Bounds on Axion Stars from Stability and Decay

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with
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For more information, see arXiv: 1412.3430, 1512.01709

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Axions are a well-motivated scalar dark matter candidate, show up in QCD and elsewhere.

Boson Stars are gravitationally bound, macroscopic condensates of scalar fields (e.g. axions).

Stability of axion stars can be analyzed through gravity, tunneling, and decay.

Open Questions and Future Work.
Axions Can Solve Two Problems

The nature of Dark Matter (DM) remains an open question

Image Credit: Symmetry Magazine
Axions Can Solve Two Problems

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The amount of $\mathcal{CP}$ observed in the QCD sector is “unnaturally” small, e.g. the neutron EDM

$$d_n \lesssim 2.9 \times 10^{-26} \text{e} \cdot \text{cm}$$

implies that

$$\theta_{QCD} \lesssim 10^{-10}$$

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(Model Dependent) \hspace{1cm} (Requires $m f_a \approx 6 \times 10^{-3} \text{ GeV}^2$)

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Observations about Boson Stars

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Important properties:
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- Typically noncompact: Wavefunctions extend to infinity
  - Define “size” $R_{99}$ inside which .99 of the mass is contained
- **Mass is bounded above** generically by gravitational stability
Historical Interlude

  - Second-quantized a free real scalar field, put all $N$ particles in ground state $|N\rangle$
  - Took expectation value of Einstein+Klein Gordon equations in ground state
  - Maximum mass for bound states: $M_{\text{max}}^{\text{free}} = 0.633 M_P^2 / m$
    Chandrasekhar limit: $M_{\text{max}} \sim M_P^3 / m^2$
Historical Interlude

  - Axion potential: \( V(\mathcal{A}) = m^2 f_a^2 \left( 1 - \cos(\mathcal{A}/f_a) \right) \approx \frac{m^2}{2} \mathcal{A}^2 + ... \)
  - Used RB method to quantize field, found numerical solutions with sizes \( R \sim 1 - 10 \text{ m} \) and masses \( M \sim 10^{13} - 10^{14} \text{ kg} \).
Historical Interlude

  - A simple variational calculation suggested to us that axion stars could have larger masses, of $O(10^{19})$ kg
  - Expanding on the work of RB and BB, we developed an expansion method which simplifies the equations of motion
  - We emphasize that the RB method is based on a Born approximation and is applicable only in the weak binding limit, where pair production and loop effects are negligible
Consider in greater detail the RB method:

1. Begin with a canonically normalized second-quantized scalar field

\[ A(r, \theta, \phi) = \sum_{n,l,m} R_{n,l}(r) \left[ e^{iE_{n,l} t} Y_l^m(\theta, \phi) a_{n,l,m} + h.c. \right] \]
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2. Build N-particle ground state

\[ |N\rangle = \left( \frac{a_{1,0,0}^\dagger}{\sqrt{N!}} \right)^N |0\rangle \]
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3. Evaluate expectation value of EKG equations

\[ \langle N | G^{\mu \nu} | N \rangle = \frac{1}{M_{P}^{2}} \langle N | T^{\mu \nu} | N \rangle \]

\[ \langle N - 1 | \left[ \Box \mathcal{A} - \frac{1}{2} W'(\mathcal{A}) \right] | N \rangle = 0 \]
Axion Stars in the Weak Binding Limit

The axion field theory can be defined by a low-energy potential, whose expectation value has the form

\[ V(\mathcal{A}) = m^2 f_a^2 \langle N | 1 - \cos(\mathcal{A}/f_a) | N \rangle \]

\[ = m^2 f_a^2 \left[ 1 - J_0 \left( 2\sqrt{NR(r)}/f_a \right) \right] \]
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The field perturbs the metric in a spherically symmetric way:

\[ ds^2 = B(r) dt^2 - A(r) dr^2 - r^2 d\Omega^2 \]
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We also require weak gravity and weak binding, parameterized by

\[ \delta \equiv \frac{f_a^2}{M_P^2} \ll 1 \quad \Delta \equiv \sqrt{1 - \frac{E_0^2}{m^2}} \ll 1 \]

We scale the axion wavefunction, radial coordinate, and metric functions accordingly:

\[ x = \Delta m r, \quad Y(x) = \frac{2\sqrt{N}R(r)}{f_a \Delta}, \quad A(r) = 1 + \delta a(x), \quad B(r) = 1 + \delta b(x) \]
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\[ A(r) = 1 + \delta a(x), \quad B(r) = 1 + \delta b(x) \]
To leading order in $\delta$ and $\Delta$ we find simplified set of equations for the axion wavefunction $Y(x)$ and metric functions $a(x)$ and $b(x)$:

\[
a'(x) = \frac{x}{2} Y(x)^2 - \frac{a(x)}{x}
\]

\[
b'(x) = \frac{a(x)}{x}
\]

\[
Y'''(x) = \left[1 + \kappa b(x)\right] Y(x) - \frac{2}{x} Y'(x) - \frac{1}{8} Y(x)^3
\]

Depends on only one free parameter, $\kappa \equiv \delta/\Delta^2$. 

Note: Leading corrections are $O(\delta)$ and $O(\kappa \delta)$. Requiring $\Delta \ll 1$ and $\kappa \delta \ll 1$ gives the range of validity: $10^{-14} \sim f_a^2 M_P^2 \ll \kappa \ll M_P^2 f_a^2 \sim 10^{14}$ for QCD axions with $m = 10^{-5} eV$. 

Leading Order Equations

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Gravitational Stability

- Maximum mass of weakly bound QCD axion stars with $m_a = 10^{-5}$ eV is $\mathcal{O}(10^{19})$ kg with radius $\mathcal{O}(100 - 1000)$ km.

- Heavier stars are gravitationally unstable and must either shed excess axions (“bosenova”), or go into some other state (black hole, strongly bound state, etc.).
Tunneling Effects

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- Using \( M(\Delta) = Nm\sqrt{1 - \Delta^2}, \)

  \[
  \frac{M_2 - M_1}{M_2} \approx \frac{1}{2} \left( \Delta^2_1 - \Delta^2_2 \right) = \frac{\delta}{2} \left( \kappa_1^{-1} - \kappa_2^{-1} \right)
  \]

Even a small mass fraction \( \mathcal{O}(\delta \cdot M) = ("many" \text{ kg}), \) and could have \textit{significant observational effects}!
Axion Star Decay

Axion stars can decay through self-interactions because:

1. Axions are Hermitian fields: their number is not conserved
2. They possess a non-trivial self-interaction potential
3. Condensed axions are not in momentum eigenstates

Wavefunctions are localized on a radius $R \sim \frac{1}{m \Delta}$

Momentum uncertainty is thus $\delta p \sim m \Delta$

The process $a \rightarrow a$ leads to a nonzero matrix element for $[\mathcal{N}_{\text{condensed}} \rightarrow (\mathcal{N} - 3_{\text{condensed}} + 1_{\text{emitted}})$ giving axion stars a finite lifetime.
Axion Star Decay

Axion stars can decay through self-interactions because:

1. Axions are **Hermitian fields**: their number is not conserved
   - In the nonrelativistic limit of real scalar fields, number changing operators are (justifiably) neglected, but they should be present in the full field theory
   - We extend our expansion in ground-state operators to include a free axion term,

\[
\mathcal{A}(r) = R(r) e^{iE_0 t} a_0 + \int \frac{d^3 p}{\sqrt{2E_p}} a_p e^{i p \cdot r - iE_p t} + \text{h.c.}
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\[ V_{int} \sim -\frac{m^2}{f_a^2} A^4 + O(A^6) \]
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The process $3a \rightarrow a$ leads to a nonzero matrix element for

$$[N \text{ condensed}] \rightarrow [(N - 3) \text{ condensed} + 1 \text{ emitted}]$$

giving axion stars a finite lifetime.
Lifetime of Axion Stars

We have calculated the leading-order decay rate (through $3 \ a \rightarrow \ a$) for axion stars in the weak binding limit:

$$\Gamma_3 = \frac{f_a^2}{4\sqrt{2}\pi m} |I_3(\Delta)|^2$$

where $I_3$ is a monotonically increasing function of $\Delta$. 

Note: The other leading-order processes, $n a \rightarrow \mu a$ with $n \geq 4$ and/or $\mu \geq 2$, are suppressed by either $\Delta \ll 1$ or the ratio $m^2/f_a^2 \ll 1$. 

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$$\frac{\partial \tau}{\partial N} \approx \frac{1}{3 \Gamma_3}$$

satisfies $\tau > \tau_U$ below a critical binding energy parameter $\Delta \lesssim .05 - .06$. 

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Combined Bounds

**Red line:** Upper bound for gravitational stability
**Green line:** Lower bound from decay analysis

---

Over a large region of the important parameter space, \( \kappa \ll 1 \). In this region, the gravitational interaction completely decouples from axion field. The attractive self-interaction is balanced by kinetic pressure.
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Current and Future Work

- We are currently working to extend our method to the case of rotating stars, requiring an expansion in spherical harmonics.

(Preliminary)

Leading-order solutions can be systematically extended to include ever higher harmonics.
Current and Future Work

- We are currently working to extend our method to the case of rotating stars, requiring an expansion in spherical harmonics $Y_0[x]$ and $Y_1[x]$

![Graph showing $Y_0[x]$ and $Y_1[x]$](image)

- Leading-order solutions can be systematically extended to include ever higher harmonics
- All of our results so far apply only to weakly bound stars. The case of strong binding requires an entirely new method.
Conclusions

- Our expansion in $\delta$ and $\Delta$ simplifies the equations and elucidates important properties of axion stars.
- The maximum mass of weakly bound QCD axion stars is $\mathcal{O}(10^{19})$ kg with $R_{99} \sim 1000$ km for $m = 10^{-5}$ eV.
- Through stability analysis, we find significant constraints on the masses of weakly bound axion stars surviving from the early universe.
- The cases of rotating or strongly bound axion stars are of great interest for the development of a realistic DM model and to obtain a full picture of the parameter space of axion stars.
- Thanks!
Backup Slides
The $\theta$-term in the QCD Lagrangian violates $CP$:

$$
L_{QCD} \ni \theta \frac{g_s^2}{32\pi^2} G^{a\mu\nu} \tilde{G}_{a\mu\nu} \tag{1}
$$

But the lack of detection of a neutron EDM constrains the free parameter $\theta$ severely:

$$
d_n \approx 5 \times 10^{-16} \theta \text{ e} \cdot \text{cm} \lesssim 10^{-25} \text{ e} \cdot \text{cm}
\Rightarrow \theta \lesssim 10^{-10} \tag{2}
$$

In principle, $\theta \sim \mathcal{O}(1)$. Why should it be so small?
Solving the Strong-CP Problem:

- Promote $\theta \rightarrow a(x)/f$, a dynamical field, whose potential is minimized at $a(x) = 0$. Naturalness saved!
- The Lagrangian for $a(x)$ has a symmetry, $U(1)_{PQ}$, which is broken at the energy scale $f$
- The physical axion field is the Goldstone boson of $U(1)_{PQ}$-breaking

$\phi$ is initially massless, but acquires a mass during the QCD phase transition due to nonperturbative effects:

$$m_a = \frac{\Lambda_{QCD}^2}{f}$$ (3)
Suppose a scalar field $\Phi$ is complex and has total energy $E_0$. In the non-relativistic limit, $|E_0 - m| = \delta E \ll m$. 

Thus $0 = (\dddot{\Phi} - \nabla^2 \Phi - m^2 \Phi) \approx (-2i m \dot{\phi} - \nabla^2 \phi) e^{-imt}$ or $i \dot{\phi} = -\frac{1}{2} m \nabla^2 \phi$. 

\[\Delta_2 \approx \frac{2 \delta E}{m} \ll 1\]
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$\Rightarrow \Delta^2 \approx 2 \delta E / m \ll 1!$
Non-Relativistic Limit of a Scalar Field

Suppose a scalar field $\Phi$ is complex and has total energy $E_0$. In the non-relativistic limit, $|E_0 - m| = \delta E \ll m$.

$$\Rightarrow \Delta^2 \approx 2 \delta E / m \ll 1!$$

Let $\Phi = e^{-imt} \phi$. Then $|\dot{\phi}| \approx \delta E \phi \ll m \phi$, so

$$\ddot{\Phi} = m^2 \left( \frac{\dddot{\phi}}{m^2} - 2 i \frac{\dot{\phi}}{m} + \phi \right) e^{-imt}$$

$$\mathcal{O}(\Delta^2) \quad \mathcal{O}(\Delta) \quad \mathcal{O}(1)$$
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Thus

$$0 = \left( \ddot{\Phi} - \nabla^2 \Phi - m^2 \Phi \right) \approx \left( - 2 i m \dot{\phi} - \nabla^2 \phi \right) e^{-imt}$$

or

$$i \dot{\phi} = - \frac{1}{2m} \nabla^2 \phi$$
Non-Relativistic Axions

Note that axions are real scalar fields, and the above procedure does not quite apply. A similar method reduces the Klein-Gordon equation by expanding the axion field $\phi$ in terms of a complex one:\(^1\)

$$\phi = \frac{1}{\sqrt{2m}} \left[ e^{-imt} \psi + e^{imt} \psi^* \right]$$

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The resulting equation of motion for the new field $\psi$ at $\Delta \ll 1$ is known as the Gross-Pitaevskii equation, coupled to gravity:

$$i \dot{\psi} = -\frac{1}{2m} \nabla^2 \psi + \frac{\lambda}{8m^2} |\psi|^2 \psi - G m^2 \psi \int d^3 x' \frac{|\psi(r')|^2}{|r - r'|}$$

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$$

**Note:** Completely equivalent to our system at leading $\Delta$.

In the non-relativistic limit, scalar field dynamics can be described by the Gross-Pitaeveskii, or nonlinear Schrödinger, equation coupled to Poisson gravity:

\[ i \dot{\psi}(r) = -\frac{1}{2m} \nabla^2 \psi(r) + m V(r) \psi(r) + \frac{\lambda}{8m^2} |\psi(r)|^2 \psi(r) \]

\[ \nabla^2 V(r) = 4\pi G m |\psi(r)|^2 \]

The GPP energy is

\[ E[\psi] = \int d^3 r \left[ -\frac{1}{2m} |\nabla \psi|^2 + V \psi + \frac{\lambda}{16m^2} |\psi|^2 \psi \right] \]
Variational Method

\[ E(R) = \frac{A}{R^2} - \frac{B N}{R} - \frac{C N}{R^3} \]
We required \( \tau > \tau_U \approx 13.6 \) billion years, but softening this requirement does not change much the lower bound on axion star masses.
Decoupling of Gravity

When \( \kappa \ll 1 \), the total gravitational energy is significantly smaller than the kinetic or interaction energies.

When \( \kappa = \mathcal{O}(1) \), the terms are of roughly the same order.
Chiral vs Instanton Potential

The instanton potential \( V(\phi) = m^2 f_a^2 \left( 1 - J_0(\phi/f_a) \right) \) is really only an approximation. Axions are more correctly described by a chiral potential which takes into account QCD effects (see e.g. 1604.00669).

In fact, updating the potential in this way does not change our results much at all:

![Graph comparing Instanton and Chiral potentials](image-url)