

Quantum reflection in the linearly downward potential

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Abstract. In this work, the motion of a particle in one dimension under the influence of the linearly downward potential well is studied within the context of the non-relativistic quantum mechanics. The attention is paid on the paradoxical phenomenon of the reflection of a particle that is in contrast between classical and quantum physics. Classically, the reflection effect occurs only at a potential barrier. To demonstrate such counter-intuitive phenomenon, the Schrödinger equation is solved to obtain the reflection coefficient in the scattering state by considering an incident particle that is represented by a monochromatic plane wave having an energy $E > 0$, propagates freely from left to right, pass through the potential well. The continuity conditions at boundaries give the desired result that is expressed in terms of the Airy functions which depends on the incident energy E , the strength $|V_0|$ and the range L of the well. The value of the reflection coefficient R lies in the interval $0 < R < 1$, and its behavior is the decreasing function with respect to the range L .

1. Introduction

Tunneling phenomenon, where an incident particle has the probability to tunnel through a potential barrier even there is forbidden classically, is the achievement of quantum mechanics which fairly found in many standard quantum mechanics textbooks [1],[2],[3] and many articles [4],[5],[6] are devoted to this effect. Besides tunneling, the paradoxical reflection is the situation where the particle can have the probability of being reflected by a potential well that classically the particle would be transmitted perfectly. Paradoxical reflection can be found both in non-relativistic and relativistic quantum mechanics. The latter is widely known as the Klein paradox [7], [8].

The main goal of this article is to consider the problem of scattering from the linear downward potential model where the width L of the step is taking into account to ensure that the paradoxical reflection is not an artificial phenomenon. Our model is the generalization of the sudden drop potential well. Another model of the potential well has been investigated recently in Ref.[9], a soft potential step with the hyperbolic tangent shape is considered and they reached the conclusion that the reflection is more likely the sharper the step. For a general potential, as shown in Ref.[10], only an upper bound of R is available and its value does not exceed the reflection coefficient for the sudden drop well. This is the reason why the specific model is still necessary to examine the features of the paradoxical reflection. In Sec. 2, we present the model of the linear downward potential well and solve the stationary Schrödinger equation to obtain the expression of R analytically. In Sec 3, we analyze some properties of R and show their

graphical solutions in an appropriate range of the reduced incident energy $E/|V_0|$. Finally, we conclude the main results in Sec. 4.

2. Model and calculation

The purpose of this section is to find the reflection coefficient R which describes quantitatively how much probability of a particle to being reflected from the region surrounded by the potential well. The strategy consists of three steps as follows: (1) Solve the Schrödinger equation to obtain the wave functions, (2) Determine the multiplicative constants of wave functions by making use the matching conditions at boundaries of the well, and (3) Calculate R from the definition $R = |j_{\text{refl}}|/j_{\text{inc}}$, where $j = (\hbar/m)\text{Im}(\psi^*\psi')$ is a probability current density.

The model of our interest can be written mathematically as

$$V(x) = \begin{cases} 0 & (x < 0) \\ -V_0x/L & (0 < x < L) \\ -V_0 & (x > L). \end{cases} \quad (1)$$

A sketch of the modeled potential is depicted in Fig. 1. It is apparent from the figure that the asymptotic regions $x < 0$ and $x > L$ of $V(x)$ stand for flat potential energies with the difference in value $-V_0$, while the intermediate region $0 < x < L$ the potential energy decreases linearly with the rate $-V_0/L$. In the limit $L \rightarrow 0$, the intermediate region vanishes and our model reduces to the sudden drop well, as expected.

An analysis of the scattering problem in the stationary state is governed by the one-dimensional time independent Schrödinger equation

$$-\frac{\hbar^2}{2m} \frac{d^2}{dx^2} \psi(x) + V(x)\psi(x) = E\psi(x), \quad (2)$$

for a particle of mass m in a potential $V(x)$. In the first step, let us solve the Schrödinger equation for $\psi(x)$ in each regions separately. The wave functions in the asymptotic regions, would be labelled as $\psi_{\text{I}}(x)$ for $x < 0$, and $\psi_{\text{III}}(x)$ for $x > L$, are in the form of plane waves having wave numbers $k = (2mE/\hbar^2)^{1/2}$ and $q = (2m(E + V_0)/\hbar^2)^{1/2}$, respectively.

$$\begin{aligned} \psi_{\text{I}}(x) &= e^{ikx} + Be^{-ikx}, \\ \psi_{\text{III}}(x) &= Ae^{iqx}. \end{aligned} \quad (3)$$

Therefore $\psi_{\text{I}}(x)$ resulting from a combination of an incoming wave e^{ikx} and a reflected wave Be^{-ikx} , whereas $\psi_{\text{III}}(x)$ has only a transmitted wave Ae^{iqx} because an incident wave is assumed to be injected from the far left with a positive energy $E > 0$. The wave function $\psi_{\text{II}}(x)$ in the intermediate region is readily solved by introduce the new variable $\xi = -(a + bx/L)/b^{2/3}$ where $a = 2mEL^2/\hbar^2$, and $b = 2mV_0L^2/\hbar^2$. Then the transformed Schrödinger equation has a dimensionless form $\psi_{\text{II}}''(\xi) - \xi\psi_{\text{II}}(\xi) = 0$ and is identical to the Airy equation [11]. Hence the general solution of ψ_{II} can be written immediately

$$\psi_{\text{II}}(x) = C_1\text{Ai}(\xi) + C_2\text{Bi}(\xi), \quad (5)$$

where Ai, Bi are the Airy functions of first and second kinds, and the range of ψ_{II} possesses between $\xi_0 = -a/b^{2/3}$, and $\xi_L = -(a + b)/b^{2/3}$.

In the second step, the requirement of the continuity of wave functions and their derivatives at boundaries of the well, $x = 0$ and L , implies a coupled set of algebraic equations (not shown here), and after we eliminate C_1 and C_2 the boundary conditions can be put in a matrix form

$$\begin{pmatrix} e^{iqL}\Gamma(\xi_L, q) & -\Gamma^*(\xi_0, k) \\ e^{iqL}\Omega(\xi_L, q) & -\Omega^*(\xi_0, k) \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} \Gamma(\xi_0, k) \\ \Omega(\xi_0, k) \end{pmatrix}. \quad (6)$$

With the help of Kramer's rule, the constants A and B are easily to be evaluated in terms of the complex functions Ω and Γ of the two real variables (ξ, p)

$$\Omega(\xi, p) = \text{Ai}'(\xi) + i(pL/b^{1/3})\text{Ai}(\xi), \quad (7)$$

$$\Gamma(\xi, p) = \text{Bi}'(\xi) + i(pL/b^{1/3})\text{Bi}(\xi). \quad (8)$$

In the final step, we employ the definition of the probability current density to calculate the incident, reflected, and transmitted currents the results are $j_{\text{inc}} = \hbar k/m$, $j_{\text{refl}} = -(\hbar k/m)|B|^2$, $j_{\text{tran}} = (\hbar q/m)|A|^2$. Since we are interested in the reflection coefficient R , then calculate the constant B from the matrix equation and use the definition of R , we obtain

$$R = \left| \frac{\Omega_L \Gamma_0 - \Gamma_L \Omega_0}{\Omega_L \Gamma_0^* - \Gamma_L \Omega_0^*} \right|^2, \quad (9)$$

which is the desired analytical formula. In the above, the subscripts L and 0 are used to indicate the arguments of the functions Ω and Γ at $x = L$ and $x = 0$, respectively. Also, it should be understood that the corresponding wave vectors are q and k in the same manner.

3. Some properties of R

The obtained reflection coefficient R formula has many features to analyze their properties. We begin by rewriting the R formula in a simplify form $R = (P - Q)/(P + Q)$, where $P = |\Omega_L|^2 |\Gamma_0|^2 + |\Gamma_L|^2 |\Omega_0|^2 - 2\text{Re}[\Gamma_L \Omega_L^*] \text{Re}[\Gamma_0 \Omega_0^*]$, and $Q = 2kqL^2/(\pi^2 b^{2/3})$. The principle of conservation of probability current density implies R is confined in an interval $0 < R < 1$. To verify the minimum of R is greater than zero, setting $R = 0$ to get $P - Q = 0$. This equality holds only when $V_0 = 0$, which leads to the conclusion that $R \neq 0$ as well as $V_0 \neq 0$. To show the maximum of R does not exceed unity let us assume $R = 1$ and obtain $Q = 0$, but Q is finite then we have to conclude $R < 1$.

The next task is to approximate the exact R formula. Since our modeled potential well coincides with the sudden drop well in the limiting case $V_0/L \rightarrow \infty$. We then expect the series expansion of R as a function of L will be an upper bound which is at least equal to the reflection coefficient for the sudden drop well in the limit $L \rightarrow 0$. A standard treatment is based on the Taylor series by expanding $\Omega(\xi_L, q)$ and $\Gamma(\xi_L, q)$ about ξ_0 up to second order. The straightforward calculation gives

$$R = \left(\frac{q - k}{q + k} \right)^2 [1 - kqL^2], \quad (10)$$

where the prefactor (independent of L) stands for the reflection coefficient in the case of the sudden drop well. The obtained result confirms our inspection.

To see clearly the paradoxical phenomenon of the quantum reflection effect. Using the exact R formula, that is expressed in terms of the dimensionless parameters $\epsilon = E/|V_0|$ and $\lambda = L(2mV_0/\hbar^2)^{1/2}$, for numerical calculations by fixing λ , the graph is presented in Fig. 2. The general feature of R is a rapidly decreasing function as ϵ increases. The paradoxical reflection occurs in a narrow range of the incident energy ($\epsilon < 0.1$). Each curve is drawn in a sequence of λ from small to large values ($\lambda = 0.5, 1, 2, 3$). Since λ is proportional to L , then for any value of ϵ with fixed $|V_0|$, we can see that the width of the potential well is an important factor for the decrease of R , i.e., the more wider the well the less reflection coefficient. In addition, R approaches zero at large ϵ , this means the particle reaches the perfect transmission for a shallow well ($|V_0|$ is small).

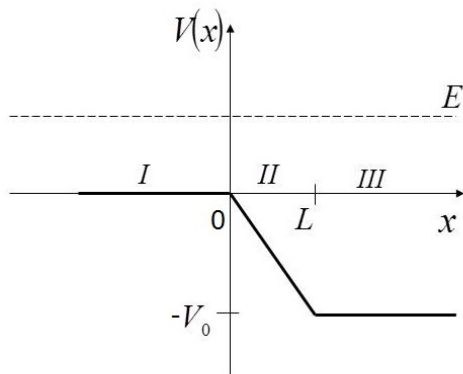


Figure 1. Sketch the model of the linearly downward potential.

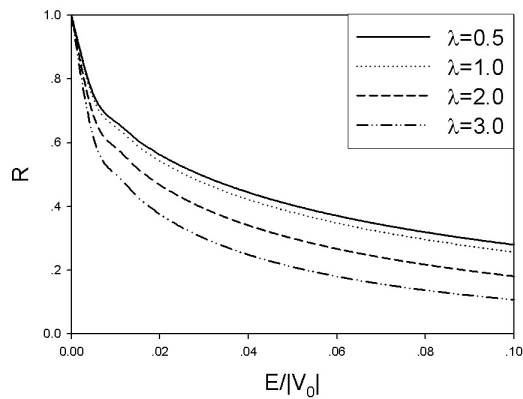


Figure 2. Graph of R versus $E/|V_0|$ for various $\lambda = L(2mV_0/\hbar^2)^{1/2}$ values.

4. Conclusion

The paradoxical phenomenon of reflection effect has been investigated in detail by considering the specific model of the linearly downward potential. The exact formula of the reflection coefficient is obtained and its general feature is a rapidly decreasing function as $E/|V_0|$ increases. An analysis leads to the conclusion that the reflection coefficient never vanishes for a finite potential depth. The numerical result also shows the width of the potential well is responsible for the reduction of the reflection coefficient. It should be noted that the conclusion obtained above is valid only with the particular potential model.

The reflection coefficient is nearly certain ($R \approx 1$) for the very low energy comparable to the depth of the potential well, as in Fig. 2. This suggests that a quantum particle can be trapped by the typical downward potential (for more detail, see [9]).

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