} &= \int_{m+1} d\sigma_{ab}^{R} + \int_m d\sigma_{ab}^{V}\end{aligned}$$

the singular parts of the QCD matrix elements for real emission, corresponding to soft and collinear emission can be isolated in a process independent manner

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- One can use this the construct a set of counterterms

$$d\sigma^{ct} = \sum_{ct} \int_m d\sigma^B \otimes \int_1 dV_{ct}$$

where $d\sigma^B$ denotes the appropriate colour and spin projection of the Born-level cross section, and the counter-terms are independent of the details of the process under consideration.

- these counterterm cancel all non-integrable singularities in $d\sigma^R$, so that one can write

$$\sigma_{ab}^{NLO} = \int_{m+1} [d\sigma_{ab}^R - d\sigma_{ab}^{ct}] + \int_{m+1} d\sigma_{ab}^{ct} + \int_m d\sigma_{ab}^V$$

The phase space integration in the first term can be performed numerically in four dimensions.

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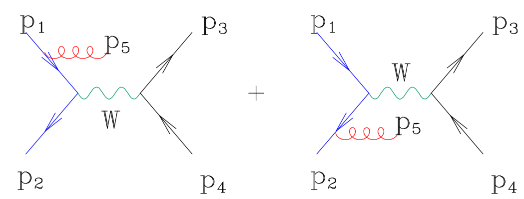

In the soft limit $p_5 \to 0$ we have

$$|M_1(p_1,p_2,p_3,p_4,p_5)|^2 = g^2 C_F \frac{p_1 \cdot p_2}{p_1 \cdot p_5 \; p_2 \cdot p_5} |M_0(p_1,p_2,p_3,p_4)|^2$$

- Eikonal factor can be associated with radiation from a given leg by partial fractioning

$$\frac{p_1 \cdot p_2}{p_1 \cdot p_5 \; p_2 \cdot p_5} = [\frac{p_1 \cdot p_2}{p_1 \cdot p_5 + p_2 \cdot p_5}][\frac{1}{p_1 \cdot p_5} + \frac{1}{p_2 \cdot p_5}]$$

- including the collinear contributions, singular as $p_1 \cdot p_5 \to 0$, the matrix element for the counter event has the structure

$$|M_1(p_1,p_2,p_3,p_4,p_5)|^2 = \frac{g^2}{x_a p_1 \cdot p_5} \hat{P}_{qq}(x_a) |M_0(x_a p_1, p_2, \tilde{p}_3, \tilde{p}_4)|^2$$

where $1 - x_a = (p_1 \cdot p_5 + p_2 \cdot p_5)/p_1 \cdot p_2$ and $\hat{P}_{qq}(x_a) = C_F(1+x^2)/(1-x)$

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- For event $q(p_1) + \bar{q}(p_2) \to W^+(\nu(p_3) + e^+(p_4)) + g(p_5)$ with $p_1 + p_2 = \sum_{i=3}^{5} p_i$
- generate a counter event $q(x_a p_1) + \bar{q}(p_2) \to W^+(\nu(\tilde{p}_3) + e^+(\tilde{p}_4))$ and $x_a p_1 + p_2 = \sum_{i=3}^{4} \tilde{p}_i$ with $1 - x_a = (p_1 \cdot p_5 + p_2 \cdot p_5)/p_1 \cdot p_2$.
- A Lorentz transformation is performed on all j final state momenta $\tilde{p}_j = \Lambda^\mu_\nu p^\nu_j, j = 3, 4$ such that $\tilde{p}^\mu_j \to p^\mu_j$ for p_5 collinear or soft.
- The longitudinal momentum of p_5 is absorbed by rescaling with x.
- The other components of the momentum, p_5 are absorbed by the Lorentz transformation.
- In terms of these variables the phase space has a convolution structure,

$$d\phi^{(3)}(p_1, p_2; p_3, p_4, p_5) = \int_0^1 dx \; d\phi^{(2)}(p_2, xp_1; \tilde{p_3}, \tilde{p_4})[dp_5(p_1, p_2, x)]$$

where

$$[dp_5(p_1, p_2, x_a)] = \frac{d^d p_5}{(2\pi)^3} \delta^+(p_5^2)\Theta(x)\Theta(1-x)\delta(x - x_a)$$

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- If k_i is the emitted parton, and p_a, p_b are the incoming momenta, define the shifted momenta

$$\widetilde{k}_j^\mu = k_j^\mu - \frac{2k_j \cdot (K + \widetilde{K})}{(K + \widetilde{K})^2} (K + \widetilde{K})^\mu + \frac{2k_j \cdot K}{K^2} \widetilde{K}^\mu ,$$

where the momenta K^μ and $\widetilde{K}^\mu$ are,

$$K^\mu = p_a^\mu + p_b^\mu - p_i^\mu \quad , \widetilde{K}^\mu = \widetilde{p}_{ai}^\mu + p_b^\mu \quad .$$

- Since $2\sum_j k_j \cdot K = 2K^2$ and $2\sum_j k_j \cdot (K + \widetilde{K}) = 2K^2 + 2K \cdot \widetilde{K} = (K + \widetilde{K})^2$ the momentum conservation constraint in the $m + 1$-parton matrix

$$p_a^\mu + p_b^\mu - \sum_j k_j^\mu - p_i^\mu = 0 \quad .$$

implies

$$\widetilde{p}_{ai}^\mu + p_b^\mu - \sum_j \widetilde{k}_j^\mu = 0 \quad .$$

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- Note also that the shifted momenta can be rewritten in the following way:

$$\widetilde{k}_j^\mu = \Lambda^\mu{}_\nu(K,\widetilde{K})\, k_j^\nu \ ,$$

$$\Lambda^\mu{}_\nu(K,\widetilde{K}) = g^\mu{}_\nu - \frac{2(K+\widetilde{K})^\mu (K+\widetilde{K})_\nu}{(K+\widetilde{K})^2} + \frac{2\widetilde{K}^\mu K_\nu}{K^2} \ ,$$

- the matrix $\Lambda^\mu{}_\nu(K,\widetilde{K})$ generates a proper Lorentz transformation on the final-state momenta.
- If the emitted parton has zero transverse momenta, the Lorentz transformation reduces to the identity.

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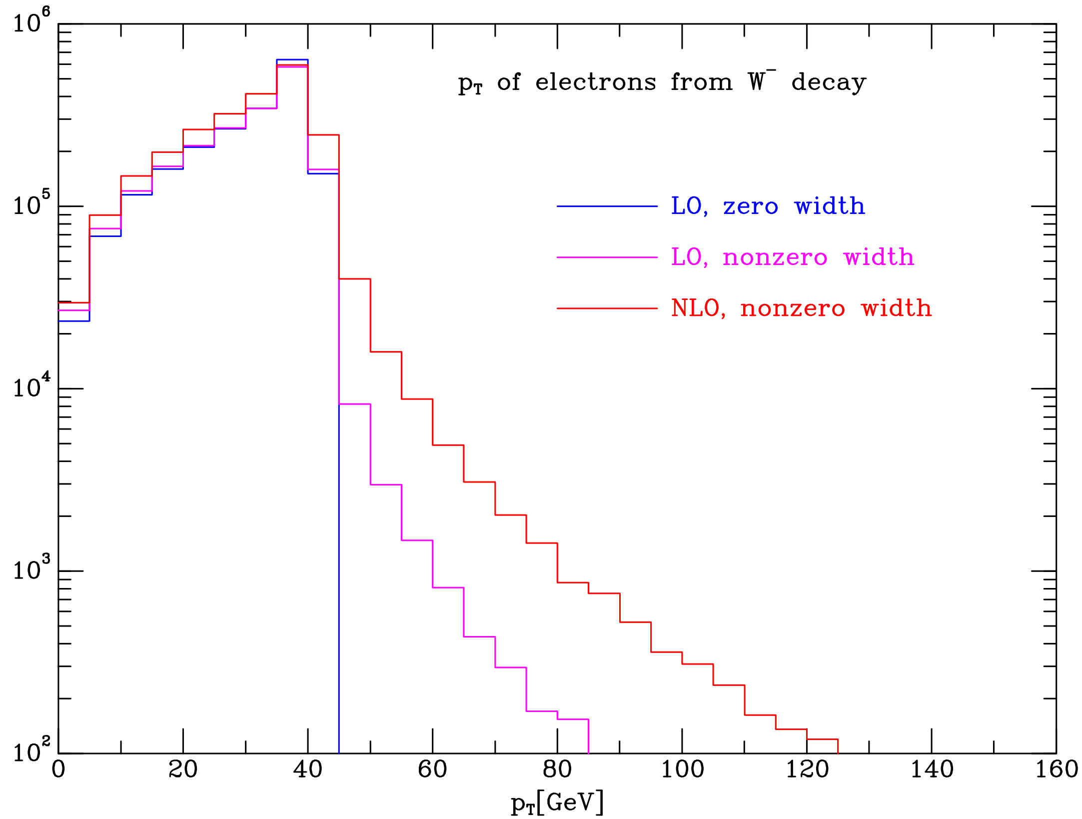

- Calculation of NLO corrections, give a better prediction for the rate.
- At NLO new parton processes can contribute.
- Extra radiation can modify kinematic distributions.

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Slicing methods

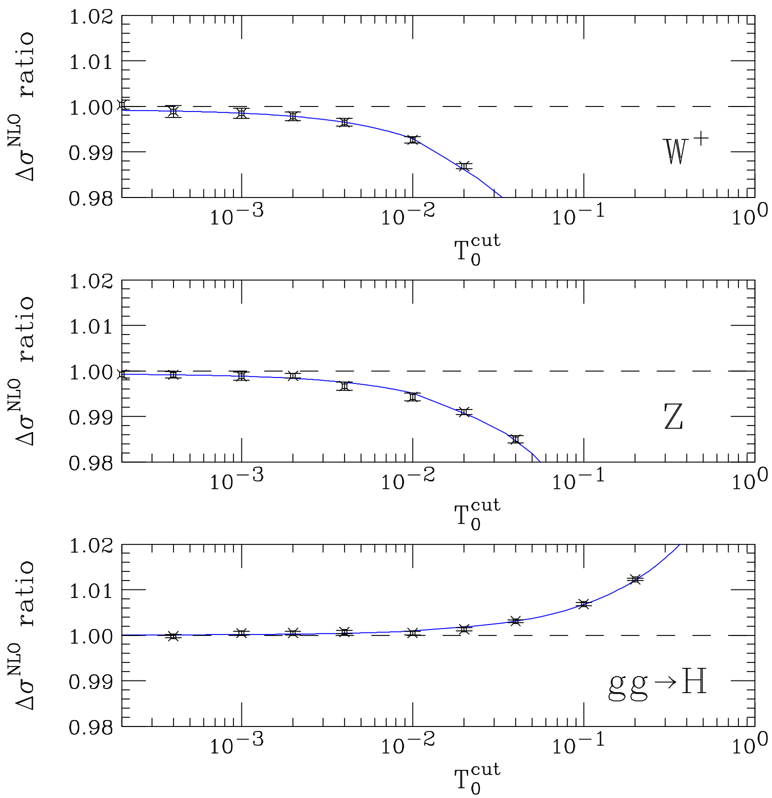
- Slicing methods have recently been applied in NNLO calculations.
- Here for illustration we show results obtained at NLO.
- The resolved region of phase space corresponds to a calculation of the process with one additional final state parton, in this case one gluon emission.
- if a suitable resolution parameter is chosen, the unresolved region can be directly calculated.
- The jettiness of parton j with momentum p_j is defined as

$$\tau(p_j) = \min_{i=a,b,1,\ldots,N} \left\{ \frac{2\, q_i \cdot p_j}{Q_i} \right\} ,$$

- The resolution parameter in the attached plots is the jettiness, and the behaviour below the cut is theoretically calculable.
- The resolution parameter should be chosen small enough, that power corrections are negligible, but not so small that numerical errors in the cancellation dominate.

Boughezal et al, 1605.08011

- Comparison of results calculated with MCFM, using subtraction and slicing.



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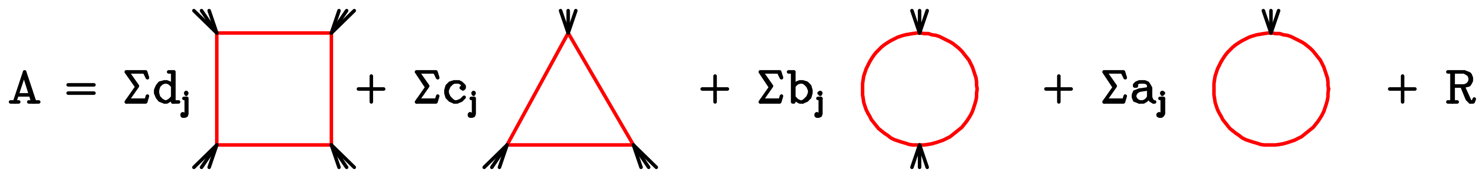
- For NLO calculations, any one-loop amplitude (no matter how many legs) can be written as a sum of sums of scalar integrals (boxes, triangles, bubbles and tadpoles)



- Scalar integrals are integrals with no numerator factors, e.g. box integral

$$I_4^D(p_1^2,p_2^2,p_3^2,p_4^2;s_{12},s_{23};m_1^2,m_2^2,m_3^2,m_4^2) = \frac{\mu^{4-D}}{i\pi^{\frac{D}{2}} r_\Gamma} \times \int d^D l$$
$$\frac{1}{(l^2-m_1^2+i\varepsilon)((l+q_1)^2-m_2^2+i\varepsilon)((l+q_2)^2-m_3^2+i\varepsilon)((l+q_3)^2-m_4^2+i\varepsilon)}$$

- The determination of the coefficients,d_j, c_j, b_j, a_j can be determined by semi-numerical methods, especially D-dimensional unitarity.
- R is a rational piece also determined by seminumerical methods
- The scalar integrals are all known analytically, see e.g. QCDLoop.fnal.gov, (RKE,Zanderighi)
- The OPP method of calculating one-loop integrals, exploits the known analytic form of the integrand, and evaluates the coefficients in that analytic form numerically.

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Example: reduction in two dimensions

- As an example, consider the reduction of a rank-two two-point integral in two dimensions Integrand and integral are,

$$\mathcal{I}(k, m_1, m_2) = \frac{(\hat{n} \cdot l)^2}{d_1 d_2}, \quad I = \int d^d l \, \mathcal{I}(k, m_1, m_2)$$

where $d_1 = l^2 - m_1^2, d_2 = (l+k)^2 - m_2^2$ and $\hat{n} \cdot k = 0, k^2 \neq 0$ and $\hat{n}^2 = 1$.

- Because of the projection onto $\hat{n}$, the momentum l in the numerator lies in the transverse space.
- Because l is a d-dimensional vector, we can decompose it as

$$l^\mu = (l \cdot n) n^\mu + (l \cdot \hat{n}) \hat{n}^\mu + n_\epsilon^\mu (l \cdot n_\epsilon),$$

where n_ϵ is the unit vector that parametrizes the $(D-2)$-dimensional vector space and n defines the physical space.

$$n^\mu = \frac{k^\mu}{\sqrt{k^2}}, \quad n^2 = 1, \quad (n_\epsilon \cdot l)^2 = \mu^2$$

$$(\hat{n} \cdot l)^2 = l^2 - (n \cdot l)^2 - (n_\epsilon \cdot l)^2 = l^2 - \frac{(l \cdot k)^2}{k^2} - \mu^2.$$

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- To proceed further, we express various scalar products through inverse Feynman propagators $d_{1,2}, l^2 = d_1 + m_1^2, \quad 2\,l\cdot k = d_2 - d_1 - r_1^2$

$$\frac{(\hat{n}\cdot l)^2}{d_1 d_2} = -\frac{(\lambda^2+\mu^2)}{d_1 d_2} + \frac{1}{4k^2}\left[\frac{r_1^2 - 2\,l\cdot k}{d_1} + \frac{r_2^2 + 2\,l\cdot k + 2k^2}{d_2}\right].$$

Note the following short-hand notations

$$r_1^2 = k^2 + m_1^2 - m_2^2, \quad r_2^2 = k^2 + m_2^2 - m_1^2, \quad \lambda^2 = \frac{k^4 - 2k^2(m_1^2+m_2^2) + (m_1^2-m_2^2)^2}{4k^2}.$$

- This is consistent with a general parametric decomposition,

$$\frac{(\hat{n}\cdot l)^2}{d_1 d_2} = \frac{b_0 + b_1(\hat{n}\cdot l) + b_2(n_\epsilon\cdot l)^2}{d_1 d_2} + \frac{a_{1,0} + a_{1,1}(n\cdot l) + a_{1,2}(\hat{n}\cdot l)}{d_1} + \frac{a_{2,0} + a_{2,1}(n\cdot l) + a_{2,2}(\hat{n}\cdot l)}{d_2}.$$

where the parameters take the values

$$b_0 = -\lambda^2, \quad b_1 = 0, \quad b_2 = -1,$$
$$a_{1,0} = \frac{r_1^2}{4k^2}, \quad a_{1,1} = -\frac{1}{2\sqrt{k^2}}, \quad a_{1,2} = 0,$$
$$a_{2,0} = \frac{r_2^2}{4k^2} + \frac{1}{2}, \quad a_{2,1} = \frac{1}{2\sqrt{k^2}}, \quad a_{2,2} = 0.$$

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- We begin by multiplying by d_1, d_2 and obtain

$$(\hat{n}\cdot l)^2 = \left[b_0 + b_1(\hat{n}\cdot l) + b_2(n_\epsilon \cdot l)^2\right] + \left[a_{1,0} + a_{1,1}(n\cdot l) + a_{1,2}(\hat{n}\cdot l)\right] d_2 + \left[a_{2,0} + a_{2,1}(n\cdot l) + a_{2,2}(\hat{n}\cdot l)\right] d_1 .$$

To see how this works, we first describe a procedure to compute the b-coefficients *only*.

- consider the loop momentum l that satisfies the constraints $d_1(l) = d_2(l) = 0$ and simultaneously, has zero projection on the d-dimensional space, $n_\epsilon \cdot l = 0$.
- We find that there are just two loop momenta l that satisfy those constraints; they can be written as

$$l_c^{\pm} = \alpha_c n \pm i\beta_c \hat{n},$$

where

$$\alpha_c = -\frac{r_1^2}{2\sqrt{k^2}}, \quad \beta_c = \lambda.$$

We substitute these two solutions and obtain two equations for the coefficients $b_{0,1}$

$$b_0 + b_1 \hat{n}\cdot l_c^+ = -\lambda^2, \quad b_0 + b_1 \hat{n}\cdot l_c^- = -\lambda^2.$$

- It follows that $b_0 = -\lambda^2$ and $b_1 = 0$.
- All other coefficients can be found numerically by iterating the procedure.
- this is the basis of the OPP method

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To find b_2 we proceed along similar lines but we require that the scalar product $l \cdot n_\epsilon$ does not vanish. Since the conditions $d_1 = 0, d_2 = 0$ are equivalent to $2l \cdot k + r_1^2 = 0$, $l^2 = m_1^2$, the loop momentum that satisfies those constraints is the same as in Eq. (40), up to a change $\hat{n} \to n_\epsilon$,

$$l^\pm = \alpha_c n \pm i\beta_c n_\epsilon .$$

Substituting $l^\pm$ into Eq. (1) and using $b_0 = -\lambda^2$, $b_1 = 0$, we obtain

$$0 = (1 + b_2)\lambda^2,$$

which implies that $b_2 = -1$, in agreement with the result stated previously

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  - Go-Sam 1404.7096
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  - Blackhat-Sherpa 1310.2808
  - Njet 1312.7140
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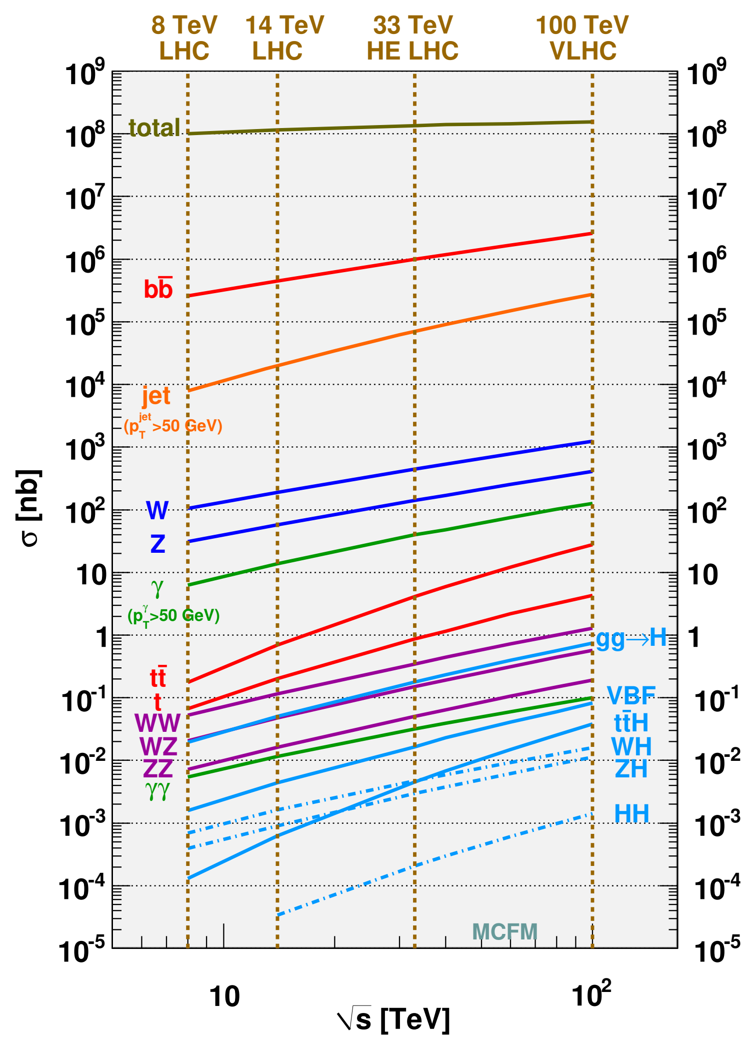
  - MCFM, hep-ph/9905386,arXiv:1503.06182
  - VBFNLO, arXiv:1404.3940



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Process	μ	n_{lf}	Cross section (pb) LO	Cross section (pb) NLO
$pp \to t\bar{t}$	m_{top}	5	123.76 ± 0.05	162.08 ± 0.12
$pp \to tj$	m_{top}	5	34.78 ± 0.03	41.03 ± 0.07
$pp \to tjj$	m_{top}	5	11.851 ± 0.006	13.71 ± 0.02
$pp \to t\bar{b}j$	$m_{top}/4$	4	31.37 ± 0.03	32.86 ± 0.04
$pp \to t\bar{b}jj$	$m_{top}/4$	4	11.91 ± 0.006	7.299 ± 0.05
$pp \to (W^+ \to) e^+ \nu_e$	m_W	5	5072.5 ± 2.9	6146.2 ± 9.8
$pp \to (W^+ \to) e^+ \nu_e\, j$	m_W	5	828.4 ± 0.8	1065.3 ± 1.8
$pp \to (W^+ \to) e^+ \nu_e\, jj$	m_W	5	298.8 ± 0.4	289.7 ± 0.3
$pp \to (\gamma^*/Z \to) e^+ e^-$	m_Z	5	1007.0 ± 0.1	1170.0 ± 2.4
$pp \to (\gamma^*/Z \to) e^+ e^-\, j$	m_Z	5	156.11 ± 0.03	203.0 ± 0.2
$pp \to (\gamma^*/Z \to) e^+ e^-\, jj$	m_Z	5	54.24 ± 0.02	54.1 ± 0.6
$pp \to (W^+ \to) e^+ \nu_e b\bar{b}$	$m_W + 2m_b$	4	11.557 ± 0.005	22.95 ± 0.07
$pp \to (W^+ \to) e^+ \nu_e t\bar{t}$	$m_W + 2m_{top}$	5	0.009415 ± 0.000003	0.01159 ± 0.00001
$pp \to (\gamma^*/Z \to) e^+ e^- b\bar{b}$	$m_Z + 2m_b$	4	9.459 ± 0.004	15.31 ± 0.03
$pp \to (\gamma^*/Z \to) e^+ e^- t\bar{t}$	$m_Z + 2m_{top}$	5	0.0035131 ± 0.0000004	0.004876 ± 0.000002
$pp \to \gamma t\bar{t}$	$2m_{top}$	5	0.2906 ± 0.0001	0.4169 ± 0.0003
$pp \to W^+ W^-$	$2m_W$	4	29.976 ± 0.004	43.92 ± 0.03
$pp \to W^+ W^-\, j$	$2m_W$	4	11.613 ± 0.002	15.174 ± 0.008
$pp \to W^+ W^+\, jj$	$2m_W$	4	0.07048 ± 0.00004	0.08241 ± 0.0004
$pp \to HW^+$	$m_W + m_H$	5	0.3428 ± 0.0003	0.4455 ± 0.0003
$pp \to HW^+\, j$	$m_W + m_H$	5	0.1223 ± 0.0001	0.1501 ± 0.0002
$pp \to HZ$	$m_Z + m_H$	5	0.2781 ± 0.0001	0.3659 ± 0.0002
$pp \to HZ\, j$	$m_Z + m_H$	5	0.0988 ± 0.0001	0.1237 ± 0.0001
$pp \to Ht\bar{t}$	$m_{top} + m_H$	5	0.08896 ± 0.00001	0.09869 ± 0.00003
$pp \to Hb\bar{b}$	$m_b + m_H$	4	0.16510 ± 0.00009	0.2099 ± 0.0006
$pp \to Hjj$	m_H	5	1.104 ± 0.002	1.333 ± 0.002

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- Parton branching gives rise to universal branching probabilities, independent of the process.
- The DGLAP equation predicts the change with scale of the parton distributions: shrinkage at large x and growth at small x.
- The branching probabilities are the basis of shower Monte Carlo programs
- The master formula predicts that hadron-hadron processes are factorized. Parton distributions measured, e.g. in deep inelastic scattering, can be used at LHC
- NLO corrections can be used to give exclusive predictions using subtraction or slicing methods.
- Virtual amplitudes can also be calculated numerically exploiting general parameterizations of the one loop amplitudes.
- If the number of partons is not too large, fully automatic procedures can be used to calculate NLO corrections.

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