The background evolution in general disformal gravity

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Abstract. We study a subclass of GLPV modified gravity theories generated by a general disformal metric $\bar{g}_{\mu\nu} = g_{\mu\nu} + D(\phi, \partial\phi)\partial_\mu\phi\partial_\nu\phi$. By analyzing the background evolution equation, we have found that such theory cannot provide the self-accelerating solution.

1. Introduction

The disformal transformation for gravity has been introduced by Bekenstein in 1992 [1]. It is the most general mapping between the metric involving one scalar field and preserve diffeomorphisms of spacetime. This transformation can be written as

$$\bar{g}_{\mu\nu} = C(\phi, Y)g_{\mu\nu} + D(\phi, Y)\partial_\mu\phi\partial_\nu\phi \ , \tag{1}$$

where $Y \equiv g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi$ a kinetic term of the scalar field ϕ. C, D are arbitrary functions of the scalar field and its kinetic terms.

These called the conformal and disformal factor, respectively. In order to study the influence of the disformal factor on the evolution of background universe, we will set $C(\phi, Y) = 1$ in this paper. We call this transformation the general disformal because the disformal factor depends also on the kinetic terms of the scalar field beside a single scalar field.

2. The disformal gravity action

The disformal gravity action is induced from disformal transformation of the metric in the Einstein-Hilbert action ,

$$S_{disf}[g_{\mu\nu}] = S_{EH}[\bar{g}_{\mu\nu}] = \frac{1}{2\kappa}\int d^4x\sqrt{-\bar{g}}R(\bar{g}_{\mu\nu}) \ , \tag{2}$$

where we have set $c = 1$ and $\kappa = 8\pi G$. From disformal metric transformation $g_{\mu\nu} \to g_{\mu\nu} + D(\phi, Y)\phi_\mu\phi_\nu$, $(\phi_\mu \equiv \nabla_\mu\phi = \partial_\mu\phi)$, we can directly obtained the basic quantities such as the inverse and the determinant of disformal metric necessary for deriving the the disformal action, which are respectively read

$$\bar{g}^{\mu\nu} = g^{\mu\nu} - \gamma^2 D(\phi, Y)\phi^\mu\phi^\nu \ , \quad \bar{g} = g(1 + D(\phi, Y)Y) \equiv g/\gamma^2 \ , \tag{3}$$

where $\gamma^2 = 1/(1 + D(\phi, Y)Y)$ and $\phi^\mu \equiv \nabla^\mu\phi = g^{\mu\nu}\partial_\nu\phi$. We then need to know the transformation of the Levi-Civita connection (defined by $\overline{\nabla}_\alpha\overline{g}_{\mu\nu} = 0$):

$$\overline{\Gamma}^\alpha_{\mu\nu} = \Gamma^\alpha_{\mu\nu} + K^\alpha_{\mu\nu} \,, \tag{4}$$

where $K^\alpha_{\mu\nu} \equiv \frac{1}{2}\overline{g}^{\alpha\beta}(\nabla_\mu\overline{g}_{\nu\beta} + \nabla_\nu\overline{g}_{\mu\beta} - \nabla_\beta\overline{g}_{\mu\nu})$. The Riemann tensor ($\overline{R}^\alpha{}_{\beta\mu\nu} \equiv R^\alpha{}_{\beta\mu\nu}(\overline{g}_{\rho\sigma})$) then can be computed from

$$\overline{R}^\alpha{}_{\beta\mu\nu} = R^\alpha{}_{\beta\mu\nu} + 2\nabla_{[\mu}K^\alpha_{\nu]\beta} + 2K^\alpha_{\gamma[\mu}K^\gamma_{\nu]\beta} \quad . \tag{5}$$

Consequently, the Ricci scalar now given by $\overline{R} \equiv \overline{g}^{\mu\nu}\overline{R}_{\mu\nu}$. After a somewhat tedious but straightforward calculation, we obtain

$$\overline{R} = R - \gamma^2 D\big(\Box\phi^2 - \phi_{\mu\nu}\phi^{\mu\nu}\big) + \gamma^2\Big[\nabla_\mu D\phi_\alpha\phi^{\alpha\mu} - \nabla_\nu D\,\Box\phi\,\phi^\nu\Big] \,, \tag{6}$$

where $\Box \equiv \nabla^\mu\nabla_\mu = g^{\mu\nu}\phi_\mu\phi_\nu$ and $\phi_{\mu\nu} \equiv \nabla_\nu\nabla_\mu\phi$. The disformal action then reads (setting $2\kappa = 1$)

$$S = \int d^4x\sqrt{-g}\frac{\overline{R}}{\gamma} = \int d^4x\sqrt{-g}\left\{\frac{R}{\gamma} - \gamma D\big(\Box\phi^2 - \phi_{\mu\nu}\phi^{\mu\nu}\big) + \gamma\left[\nabla_\mu D\phi_\alpha\phi^{\alpha\mu} - \nabla_\nu D\,\Box\phi\,\phi^\nu\right]\right\}.$$

This action can be recast into the form of covariant GLPV action [2]: $S = \int d^4x\sqrt{-g}\sum_{i=2}^{5}\mathcal{L}_i$, where

$$\mathcal{L}_3 = \big(C_3 + 2Y\,C_{3Y}\big)\Box\phi \,, \tag{7}$$

$$\mathcal{L}_4 = \big(B_4 + C_5\big)\,R - 2\big(B_4 + C_5\big)_Y(\Box\phi^2 - \phi^2_{\mu\nu}) + F_4\epsilon^{\mu\nu\rho\sigma}\epsilon_{\mu\beta\gamma\delta}\phi_\nu\phi^\beta\phi^\gamma_\rho\phi^\delta_\sigma \,, \tag{8}$$

$$\mathcal{L}_5 = \tilde{G}_5\,G_{\mu\nu}\phi^{\mu\nu} + \frac{1}{3}\tilde{G}_{5Y}\Big(\Box\phi^3 - 3\Box\phi\,\phi^2_{\mu\nu} + 2\,\phi^3_{\mu\nu}\Big) + F_5\epsilon^{\mu\nu\rho\sigma}\epsilon_{\alpha\beta\gamma\delta}\phi^\alpha\phi_\mu\phi^\beta_\nu\phi^\gamma_\rho\phi^\delta_\sigma \,, \tag{9}$$

where $\epsilon^{\mu\nu\rho\sigma}$ is a Levi-Civita pseudotensor. In our case, it can be shown that $B_4 = 1/\gamma$, $C_3 = -\frac{1}{2}\int\gamma D_\phi dY$, $A_2 = -\frac{1}{2}D_\phi Y^2 - YC_{3\phi}$, $A_4 = \gamma DY - \frac{1}{\gamma}$, $C_5 = \tilde{G}_5 = F_5 = 0$, $F_4 \equiv Y^{-2}(B_4 + A_4 - 2YB_{4Y}) = -\gamma D_Y$.

3. The dynamics of background spacetime

For simplicity we will study the disformal gravity obtained from the disformal transformation of FLRW metric

$$ds^2 = -N^2(t)dt^2 + a^2(t)\delta_{ij}dx^i dx^j \,. \tag{10}$$

From this metric we can obtained the non-zero components of Levi-Civita connection. They read

$$\Gamma^0_{00} = \frac{\dot{N}}{N},\; \Gamma^0_{ij} = \frac{a^2H}{N^2}\delta_{ij},\; \Gamma^i_{0j} = H\delta^i_j \,. \tag{11}$$

We then consequently obtained

$$R = 6\left(\frac{\ddot{a}}{N^2a} + \frac{\dot{a}^2}{N^2a^2} - \frac{\dot{a}\dot{N}}{a\,N^3}\right) \,,\; \Box\phi = -\frac{\ddot{\phi}}{N^2} + \frac{\dot{N}\dot{\phi}}{N^3} - \frac{3H\dot{\phi}}{N^2} \,,\; Y = -\frac{\dot{\phi}^2}{N^2} \,,$$

$$\phi_{\mu\nu}\phi^{\mu\nu} = \frac{\ddot{\phi}^2}{N^4} - \frac{2\dot{N}\dot{\phi}\ddot{\phi}}{N^5} + \frac{\dot{N}^2\dot{\phi}^2}{N^6} + \frac{9\dot{\phi}^2H^2}{N^4} \,,\; (\Box\phi^2 - \phi_{\mu\nu}\phi^{\mu\nu}) = \frac{6H\dot{\phi}\ddot{\phi}}{N^4} - \frac{6H\dot{N}\dot{\phi}^2}{N^5} \,,$$

$$
\begin{aligned}
&\phi_{\mu\nu}\phi^\mu\phi^\nu = \frac{\ddot\phi\dot\phi^2}{N^4} - \frac{\dot N\dot\phi^3}{N^5}\ , \ \phi^\mu\phi_{\mu\alpha}\phi^{\alpha\beta}\phi_\beta = -\frac{1}{N^6}\Big\{\ddot\phi^2\dot\phi^2 + 2\ddot\phi\dot\phi^3\frac{\dot N}{N} + \frac{\dot N^2}{N^2}\dot\phi^4\Big\}\ ,\\
&\phi_{\beta\nu}\phi^\beta\phi^\nu\,\Box\phi = -\frac{\ddot\phi^2\dot\phi^2}{N^6} - \frac{3H}{N^6}\ddot\phi\dot\phi^3 + \frac{\dot N^2\dot\phi^4}{N^8} - \frac{3H\dot\phi^4\dot N}{N^7}\ ,\\
&\Big(\phi_{\beta\nu}\phi^\beta\phi^\nu\,\Box\phi - \phi^\mu\phi_{\mu\alpha}\phi^{\alpha\beta}\phi_\beta\Big) = -\frac{3H}{N^6}\ddot\phi\dot\phi^3 + \frac{2\dot N^2\dot\phi^4}{N^8} - \frac{3H\dot\phi^4\dot N}{N^7} + 2\ddot\phi\dot\phi^3\frac{\dot N}{N^7}\ .
\end{aligned}
$$

The disformal Lagrangian now can be written as

$$
\begin{aligned}
\mathcal{L} = {}& 6\sqrt{1-\frac{D}{N^2}\dot\phi^2}\left(\frac{\ddot a}{N^2 a} + \frac{\dot a^2}{N^2a^2} - \frac{\dot a\dot N}{a\,N^3}\right) - \frac{D}{\sqrt{1-\frac{D}{N^2}\dot\phi^2}}\left(\frac{6H\dot\phi\ddot\phi}{N^4} - \frac{6H\dot N\dot\phi^2}{N^5}\right) \qquad (12)\\
&- \frac{2D_y}{\sqrt{1-\frac{D}{N^2}\dot\phi^2}}\left(-\frac{3H}{N^6}\ddot\phi\dot\phi^3 + \frac{2\dot N^2\dot\phi^4}{N^8} - \frac{3H\dot\phi^4\dot N}{N^7} + 2\ddot\phi\dot\phi^3\frac{\dot N}{N^7}\right)\\
&+ \frac{D_\phi\dot\phi^2}{N^2\sqrt{1-\frac{D}{N^2}\dot\phi^2}}\left(-\frac{\ddot\phi}{N^2} + \frac{\dot N\dot\phi}{N^3} - \frac{3H\dot\phi}{N^2}\right) + A_2 + Y\,C_{3\phi} + C_3\left(-\frac{\ddot\phi}{N^2} + \frac{\dot N\dot\phi}{N^3} - \frac{3H\dot\phi}{N^2}\right),
\end{aligned}
$$

where the first two lines are $\mathcal{L}_4$ and the third line is $\mathcal{L}_2 + \mathcal{L}_3$. The evolution equations of the background spacetime can be obtained from varying the action with respect to the metric. In this case we must vary the action with respect to $N(t)$ and $a(t)$. For simplicity, we will set $N(t) = 1, \dot N(t) = 0$ after calculate the Euler-Lagrange equations

$$
\frac{\partial\tilde{\mathcal{L}}}{\partial N} - \frac{d}{dt}\left(\frac{\partial\tilde{\mathcal{L}}}{\partial\dot N}\right) = 0\ ,\ \ \frac{\partial\tilde{\mathcal{L}}}{\partial a} - \frac{d}{dt}\left(\frac{\partial\tilde{\mathcal{L}}}{\partial\dot a}\right) + \frac{d^2}{dt^2}\left(\frac{\partial\tilde{\mathcal{L}}}{\partial\ddot a}\right) = 0\ , \qquad (13)
$$

where $\tilde{\mathcal{L}} := \sqrt{-g}\mathcal{L} \equiv Na^3\mathcal{L}$, we respectively obtain

$$
\begin{aligned}
0 &= \big(A_2 - 2YA_{2,\mathrm{Y}}\big) - \rho_m + 3H^2\gamma\frac{1-Y^2D_{,Y}}{1+DY}\ , &(14)\\
0 &= H\gamma^3\dot\phi\big(D_{,\phi}Y - 2\big(D+YD_{,Y}\big)\ddot\phi\big) + \gamma\big(2\frac{\ddot a}{a} + H^2\big) + A_2 + p_m\ , &(15)
\end{aligned}
$$

where $H = \dot a/a$ is the Hubble parameter, ρ_m and p_m are the energy density and pressure of matter respectively, they are given by $\rho_m = T_{00} = -\frac{1}{a^3}\frac{\delta\tilde{\mathcal{L}}_m}{\delta N}$, $p_m\delta_{ij} = T_{ij} = \frac{\delta\tilde{\mathcal{L}}_m}{\delta a}\delta_{ij}$. Since the disformal gravity considered in this work is a sub class of the GLPV theory which is the covariantized Galileon theory [3], we check whether the acceleration of the universe can be driven by the kinetic terms of scalar field as in the Galileon theory[4, 5]. In the flat FLRW background, we have $\gamma = (1 - D\dot\phi^2)^{-1/2}$, so that $D\dot\phi^2$ should lie within the range $(-\infty, 1)$ or $(1, \infty)$. However, it follows from the above equations that γ should be unity during matter dominated epoch and should be larger than unity during the acceleration of the universe. Hence, $0 \le D\dot\phi^2 < 1$ throughout the evolution of the universe. Since $D\dot\phi^2$ is always less than unity, the main contribution in the above equation that can make $\ddot a > 0$ is expected to be proportional to $\rho_\phi/3 + A_2$, where $\rho_\phi \equiv 2YA_{2,\mathrm{Y}} - A_2$. From this rough analysis, we expect that for the disformal gravity considered here, the accelerated expansion of the universe cannot be driven by kinetic terms of the scalar field. To confirm this analysis we solve the equations of motion for the background universe numerically. By varying the action with respect to ϕ we obtain the equation of motion for ϕ

$$
0 = \ddot\phi\big[A_{2,\mathrm{Y}} + 2YA_{2,\mathrm{YY}} + \frac{3}{2}H^2\gamma^5\big[D\big(1 - Y^2D_{,Y} + 2Y^3D_{,YY}\big) - 2YD^2\big]\big]
$$

$$+Y\left(5D_{,Y}-3Y^2D_{,Y}^2+2YD_{,YY}\right)+3H\dot{\phi}\bigg(A_{2,\mathrm{Y}}-\gamma^3Y\left(D+YD_{,Y}\right)\left(\frac{1}{2}H^2+\frac{\ddot{a}}{a}\right)\bigg)$$

$$+\frac{1}{2}\bigg(A_{2,\phi}-2YA_{2,\mathrm{Y}\phi}+\frac{3}{2}H^2\gamma^3\left[3Y^2D_{,\phi}\frac{D+YD_{,Y}}{1+DY}-2Y^2D_{,\phi Y}-YD_{,\phi}\right]\bigg)\,. \tag{16}$$

For concreteness, we choose the disformal coupling and A_2 as

$$D\equiv M^{-4\lambda_2-4}\mathrm{e}^{-\lambda_1\phi}(-Y)^{\lambda_2}\,,\;A_2\equiv\frac{1}{2}M_k^{4-4\lambda_3}(-Y)^{\lambda_3}-M_v^4\mathrm{e}^{-\lambda_4\phi}\,. \tag{17}$$

Here, M, M_k and M_v are the constant parameter with dimension of mass, while λ_1, λ_2, λ_3 and λ_4 are the dimensionless constant parameters. My setting $M^2=M_k^2=M_v^2=H_0$ where H_0 is

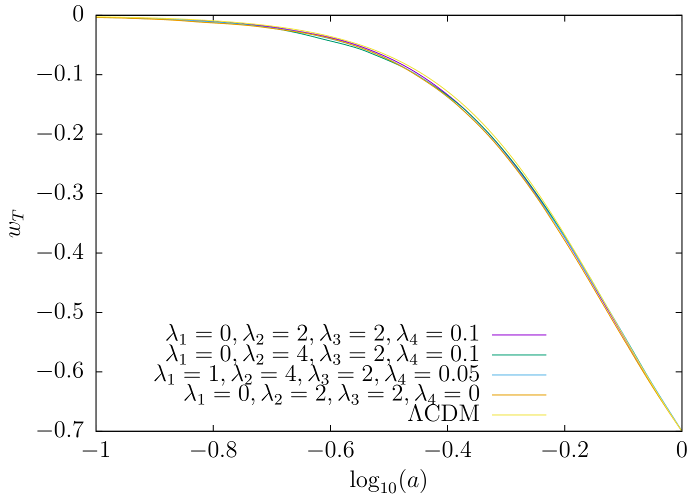

Figure 1. The equation of state parameter w_T as a function of $\log_{10}a$ for various values of λ_1, λ_2, λ_3 and λ_4.

the present value of the Hubble parameter, we have found that the acceleration of the universe at late time can occur only if scalar field ϕ slowly evolves, i.e. $\dot{\phi}\ll H$. Hence, the accelerated expansion of the universe is driven by the potential terms rather than the kinetic terms of the scalar field. As a result the evolution of $w_T\equiv-2\dot{H}/(3H^2)-1$ always mimics the evolution of w_T for ΛCDM model as shown in figure 1.

From this analysis, we conclude that for the disformal gravity considered here, the accelerated expansion of the universe cannot be driven by kinetic terms of the scalar field.

4. Conclusion

In this work we have studied the gravity theory generated by general disformal transformation which can be shown that it fits into the class of GLPV theories. By analyzing the background evolution equations we have found that this theory does not provide the self-accelerating solution as generally expected from GLPV theories.

References

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