

Supersymmetrizing a Quantum Mechanical System

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Abstract. Supersymmetry (SUSY) is one of the most active fields of research in high-energy physics and particle physics. However, SUSY is not so easily accessible for students wishing to get started on this field of research due to complexities often associated with quantum field theories. In this article, I'll discuss some aspects of SUSY in the context of quantum mechanics. In particular, I'll demonstrate how to supersymmetrize a familiar quantum system in the case of a square well and a harmonic oscillator. The harmonic oscillator clearly demonstrate the hallmark of SUSY which link a bosonic degree of freedom and a fermionic degree of freedom. I'll briefly discuss the applications of SUSY in particle physics.

1. Introduction

Symmetry plays an important role in our understanding of modern physics. For every continuous symmetry there is a conservation law associated with it [1]. To a particle physicist, an object of particular significance is a scattering matrix (S-matrix). Thus it is interesting to identify the maximal symmetry that an S-matrix can possess. In their famous 1967 paper, Coleman and Mandula showed that the maximal symmetry of a non-trivial S-matrix is the product of Poincare symmetry and internal symmetries [2]. However, in 1971 a loop-hole in the Coleman and Mandula's proof had been identified [3, 4]. This loop-hole allowed for a symmetry generator with spin 1/2 [5] which transforms a boson to a fermion and vice versa. This marked the discovery of supersymmetry (SUSY).

Since its discovery, SUSY has generated enormous research activity. It provides possible solutions to many puzzles in particle physics. First, supersymmetric partners of ordinary particles cure an ultraviolet divergence in the Higgs boson mass, rendering it technically natural [6]. Second, the lightest stable supersymmetric partner could serve as a dark matter [7]. Last, but not least, SUSY allows for the unification of the strong, the weak, and the electromagnetic interactions [8].

Because SUSY is an extension of Poincare symmetry, it's necessarily a symmetry of a quantum field theory (QFT). As a result, SUSY is generically inaccessible for students whom have never took a course in QFT. However, it is interesting to point out that a QFT in $0 + 1$ dimension is equivalent to a 1-dimensional quantum mechanics. Specifically the operators x and p can be identified with the field operator $\phi(t)$ and its conjugate momentum. Therefore, one could glean some of the features of SUSY in the context of quantum mechanics.

The purpose of this article is to introduce SUSY to a non-particle physicist in a friendly setting of quantum mechanics. Throughout this article, the natural unit system has been employed, ie., $\hbar = c = 1$. For an extensive review on the subject of SUSY in quantum mechanics and its applications, see Ref. [9] and references therein.

2. Formulation of SUSY QM

In this section I will formulate SUSY QM in the Hamiltonian framework. Suppose we have a quantum mechanical system H_1 that we know its ground state, $\psi_0^{(1)}$. Without loss of generality, we can set the ground state energy to be 0. This gives a relation between the ground state wavefunction and the potential V_1 as $V_1(x) = 1/(2m) \psi_0''^{(1)}/\psi_0^{(1)}$. Then one can factorize the Hamiltonian as

$$H_1 = \frac{p^2}{2m} + V_1(x) = A^\dagger A, \quad \text{where} \quad (1)$$

$$A = \frac{ip}{\sqrt{2m}} + W(x) = \frac{1}{\sqrt{2m}} \frac{d}{dx} + W(x), \quad A^\dagger = \frac{-ip}{\sqrt{2m}} + W(x) = -\frac{1}{\sqrt{2m}} \frac{d}{dx} + W(x).$$

The function $W(x)$ is called the superpotential. It is related to the original potential V_1 by $V_1(x) = W^2(x) - W'(x)/\sqrt{2m}$. The superpotential $W(x)$ can also be written in term of the ground state wavefunction $\psi_0^{(1)}$ by noting that $A\psi_0^{(1)} = 0$. Thus $W(x) = -1/\sqrt{2m} \psi_0'^{(1)}/\psi_0^{(1)}$.

Now we can define a partner Hamiltonian $H_2 = AA^\dagger$. The spectrum of H_2 can be easily obtained from those of H_1 . Let $\psi_n^{(1)}$ be an eigenstate of H_1 with eigenenergy $E_n^{(1)}$. Then

$$AH_1\psi_n^{(1)} = AE_n^{(1)}\psi_n^{(1)} \implies H_2A\psi_n^{(1)} = E_n^{(1)}A\psi_n^{(1)}. \quad (2)$$

Thus $A\psi_n^{(1)}$ is an eigenstate of H_2 with the same eigenenergy $E_n^{(1)}$. Similarly for an eigenstate $\psi_n^{(2)}$ of H_2 with an eigenenergy $E_n^{(2)}$, $A^\dagger\psi_n^{(2)}$ is an eigenstate of H_1 with the same eigenenergy. Figure 1 shows the spectra of both H_1 and H_2 .

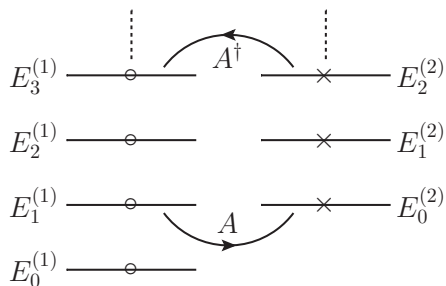


Figure 1. The spectra of H_1 and H_2 . The eigenstates of H_1 are marked by \circ , while those of H_2 are marked by \times . Note $E_{n+1}^{(1)} = E_n^{(2)}$ and there is no partner state for $\psi_0^{(1)}$ because $A\psi_0^{(1)} = 0$.

2.1. SUSY Algebra

The degeneracy in H_1 and H_2 spectra is the consequence of an underlying SUSY. To see this, we introduce a SUSY Hamiltonian encompassing both H_1 and H_2 , as well as two new operators

$$H = \begin{pmatrix} H_1 & 0 \\ 0 & H_2 \end{pmatrix} = \begin{pmatrix} A^\dagger A & 0 \\ 0 & AA^\dagger \end{pmatrix}, \quad Q = \begin{pmatrix} 0 & 0 \\ A & 0 \end{pmatrix}, \quad Q^\dagger = \begin{pmatrix} 0 & A^\dagger \\ 0 & 0 \end{pmatrix}. \quad (3)$$

It is easy to see that these three operators satisfy

$$[H, Q] = [H, Q^\dagger] = 0, \quad \{Q, Q^\dagger\} = H, \quad \{Q, Q\} = \{Q^\dagger, Q^\dagger\} = 0, \quad (4)$$

which form the SUSY algebra. We can see that Q and Q^\dagger are symmetry generators responsible for the degeneracy in the spectrum of H_1 and H_2 . Q and Q^\dagger are called the supercharge. Moreover, since Q and Q^\dagger are fermionic operators, they change a boson into a fermion and vice versa.

Now consider an eigenstate of H which can be written as $\Psi^n \equiv \begin{pmatrix} \psi_n^{(1)} \\ 0 \end{pmatrix}$ and $\Psi_n \equiv \begin{pmatrix} 0 \\ \psi_n^{(2)} \end{pmatrix}$.

The action of the supercharges are

$$Q\Psi^{n+1} = \begin{pmatrix} 0 \\ A\psi_n^{(1)} \end{pmatrix} \sim \Psi_n, \quad Q\Psi_n = 0, \quad Q^\dagger\Psi^n = 0, \quad \text{and } Q^\dagger\Psi_n = \Psi^{n+1}. \quad (5)$$

Since the supercharges are fermionic operators, one can think of Ψ^{n+1} and Ψ_n as having different fermion number. To make this concrete, one can define a fermion number operator. To do this, first note that the ground state of H is Ψ^0 . This state must have fermion number 0. Thus one can define the fermion number operator as $N_f = \text{diag}(0, 1)$. Hence all the eigenstates Ψ^n have fermion number 0 while the eigenstates Ψ_n have fermion number 1. Notice there is no eigenstate with fermion number greater than 1 because the anticommutation relation of the operator Q , see Eq. (4). We will return to this concept in Sec. 3.2.

3. Examples

3.1. Square Well

Let's consider a familiar infinite square well potential, $V_1(x) = -\pi^2/(2ma^2)$ for $0 \leq x \leq a$, and $V_1(x) = \infty$ otherwise. The spectrum of H_1 can be written as

$$E_n^{(1)} = \frac{n(n+2)\pi^2}{2ma^2}, \quad \psi_n^{(1)}(x) = \sqrt{\frac{2}{a}} \sin \frac{(n+1)\pi x}{a} \quad \text{for } 0 \leq x \leq a. \quad (6)$$

The superpotential, $W(x)$, and the potential, $V_2(x)$, can then be computed. One gets

$$W(x) = -\frac{1}{\sqrt{2m}} \frac{\pi}{a} \cot \frac{\pi x}{a}, \quad V_2(x) = \frac{\pi^2}{2ma^2} \left(2 \csc^2 \frac{\pi x}{a} - 1 \right). \quad (7)$$

Finally, one can compute the eigenstate of H_2 . For example

$$\psi_0^{(2)} \sim A\psi_1^{(1)} \sim \left(\frac{d}{dx} \sin \frac{2\pi x}{a} - \frac{\pi}{a} \cot \frac{\pi x}{a} \sin \frac{2\pi x}{a} \right) \sim \sin^2 \frac{\pi x}{a}. \quad (8)$$

The expression for the eigenstate of H_2 gets more complicated with increasing value of n :

$$\psi_1^{(2)} \sim \sin \frac{\pi x}{a} \sin \frac{2\pi x}{a}, \quad \psi_2^{(2)} \sim \sin^2 \frac{2\pi x}{a} + \sin^2 \frac{\pi x}{a} \cos \frac{2\pi x}{a}. \quad (9)$$

One can repeat the same exercise for the finite square well potential $V_1(x) = -V_0$ for $-a \leq x \leq a$, and $V_1(x) = 0$ otherwise. It's an elementary exercise to determine the ground state energy, $E_0^{(1)}$, and the ground state wavefunction, $\psi_0^{(1)}$. One can then shift the potential $V_1(x)$ so that the ground state energy become zero and apply the steps outlined at the beginning of this section to supersymmetrize the finite square well. Then from

$$\psi_0^{(1)}(x) \sim e^{\sqrt{-2mE_0^{(1)}}|x|} \text{ for } |x| > a, \quad \text{and } \psi_0^{(1)}(x) \sim \cos \left(\sqrt{2m(V_0 + E_0^{(1)})} x \right), \text{ for } |x| \leq a, \quad (10)$$

one determines the superpotential, $W(x)$, and the potential, $V_2(x)$, to be

$$W(x) = \begin{cases} -\sqrt{-E_0^{(1)}}, & x < -a, \\ \sqrt{V_0 + E_0^{(1)}} \tan \sqrt{2m(V_0 + E_0^{(1)})} x, & -a \leq x \leq a, \\ \sqrt{-E_0^{(1)}}, & x > a, \end{cases} \quad (11)$$

$$V_2(x) = -E_0^{(1)} \text{ for } |x| > a, \quad \text{and } V_2(x) = (V_0 + E_0^{(1)}) \left[2 \tan^2 \sqrt{2m(V_0 + E_0^{(1)})} x + 1 \right] \text{ for } |x| \leq a. \quad (12)$$

3.2. Harmonic Oscillator

For a harmonic oscillator, one can write a Hamiltonian as $H_1 = a_+ a_- \omega$ where a_+ is the creation operator and a_- is the annihilation operator. These operators satisfy the algebra $[a_-, a_+] = 1$ and $[a_+, a_+] = [a_-, a_-] = 0$. One can describe the eigenstate of H_1 in terms of the bosonic occupation number, n_b , as $H_1 |n_b\rangle = n_b \omega |n_b\rangle$. Notice that n_b is just an eigenvalue of the bosonic number operator $N_b = a_+ a_-$.

To supersymmetrize the harmonic oscillator, one starts from identifying $A^\dagger = \sqrt{\omega} a_+$ and $A = \sqrt{\omega} a_-$. This allows one to write the Hamiltonian H_2 as $H_2 = AA^\dagger = H_1 + \omega$. Then one can define the SUSY Hamiltonian and the supercharges as in Eq. (3). The eigenstate of H are Ψ^n and Ψ_n as defined in Sec. 2.1. One can label the eigenstate of the SUSY harmonic oscillator in terms of its boson number and fermion number, $|n_b, n_f\rangle$. For example, $|0, 0\rangle = \Psi^0$, $|1, 0\rangle = \Psi^1$, $|0, 1\rangle = \Psi_1$, etc. Hence one can see that $Q |n_b, n_f\rangle \sim |n_b - 1, n_f + 1\rangle$ and $Q^\dagger |n_b, n_f\rangle \sim |n_b + 1, n_f - 1\rangle$. This makes explicit the fact that a SUSY transformation turns a bosonic degree of freedom into a fermionic one and vice versa.

4. Discussion and Final Remarks

SUSY predicts that there should be a superpartner of the electron, selectron. The selectron has spin 0, electric charge $-e$ and mass m_e . However, such a particle hasn't been observed in nature. Thus if SUSY is to be a symmetry of nature, it must be spontaneously broken. In fact, SUSY QM was originally introduced as a toy model for studying SUSY breaking [10, 11] to avoid unnecessary complications of QFT. In addition to serving as a toy model for studying aspect of SUSY in QFT, SUSY QM generates a large areas of research activity of its own. Many of them are reviewed in Ref. [9].

In the context of QFT, SUSY has found many application in particle physics. In the minimal supersymmetric Standard Model (MSSM) of particle physics, every particle comes with a superpartner. In MSSM the lightest superpartner (LSP) is stable. If in addition the LSP is neutral, it can serve as a dark matter candidate. Moreover, interactions of the Higgs boson with superpartners cancel the quantum corrections to the Higgs boson mass from ordinary particles. In the absence of quantum corrections, the Higgs mass can naturally be at 125 GeV as measured by the ATLAS and CMS experiments [12, 13]. Lastly, the quantum corrections lead to renormalization [14, 15]. Renormalization implies the strength of the strong, the weak and the electromagnetic interactions change with energy scale. As it turns out, in the MSSM, renormalization unifies the three interaction at the high energy scale, dubbed the grand unification scale.

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