Born approximation in linear-time invariant system

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Abstract. An alternative way of finding the LTI's solution with the Born approximation, is investigated. We use Born approximation in the LTI and in the transformed LTI in form of Helmholtz equation. General solution are considered as infinite series or Feynman graph. Slow-roll approximation are explored. Transforming the LTI system into Helmholtz equation, approximated general solution can be found for any given forms of force with its initial value.

1. Introduction

Oscillations with time-independent properties are known as linear-time invariant (LTI) systems, e.g. spring and RLC circuits. With external force $F(t)$ (input signal), the DE becomes inhomogeneous and general solution is sum of complementary $y_{\rm c}$ and particular $y_{\rm p}$ solutions. For periodic $F(t)$, $y_{\rm p}$ is obtained with Fourier series and for arbitrary $F(t)$, $y_{\rm p}$ is found with Green's function (see [1]). Initial (boundary) conditions (I.C.), $F(t_0)$ must be known, since $F(t_0)$ is embedded in $y_{\rm p}$. We know Green's functions of the second-order LTI and the initial $F(t_0)$ can be setup. This enables us to find $y_{\rm p}$ for an arbitrary $F(t)$. In quantum mechanics, particle scattered by a potential is described with Schrödinger equation in form of Helmholtz equation. In LTI, there is a damping term which absences in Helmholtz equation. Both LTI and Helmholtz equations are second-order linear DE with source term. In the Helmholtz equation, the source term $Q(\mathbf{r}) = V(\mathbf{r})\psi(\mathbf{r})$ where $\psi(\mathbf{r})$ is the general solution. Hence $Q(\mathbf{r})$ is non-designable, unlike $F(t)$ of the LTI system, which can be setup. Solving Helmholtz equation is non-trivial since general solution $\psi(\mathbf{r})$ has $\psi(\mathbf{r}_0)$ in itself (see e.g. [2] or [3]). We use Born approximation to find particular solution perturbatively. Under the approximation, we do not need to know form of $\psi(\mathbf{r})$. We can approximate $\psi_{\rm c}(\mathbf{r}) \approx \psi(\mathbf{r})$ at boundary, for $\psi_{\rm c}(\mathbf{r})$ is not much altered by the source. We will explore this possibility in the LTI system.

2. Linear-Time Invariant system

Equation of motion can be viewed as $\hat{\mathcal{L}}y(t) = \mathcal{F}(t)$, with principle of superposition, $\hat{\mathcal{L}}\left[\sum_{i=1}^{N} c_i y_i(t)\right] = \sum_{i=1}^{N} c_i \mathcal{F}_i(t)$, for N solutions. Any solution $y(t)$ and inhomogeneous part $F(t)$ can be expressed as $y(t) = \sum_{i=1}^{N} c_i y_i(t)$, and $\mathcal{F}(t) = \sum_{i=1}^{N} c_i \mathcal{F}_i(t)$. The second-order system is $[{\rm d}^2/{\rm d}t^2 + (a_1/a_2){\rm d}/{\rm d}t + (a_0/a_2)]y(t) = \mathcal{F}(t)/a_2 \equiv F(t)$, where a_0, a_1, a_2 are time-independent properties. $a_1/a_2 \equiv 2\beta$ and $a_0/a_2 \equiv \omega_0^2$. Detail discussions referred to textbooks e.g. [4], [5] and [6]. The system is hence

$$\frac{{\rm d}^2 y}{{\rm d}t^2} + 2\beta\frac{{\rm d}y}{{\rm d}t} + \omega_0^2 y = F(t)\,. \tag{1}$$

This system has general solution, $y(t) = y_{\rm c}(t) + y_{\rm p}(t)$. The complementary solution $y_{\rm c}$ is a solution of homogeneous system ($F(t) = 0$) and the particular solution ($y_{\rm p}(t)$) is of the inhomogeneous case (non-zero $F(t)$). As well-known that for harmonic function, $F(t) = F_0\cos(\omega t - \phi)$, the particular solution, which is of steady-state, is $y_{\rm p}(t) = F_0[(\omega_0^2 - \omega^2)^2 + 4\omega^2\beta^2]^{-1/2}\cos(\omega t - \phi - \xi)$, with $\xi = \arctan\left[2\omega\beta/(\omega_0^2 - \omega^2)\right]$ and ϕ is the initial phase. For any periodic functions $F(t)$, Fourier series method can help finding $y_{\rm p}(t)$ but for arbitrary $F(t)$, Green's function method is applicable. When $F(t)$ is Dirac's delta function $\delta(t)$ as $[({\rm d}^2/{\rm d}t^2) + 2\beta({\rm d}/{\rm d}t) + \omega_0^2]G(t) = \delta(t)$, the general solution $y(t)$ is $G(t)$. Hence arbitrary force function is summation of impulse forces. The $y_{\rm p}(t)$ of an arbitrary $F(t)$ is therefore, $y_{\rm p}(t) = \int_{-\infty}^{t} G(t - t_0)F(t_0)\,{\rm d}t_0$, and the Green's function for the LTI system is,

$$G(t - t_0) = \frac{1}{\omega_{\rm d}} e^{-\beta(t-t_0)}\sin\left[\omega_{\rm d}(t - t_0)\right], \quad \text{for } \ t \geq t_0 \ \text{ otherwise zero.} \tag{2}$$

Here $\omega_{\rm d} \equiv (\omega_0^2 - \beta^2)^{1/2}$ (see e.g. [7]). This method is valid for any second-order LTI systems in form of (1).

3. Time-independent Schrödinger Equation: Helmholtz equation

3.1. Helmholtz equation

The Schrödinger equation, $-[\hbar^2/(2m)]\nabla^2\psi(\mathbf{r}) + V(\mathbf{r})\psi(\mathbf{r}) = E\psi(\mathbf{r})$, of non-relativistic quantum mechanics, with spatial-dependent wave function, $\psi = \psi(\mathbf{r})$, can be expressed as Helmholtz equation (see, e.g. [1], [2] or [3])

$$\left(\nabla^2 + k^2\right)\psi(\mathbf{r}) \ = \ \frac{2m}{\hbar^2}V(\mathbf{r})\psi(\mathbf{r}) \ \equiv \ Q(\mathbf{r})\,, \tag{3}$$

with $k^2 \equiv 2mE/\hbar^2$. The LTI system with $\beta = 0$ has similar to (3) but with spatial dependency instead of temporal dependency. If there is a response solution $G(\mathbf{r})$ to delta function $\delta^3(\mathbf{r})$ such that $\left(\nabla^2 + k^2\right)G(\mathbf{r}) = \delta^3(\mathbf{r})$, hence for an arbitrary inhomogeneous "source" $Q(\mathbf{r})$, the particular solution is $\psi_{\rm p}(\mathbf{r}) = \int_{-\infty}^{\mathbf{r}} G(\mathbf{r} - \mathbf{r}_0)Q(\mathbf{r}_0){\rm d}^3\mathbf{r}_0$. Green's function of the equation $\left(\nabla^2 + k^2\right)G(\mathbf{r}) = \delta^3(\mathbf{r})$, is well known, $G(\mathbf{r}) = -e^{ikr}/(4\pi r)$. When inhomogenous part is absence, i.e. $V(\mathbf{r}) = 0$, the Green's function is $G_0(\mathbf{r})$, hence $\left(\nabla^2 + k^2\right)G_0(\mathbf{r}) = 0$. Adding these two equations, we have $\left(\nabla^2 + k^2\right)[G(\mathbf{r}) + G_0(\mathbf{r})] = \delta^3(\mathbf{r})$. One can find $\psi_{\rm c}(\mathbf{r})$ and $\psi(\mathbf{r}) = \psi_{\rm c}(\mathbf{r}) + \psi_{\rm p}(\mathbf{r})$ of the system. The general solution of Eq. (3) is hence $\psi(\mathbf{r}) = \psi_{\rm c}(\mathbf{r}) + [-m/(2\pi\hbar^2)]\int_{-\infty}^{\mathbf{r}} (e^{ik|\mathbf{r}-\mathbf{r}_0|}/|\mathbf{r} - \mathbf{r}_0|)V(\mathbf{r}_0)\psi(\mathbf{r}_0)\,{\rm d}^3\mathbf{r}_0$. This is the integral form of the Schrödinger equation. $\psi_{\rm c}(\mathbf{r})$ is a plane wave $\psi_0(\mathbf{r})$ of an incoming particle to a massive point of scattering at $\mathbf{r} = \mathbf{r_0}$ with scattering potential $V(\mathbf{r}_0)$. After scattering, $\psi_{\rm p}(t)$ is a "response" wave function at far distance from the scattering point.

3.2. Born Approximation in quantum mechanics

$\psi_{\rm p}(t)$ can be analyzed perturbatively with well-known Born approximation (see e.g. [2] and [3]). Let $g(\mathbf{r}) \equiv -[m/(2\pi\hbar^2)](e^{ikr}/r)$, Therefore $\psi(\mathbf{r}) = \psi_0(\mathbf{r}) + \int_{-\infty}^{\mathbf{r}} g(\mathbf{r}, \mathbf{r}_0)V(\mathbf{r}_0)\psi(\mathbf{r}_0)\,{\rm d}^3\mathbf{r}_0$, where $g(\mathbf{r}, \mathbf{r}_0) \equiv g(\mathbf{r} - \mathbf{r}_0)$. At $\mathbf{r}_0$ the incoming plane wave is approximately not much affected by the potential, i.e. $\psi(\mathbf{r}_0) \approx \psi_0(\mathbf{r}_0)$, hence it is approximated that $\psi(\mathbf{r}) \approx \psi_0(\mathbf{r}) + \int_{-\infty}^{\mathbf{r}} g(\mathbf{r}, \mathbf{r}_0)V(\mathbf{r}_0)\psi_0(\mathbf{r}_0)\,{\rm d}^3\mathbf{r}_0$. Considering $\psi_0(\mathbf{r}_0)$ as a scattered wave from $\mathbf{r}_{00}$ with incoming wave $\psi_{00}(\mathbf{r_0})$, hence $\psi_0(\mathbf{r}_0) = \psi_{00}(\mathbf{r_0}) + \int_{-\infty}^{\mathbf{r}_0} g(\mathbf{r}_0, \mathbf{r}_{00})V(\mathbf{r}_{00})\psi_0(\mathbf{r}_{00})\,{\rm d}^3\mathbf{r}_{00}$. The plane wave was scattered once at $\mathbf{r_{00}}$ by $V(\mathbf{r}_{00})$ before arriving at $\mathbf{r_0}$. Using this expression of $\psi_0(\mathbf{r}_0)$ in $\psi(\mathbf{r})$, we obtain $\psi(\mathbf{r})$ as $\psi(\mathbf{r}) \approx \psi_0(\mathbf{r}) + \int_{-\infty}^{\mathbf{r}} g(\mathbf{r}, \mathbf{r}_0)V(\mathbf{r}_0)\psi_{00}(\mathbf{r}_0){\rm d}^3\mathbf{r}_0 +$ third term. Under Born approximation, $\psi_{00}(\mathbf{r}_0) \approx \psi_0(\mathbf{r}_0)$, hence at second order,

$$\psi(\mathbf{r}) \ \approx \ \psi_0(\mathbf{r}) \ + \int_{-\infty}^{\mathbf{r}} g(\mathbf{r}, \mathbf{r}_0)V(\mathbf{r}_0)\psi_0(\mathbf{r}_0)\,{\rm d}^3\mathbf{r}_0$$

$$+\int_{-\infty}^{\mathbf{r}}\int_{-\infty}^{\mathbf{r}_0}\Big[g(\mathbf{r},\mathbf{r}_0)V(\mathbf{r}_0)\Big]\Big[g(\mathbf{r}_0,\mathbf{r}_{00})V(\mathbf{r}_{00})\Big]\psi_0(\mathbf{r}_{00})\,\mathrm{d}^3\mathbf{r}_{00}\,\mathrm{d}^3\mathbf{r}_0\,. \tag{4}$$

$\psi_0(\mathbf{r})$ is a plane wave. The second term implies ψ_0 scattered at $\mathbf{r}_0$. The third term represents incoming plane wave ψ_0 scattered twice, at $\mathbf{r}_{00}$ and later at $\mathbf{r}_0$. This makes infinite Born series - the Feynman graphs.

4. Born approximation for LTI system: estimated Helmholtz equation

Considering $F(t)$ as a product of function $f(t)$ and general solution $y(t)$, $F(t) \equiv f(t)y(t)$. The $f(t)$ will have the same role as $V(\mathbf{r})$ in quantum mechanics, i.e. $f(t)$ represents external influence on the LTI system in similar manner as $V(\mathbf{r})$ in scattering problem. One can view that $F(t)$ is a displacement $y(t)$ modulated with $f(t)$. The LTI system hence is rewritten as Schrödinger-like equation but with an extra term $-2\beta(\mathrm{d}y/\mathrm{d}t)$. This is, $-(\mathrm{d}^2y/\mathrm{d}t^2) - 2\beta(\mathrm{d}y/\mathrm{d}t) + fy = \omega_0^2 y$. Analogous quantities are, $-(\mathrm{d}^2y/\mathrm{d}t^2) \Leftrightarrow -[\hbar^2/(2m)]\nabla^2\psi(\mathbf{r})$, $f(t)y(t) \Leftrightarrow V(\mathbf{r})\,\psi(\mathbf{r})$ and $\omega_0^2\,y(t) \Leftrightarrow E\,\psi(\mathbf{r})$. In the limit of $\beta \to 0$, the LTI and Schrödinger equations (as Helmholtz equation) are estimably analogous. At time $t \geq t_0$, system is under influence of $F(t)$. General solution is hence $y(t) = y_0(t) + \int_{-\infty}^{t} G(t,t_0)f(t_0)y(t_0)\mathrm{d}t_0$, where $y_\mathrm{c}(t)$ renamed to $y_0(t)$. Born approximation for the LTI case is $y(t_0) \approx y_0(t_0)$, implying that at time $t = t_0$ the complementary solution is not much altered. To second order, the Born series for LTI system is

$$\begin{aligned} y(t) \quad &\approx \quad y_0(t) \;+\; \int_{-\infty}^{t} G(t,t_0)f(t_0)y_0(t_0)\,\mathrm{d}t_0 \\ &\quad + \int_{-\infty}^{t}\int_{-\infty}^{t_0}\Big[G(t,t_0)f(t_0)\Big]\Big[G(t_0,t_{00})f(t_{00})\Big]y_0(t_{00})\,\mathrm{d}t_{00}\,\mathrm{d}t_0 \end{aligned} \tag{5}$$

The Feynman graphs are a series of straight line in Fig. 1. Consider harmonic driving force, $F = F_0 e^{i\omega t}$, the y_c and y_p are known. When it is homogenous ($F = 0$), complementary solution is $y_0(t) = Ae^{pt}$, where $p = -\beta \pm (\beta^2 - \omega_0^2)^{1/2}$. Born approximation is $y(t_0) \approx y_0(t_0)$. Damping needs to be small to estimate LTI as Helmholtz equation. Hence it works only for light damping, $\beta < \omega_0$, i.e. $p = -\beta + i\,\omega_\mathrm{d}$, choosing, $\omega_\mathrm{d} = (\omega_0^2 - \beta^2)^{1/2}$. $y_0(t) = Ae^{-\beta t + i\,\omega_\mathrm{d} t}$, where $A = \exp(\phi_0)$. With Born approximation, $f(t_0) = F(t_0)/y(t_0) \approx F(t_0)/y_0(t_0) = (F_0/A)e^{\beta t_0}\,e^{i(\omega - \omega_\mathrm{d})t_0}$. Hence the solution is Born series, approximated to second order,

$$\begin{aligned} y(t) \approx y_0(t) \;+\; &\int_{-\infty}^{t}\left(\frac{F_0}{\omega_\mathrm{d}}\right) e^{-\beta(t-t_0)}\Big\{\sin\left[\omega_\mathrm{d}(t-t_0)\right]e^{i\omega t_0}\Big\}\,\mathrm{d}t_0 \\ + \int_{-\infty}^{t}\int_{-\infty}^{t_0}\frac{1}{A}\left(\frac{F_0}{\omega_\mathrm{d}}\right)^2 &e^{-\beta(t-t_0-t_{00})}\Big\{\sin\left[\omega_\mathrm{d}(t-t_0)\right]\sin\left[\omega_\mathrm{d}(t_0-t_{00})\right]e^{i[(\omega-\omega_\mathrm{d})t_0+\omega t_{00}]}\Big\}\mathrm{d}t_{00}\mathrm{d}t_0,. \end{aligned} \tag{6}$$

First order shows transient "beats". Higher terms represent more complex modes with less contributions.

5. Born and slow-roll approximations in LTI system

For the system in Eq. (1) under harmonic driving force, complementary solution is $y_0(t) = Ae^{pt}$. If the LTI system initially moves very slow under very small initial magnitude of force. At t_0, the complementary solution is not much altered, hence Born approximation, $y(t_0) \approx y_0(t_0)$ is valid. Applying *slow-roll* approximation, $\ddot{y}(t_0) \approx 0$ to the LTI system in Eq. (1), hence $2\beta Ae^{pt_0}p + \omega_0^2 Ae^{pt_0} \approx F_0 e^{i\omega t_0}$, given a condition $\omega_\mathrm{d}^2 + \beta^2 \approx (F_0/A)e^{\beta t_0}$, which depends much on the initial phase $A = e^{\phi}$. For light damping case, $\omega_\mathrm{d}^2 = \omega_0^2 - \beta^2$ hence the initial time for both approximations to be valid is $t_0 \approx \beta^{-1}\ln\left(A\omega_0^2/F_0\right)$. If the damping is critical, $\omega_0 = \beta$, i.e. $\omega_\mathrm{d} = 0$, hence $t_0 \approx \beta^{-1}\ln\left(A\beta_0^2/F_0\right)$. For heavy damping $\beta > \omega_0$, therefore $\omega_\mathrm{d} = i(\beta^2 - \omega_0^2)^{1/2}$. Using $\omega_\mathrm{d}^2 + \beta^2 \approx (F_0/A)e^{\beta t_0}$, we obtain the same t_0 as of light damping case.

6. Transformation of the LTI system to the Helmholtz equation

LTI system, e.g. series RLC, has been found that, under a transformation, $i(t) = I(t)\,e^{-tR/2L} = I(t)\,e^{-\beta t}$, can be transformed into the Helmholtz equation. As a result, Fourier and Laplace transforms can be applied to derive transient solution with a setup initial conditions (Sumichrast (2012) [8]). Considering LTI system (1), under transformation $y \equiv \tilde{y}e^{-\beta t}$, the Eq. (1) becomes 1-dim. Helmholtz equation, $\left[(\mathrm{d}^2/\mathrm{d}t^2) + \omega_\mathrm{d}^2\right]\tilde{y} = F(t) \equiv \tilde{f}(t)\tilde{y}(t)$ where $\omega_\mathrm{d}^2 = \omega_0^2 - \beta^2$. Unlike Sec. 4, here the Schrödinger-like expression of the LTI does not contain damping term, $-(\mathrm{d}^2\tilde{y}/\mathrm{d}t^2) + \tilde{f}(t)\tilde{y}(t) = \omega_\mathrm{d}^2\tilde{y}$. Green's function of the system is solution of $[(\mathrm{d}^2/\mathrm{d}t^2) + \omega_\mathrm{d}^2]G(t) = \delta(t)$, and it is $G(t, t_0) = -ie^{i\omega_\mathrm{d}(t-t_0)}/(2\omega_\mathrm{d})$. Hence the general solution is $\tilde{y}(t) = \tilde{y}_\mathrm{c}(t) + \int_{-\infty}^{t} G(t, t_0)\tilde{f}(t_0)\tilde{y}(t_0)\mathrm{d}t_0$. With harmonic driving force $F(t) = F_0e^{i\omega t}$, complementary solution is $\tilde{y}_0(t) = Be^{i\omega_\mathrm{d}t}$. From $F(t) = \tilde{f}(t)\tilde{y}(t)$, using Born approximation $\tilde{y}(t_0) \approx \tilde{y}_0(t_0)$, hence $F(t_0) \approx \tilde{f}(t_0)\tilde{y}_0(t_0)$ and $\tilde{f}(t_0) = (F_0/B)e^{i(\omega-\omega_\mathrm{d})t_0}$. The Born series for the LTI system under harmonic driving force is written at second order as

$$\begin{aligned} \tilde{y}(t) \quad &\approx \quad \tilde{y}_0(t) + \int_{-\infty}^{t} \left(\frac{-iF_0}{2\omega_\mathrm{d}}\right)\left[e^{i\omega_\mathrm{d}(t-t_0)+i\omega t_0}\right]\mathrm{d}t_0 \\ &\quad + \int_{-\infty}^{t}\int_{-\infty}^{t_0}\left(\frac{-iF_0}{2\omega_\mathrm{d}}\right)^2\left(\frac{1}{B}\right)\left[e^{i\omega_\mathrm{d}(t-t_0-t_{00})}\,e^{i\omega(t_0+t_{00})}\right]\mathrm{d}t_{00}\,\mathrm{d}t_0\,. \end{aligned} \tag{7}$$

One only needs to know $F(t)$ and its initial value, then find $\tilde{f}(t_0)$ and approximate $\tilde{y}(t_0) \approx \tilde{y}_0(t_0)$.

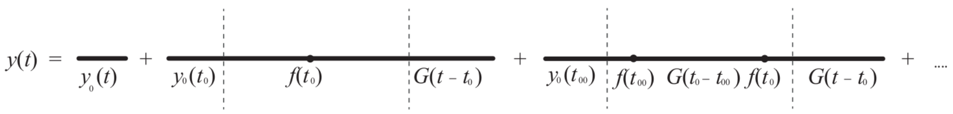

Figure 1. Feynman graphs for Born series of a second order LTI system

7. Conclusion

Transforming LTI to Helmholtz equation enables us to use Born approximation of finding general solution in the Helmholtz case, in LTI case. Born condition is that initial value of force is known and is not large so that y_c at beginning is not much altered. Initial time for the validity of Born and slow-roll approximations is derived. Directly applying Born approximation to LTI system, it is valid only for light damping limit. Transforming LTI system into Helmholtz form before applying Born approximation can avoid the limit. If any $F(t)$ and $F(t_0)$ are given, general solution can be found with Born approximation.

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References

[1] Barton G 1995 *Elements of Green's Functions and Propagation* (Oxford: Clarendon Press)
[2] Griffiths D J 1994 *Introduction to Quantum Mechanics* (Englewood Cliffs, NJ: Prentice-Hall)
[3] Schiff L I 1968 *Quantum Mechanics* 3rd ed (Singapore: McGraw-Hill)
[4] Graham Kelly S 2009 *System Dynamics and Response* (Stamford, CT: Cengage Learning)
[5] Cha P D and Molinder J I 2006 *Fundamentals of Signals and Systems* (Cambridge: CUP)
[6] Main I G 1998 *Vibrations and Waves in Physics* 3rd ed (Cambridge: CUP)
[7] Kibble T W B and Berkshire F H 1996 *Classical Mechanics* 4th ed (Harlow, Longman)
[8] Sumichrast L 2012 Unified approach to the impulse response and Green function in circuit and field theory, Part I: one-dimensional case *J. Elect. Eng.* **63** 273