Mass Reconstruction for High Multiplicity Final States Using the Boundary of Phase Space

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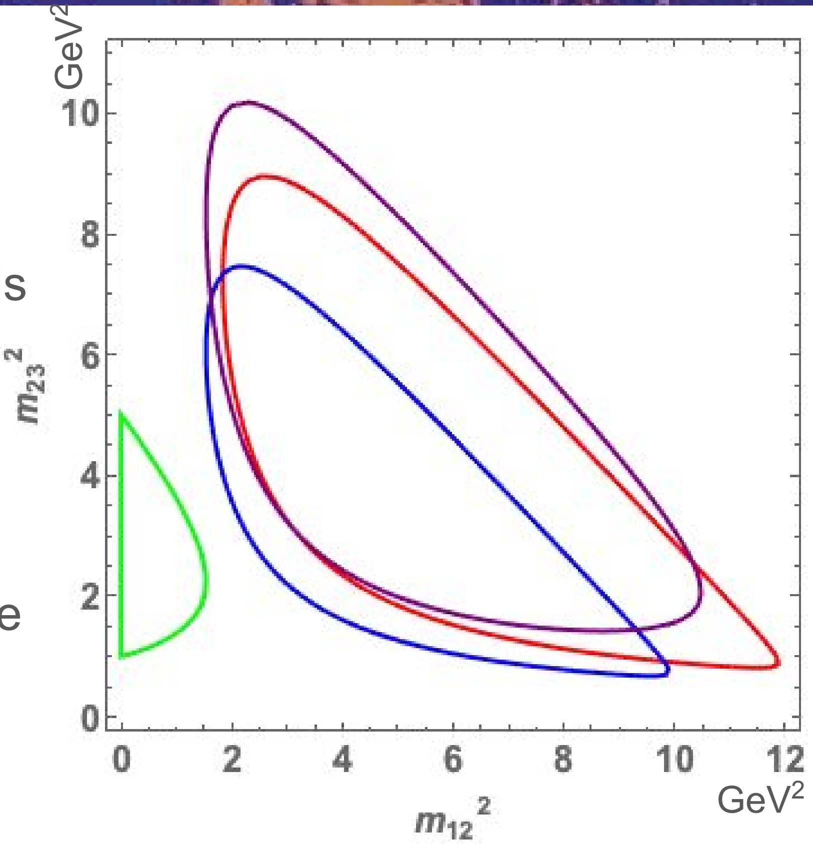
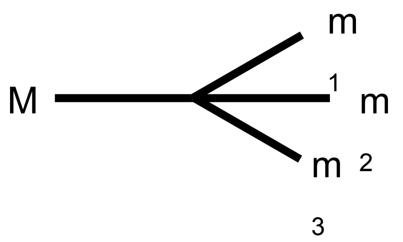
Motivation

- We will demonstrate a technique that allows for the determination of masses in cascade decays by utilizing correlations in the full phase space distribution with small samples
  - No (conclusive) BSM discoveries at LHC yet
  - Discoveries are likely to come with limited numbers of events
  - Essential to be able to make maximal use of small samples
  - This technique will outperform traditional edge/endpoints mass determination
  - Along the way, we'll see some interesting structure in phase space

Introduction: the Dalitz plot (3-body decay)

- Kinematic invariants used as coordinates
- Phase space weight is *flat* in these coordinates
- Mass spectrum is encoded by the shape of the boundary





$(M^2, m_1{}^2, m_2{}^2, m_3{}^2)$ =

(6, 3.6, 2.2, 1.2) GeV^2

(5, 3, 2, 1) GeV^2

(5, 4.8, 3.2, 1) GeV^2

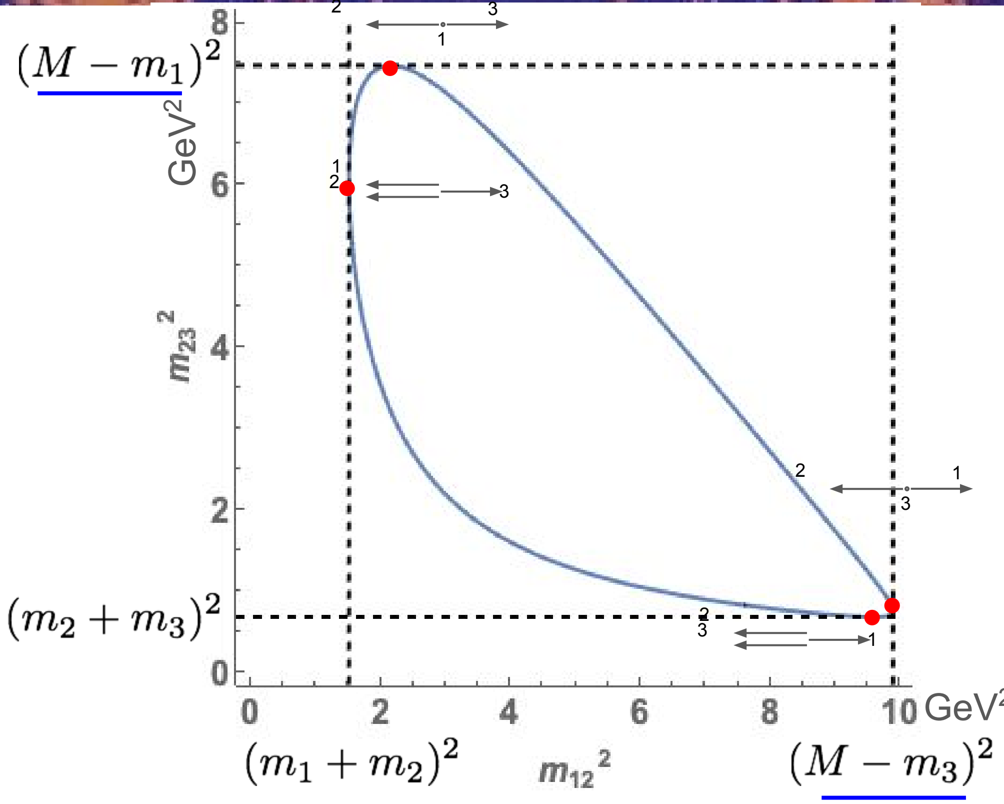
(5, 0, 0, 1) GeV^2

Introduction: the Dalitz plot (3-body decay)

Mass information on the boundary is typically expressed in terms of *endpoints.*

Endpoints are 1-d *projections* of the full phase space.

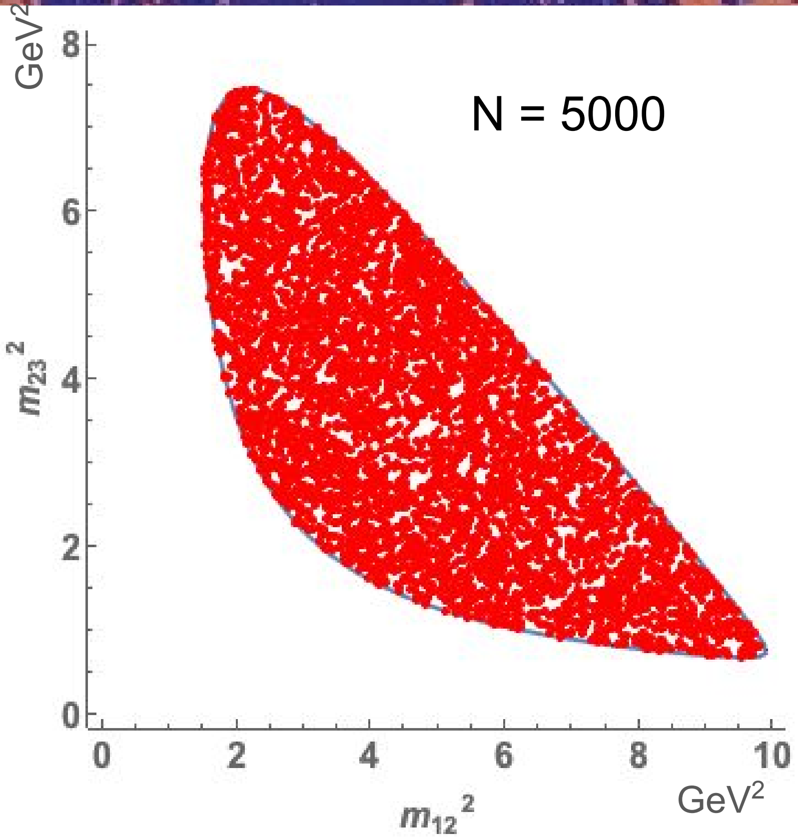
Endpoints correspond to momentum configurations which span a *lower dimensional* subspace.
(Collinear in this case.)



Note that the endpoints are mass *differences.*

Introduction: the Dalitz plot (3-body decay)

- Phase space weight is *flat*
- With a large number of events, the corners of the Dalitz region are well populated



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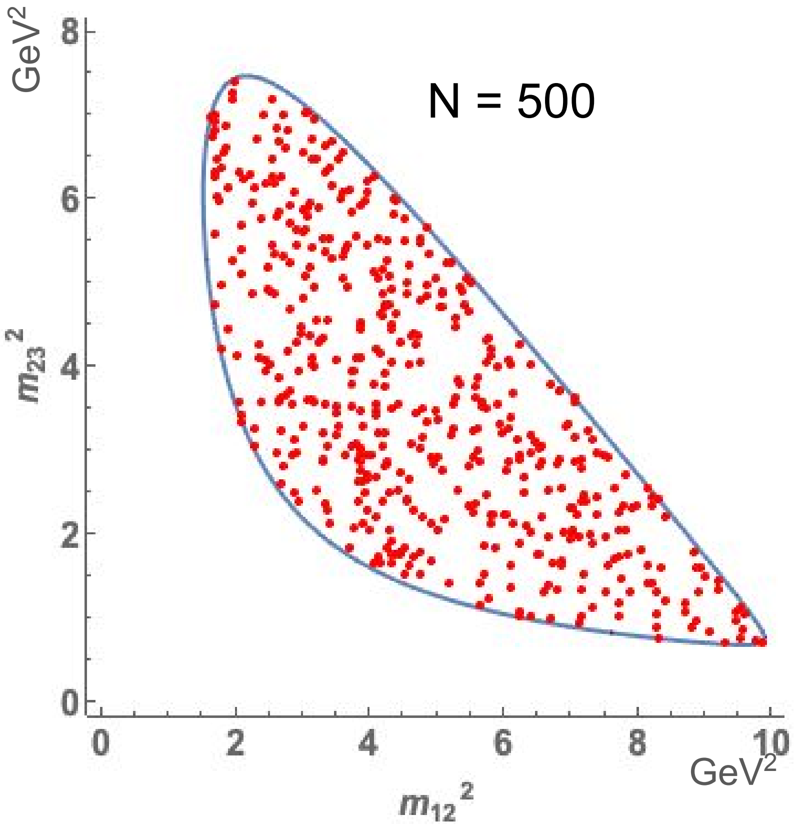
- But with fewer events, the small area that defines the endpoints will not contain many events



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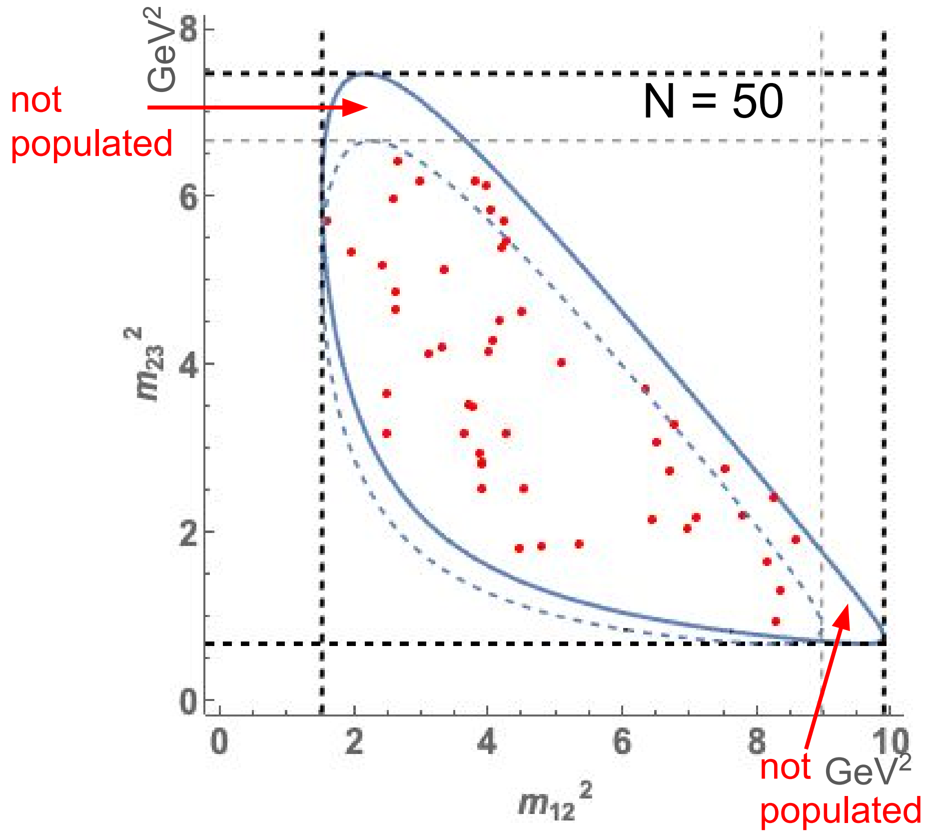
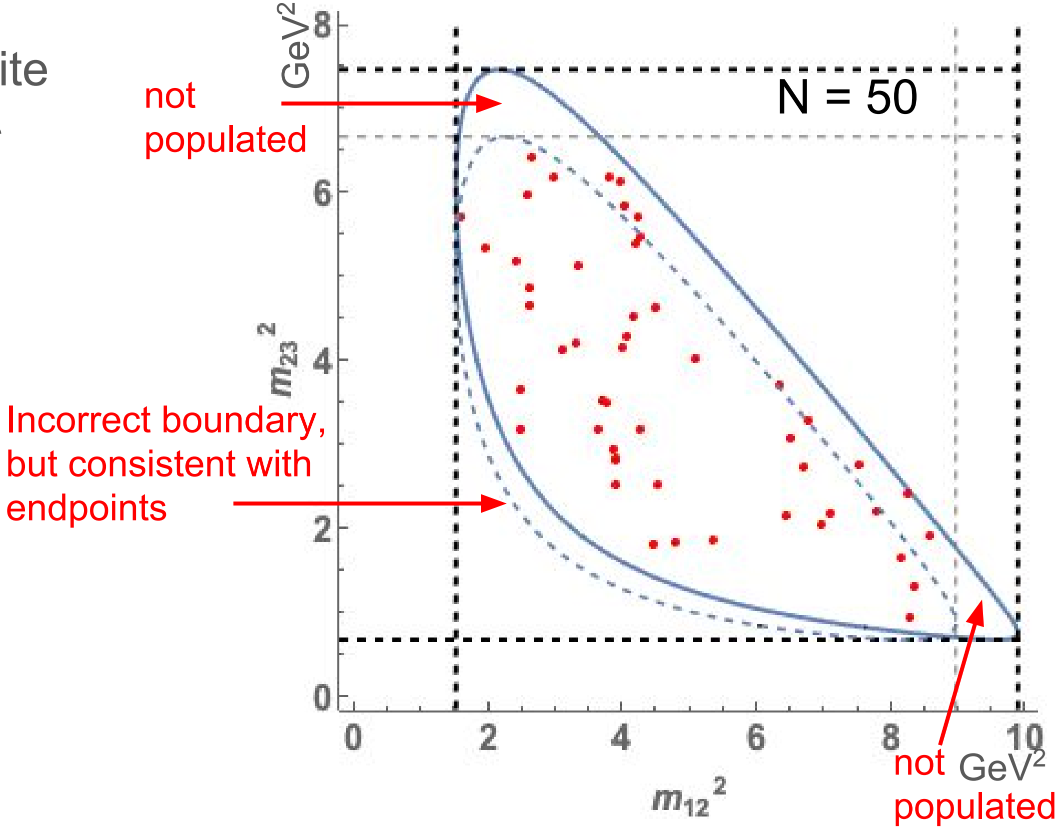


The full phase space advantage

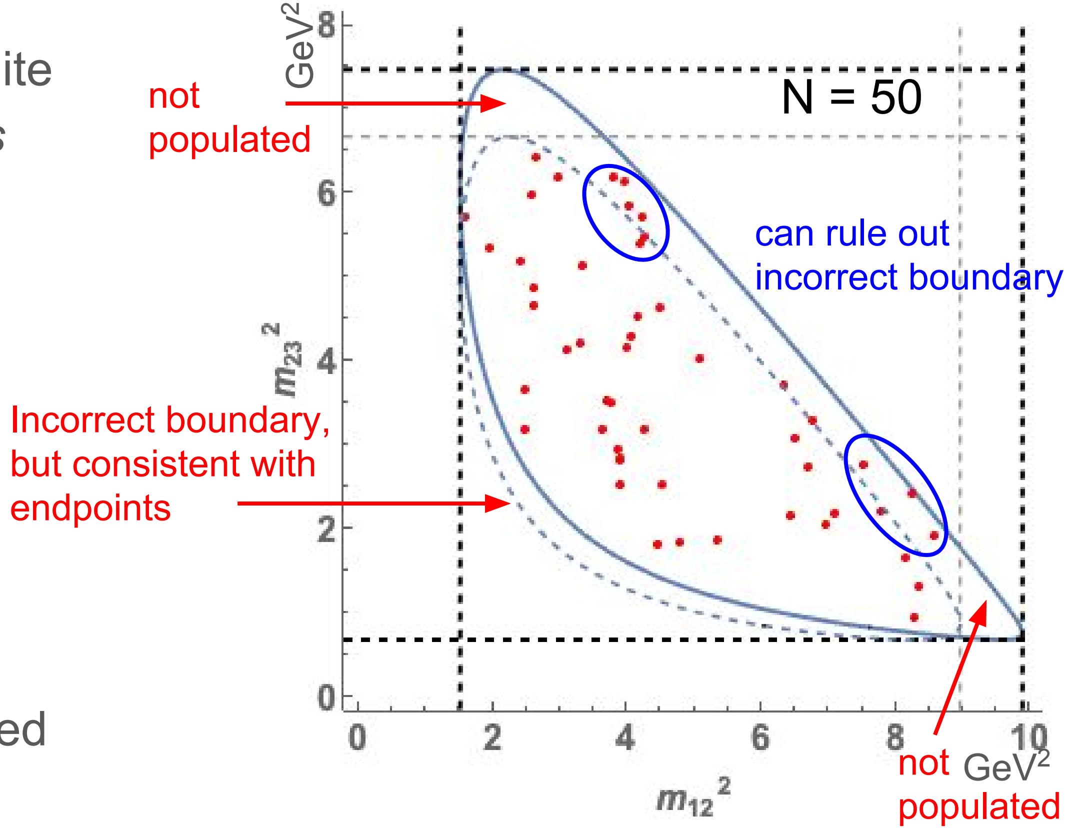
- The *endpoints* inferred from finite samples only set *lower bounds* on the true endpoints

- There are *incorrect* mass spectrum hypotheses that the endpoints *cannot rule out*



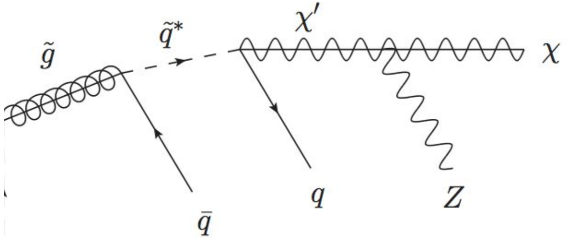
The full phase space advantage

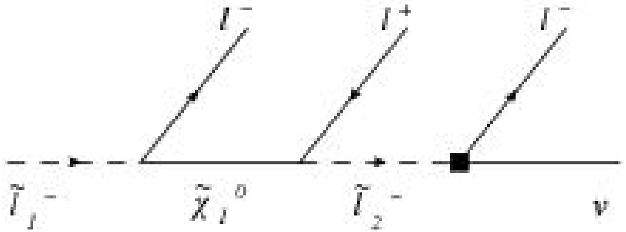
- The *endpoints* inferred from finite samples only set *lower bounds* on the true endpoints
- There are *incorrect* mass spectrum hypotheses that the endpoints *cannot rule out*
- There are events near the *boundary,* but *not* near any endpoint, which could have ruled it out



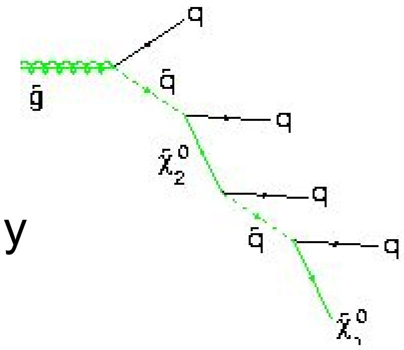
Higher multiplicity decays

4 body





5 body



- One invisible particle
- Depicted as chain of 2-body decays, but intermediate particles could be off-shell giving a 3-body vertex

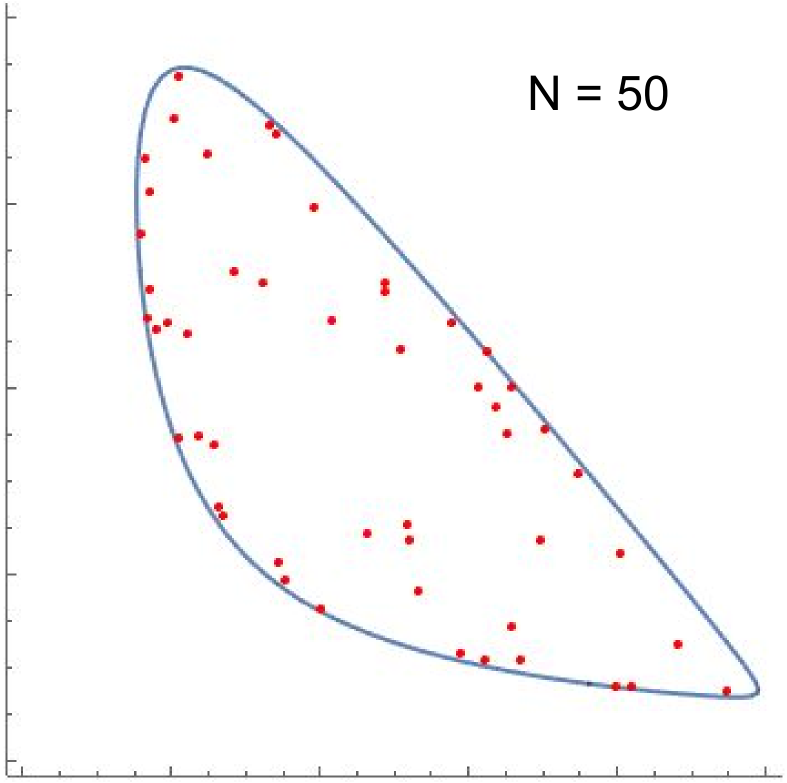
4-body boundary enhancement

Surprise: For 4-body decays, *the phase space weight is greater near the boundary*!

The *most useful* events are the *most plentiful.*

Fitting a boundary to the data, even with a limited number of events, should reconstruct the masses efficiently.

Yang & Byers 1964
Agrawal, Kilic, White, Yu 2013



schematic only

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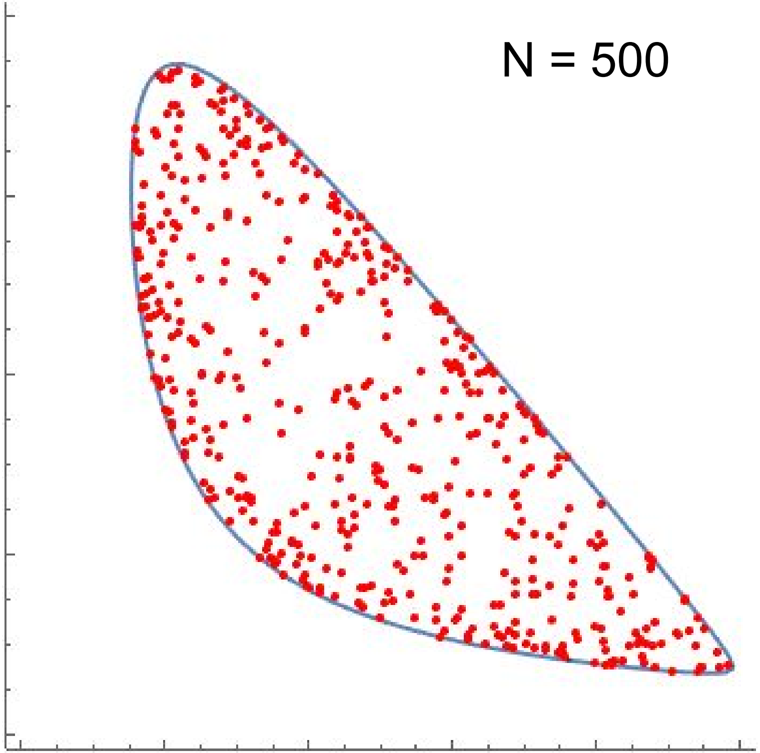

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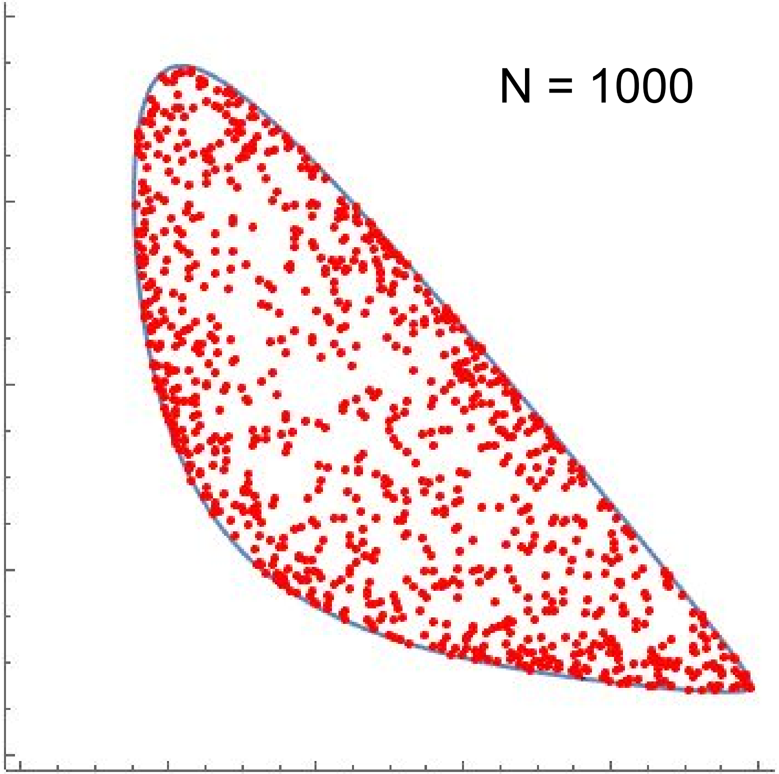

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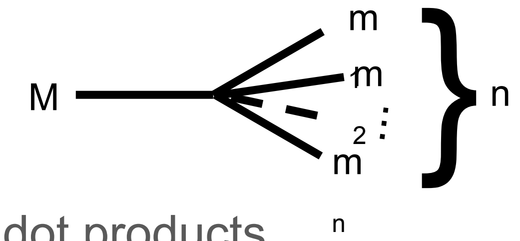
Invariant formalism for 1 → n decay

In order to describe this boundary enhancement quantitatively, we will need to introduce some notation.



Kinematics will be described in terms of Lorentz invariant dot products.

$$\{i, j\} \equiv p_i \cdot p_j = (m_{ij}^2 - m_i^2 - m_j^2)/2$$

$$M_n = \begin{pmatrix} m_1^2 & \{1,2\} & \cdots \\ \{1,2\} & m_2^2 & \\ \vdots & & \ddots \end{pmatrix}$$

Form a *n x n* matrix out of the dot products. Diagonal elements are just the masses squared of the daughter particles.

$$M^2 = \sum_{i=1}^{n} m_i^2 + 2 \sum_{i<j} \{i, j\}$$

Overall energy conservation of the parent particle must be imposed.

Invariant formalism for $1 \rightarrow 4$ decay

In order to express the enhancement near the boundary, we need a sort of "radial" coordinate that measures our distance from the boundary.

The boundary is composed of kinematic configurations that span lower dimensional space (a plane for 4-body decays).

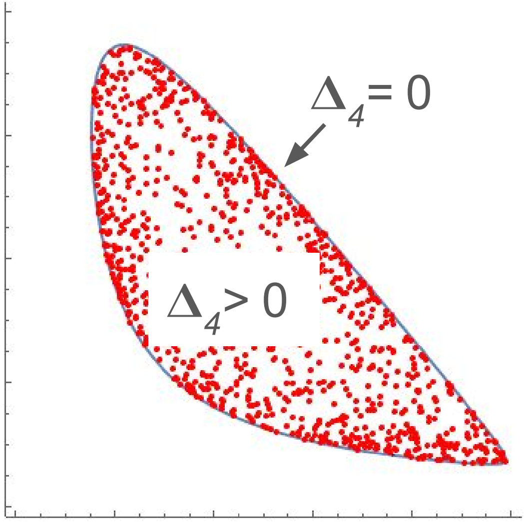
The matrix M_4 becomes singular when the momentum vectors are linearly dependent.

$\Delta_4 \equiv \det M_4 = 0$ if M_4 is singular, so this serves as a good radial coordinate.

$$\Delta_4 = 0 \text{ on the boundary,} \qquad \Delta_4 > 0 \text{ inside}$$

Invariant formalism for 1 → 4 decay

$\Delta_4 = 0$ on the boundary, $\Delta_4 > 0$ inside



Yang & Byers 1964

$$dPS_4 = \frac{16\pi^2}{M^2}\Delta_4^{-1/2}\prod_{i<j} d\{i,j\}\delta\left(\sum\{i,j\} - (M^2 - \sum m_i^2)\right)$$

In terms of our "radial coordinate" there is a power law enhancement in the phase space weight approaching the boundary

(This can be traced back to the singular [linearly dependent] nature of the kinematic configurations that define the boundary.)

Still true, even with intermediate resonances in the decay.

4-body likelihood functions

➔ *To find the masses, we are really only interested in the boundary.*

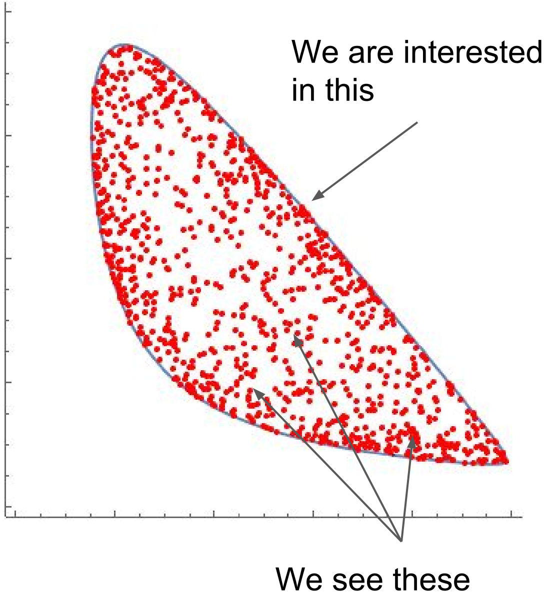
➔ But what we see are events inside, but mostly close to, the boundary. (However, see the previous talk □)

We will use a likelihood function to *fit* the *observed events* to an *optimal hypothesis* for the shape of the boundary (*i.e.* mass spectrum)

The likelihood function must satisfy the following two conditions:

❑ Disfavor spectrum hypotheses that leave events outside its boundary

❑ Prefers spectrum hypotheses when events lie inside and near its boundary



4-body likelihood functions

$$\mathcal{L}(\tilde{m}) \sim \prod_{\text{events}} \Delta_4^{-1/2}(\tilde{m})\,\Theta(\Delta_4(\tilde{m}))$$

- Event outside → excluded
- Otherwise, favored if events near boundary
- The spectrum with the highest likelihood is the winner!

$$Q = \left(\sum_{i=\text{endpts.}} \left(\frac{\mathcal{O}_{i,predicted} - \mathcal{O}_{i,measured}}{\mathcal{O}_{i,measured}} \right)^2 \right)$$

For comparison, a likelihood function based on 1-d distributions, aka endpoints.

Endpoints are computed for each mass spectrum hypothesis.

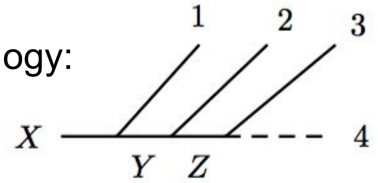
Spectrum excluded if any event is beyond any endpoint predicted by the hypothesis.

Otherwise, spectrum favored if observed endpoint is near predicted endpoint.

4-body results

Agrawal, Kilic, White, Yu 2013

Topology:



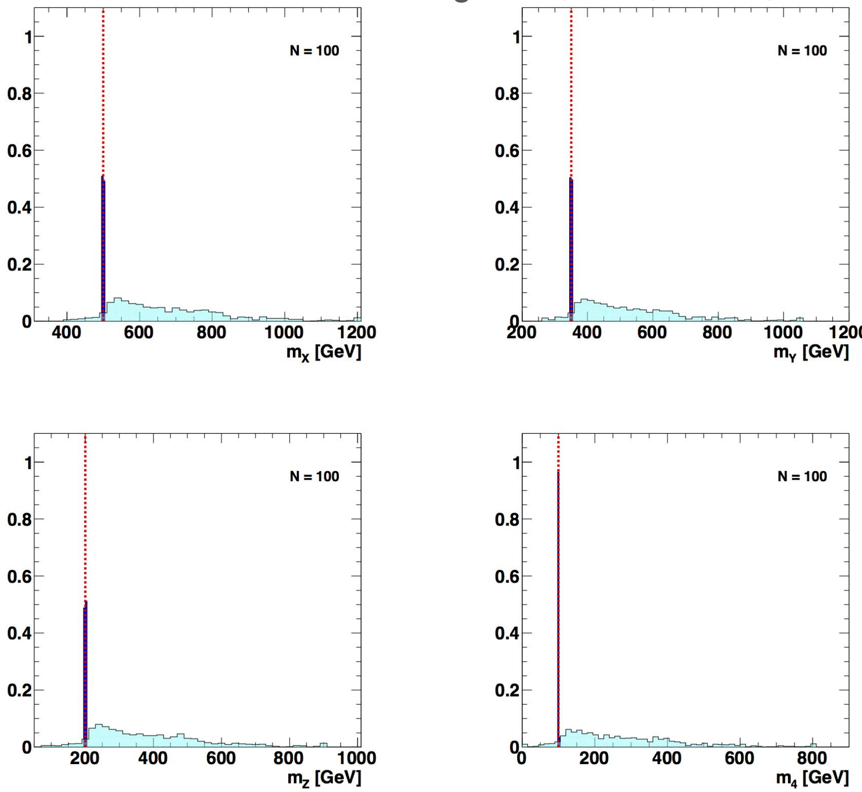
(invisible)

Histogram of winning mass spectra for many samples of Monte Carlo data, each with 100 events.

True value indicated by red.
Phase space boundary winners in blue.
Endpoint winners in cyan.

Note that the spread in the endpoint results is correlated: it reflects the insensitivity of endpoints to overall scale (they measure differences).

The boundary method gets both the scale and the differences! (Not visible here.)



What happens for *n* > 4 ?

n = 3 → flat. n = 4 → boundary enhanced. n = 5 → ???

Cartesian: *3 x 5* momentum coordinates

- *4* conservation constraints

- *3* irrelevant rotations

= *8* degrees of freedom

Invariants:

$$M_5 = \begin{pmatrix} m_1^2 & \{1,2\} & \{1,3\} & \{1,4\} & \{1,5\} \\ \{1,2\} & m_2^2 & \{2,3\} & \{2,4\} & \{2,5\} \\ \{1,3\} & \{2,2\} & m_3^4 & \{3,4\} & \{3,5\} \\ \{1,4\} & \{2,4\} & \{3,4\} & m_4^2 & \{4,5\} \\ \{1,5\} & \{2,5\} & \{3,5\} & \{4,5\} & m_5^2 \end{pmatrix}$$

symmetric

10 invariants

- 1 constraint

= ***9*** degrees of freedom

9 *would* be the right number *if* we were in 5 dimensions. In 4 dimensions, *one more constraint* is needed.

What happens for *n* > 4 ?

Generalization of the formalism:

We now have 5 particles, so define

$\Delta_5 \equiv \det M_5$

We will *require* $\Delta_5 = 0$, ensuring the event will be realizable in 4 dimensions.
This is our extra constraint.

The boundary of phase space is still given by $\Delta_4 = 0$ where now Δ_4 is a sum over all choices of 4 out of the 5 particles.

What happens for *n* > 4 ?

Yang & Byers 1964

$$dPS_5 = \frac{32\pi^2}{M^2}\delta(\Delta_5)\prod_{i<j} d\{i,j\}\,\delta\left(2\sum_{i<j}\{i,j\} - M^2 + \sum m_i^2\right)$$

No Δ_4 dependence? Phase space weight is flat again?

Not so fast… we are working in the coordinate system of the {i, j}.

The $\Delta_5 = 0$ constraint is *non-linear* in these coordinates.

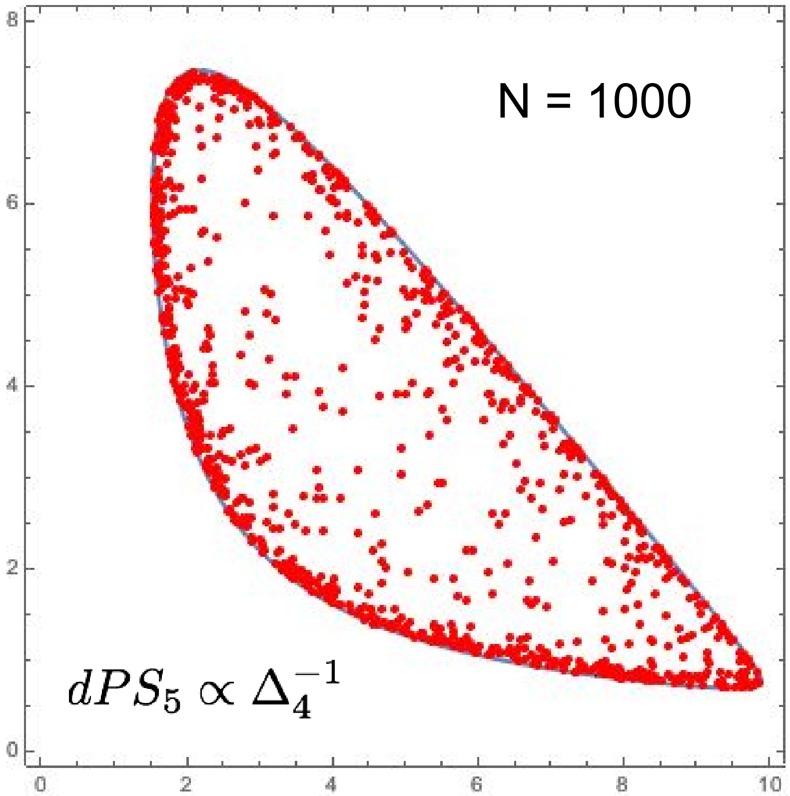
Evaluating the delta function gives a Jacobian $|d\Delta_5/d\{i,j\}|^{-1}_{\Delta_5=0}$

What happens for $n > 4$?

Punch line: *This Jacobian restores the boundary enhancement, and makes it even stronger than the 4-body case!*

$$dPS_5 \propto \boxed{\Delta_4^{-1}}$$

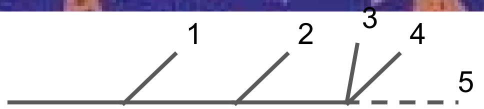
Intuition: when all 5 particles span a plane, the matrix M_5 is even more singular.



schematic only

5-body likelihood functions

Example topology



$$\mathcal{L}(\tilde{m}) \sim \prod_{\text{events}} \int \Theta\left(\text{disc}(\Delta_5(\tilde{m}))\right) \underline{\delta(\Delta_5(\tilde{m}))\, d\{4,5\}} = \prod_{\text{events}} \underline{\frac{\Theta\left(\text{disc}(\Delta_5(\tilde{m}))\right)}{\sqrt{\text{disc}(\Delta_5(\tilde{m}))}}}$$

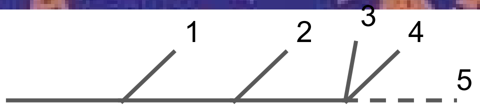
- Event outside → excluded
- Otherwise, favored if events near boundary
- The spectrum with the highest likelihood is the winner!

The extra delta function let's us take one integral over an unobserved variable

Which gives the enhancement in terms of the Jacobian of Δ_5

5-body likelihood functions

Example topology



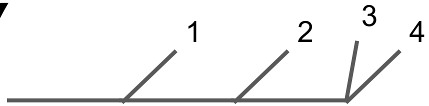
$$\mathcal{L}(\tilde{m}) \sim \prod_{\text{events}} \int \Theta\left(\text{disc}(\Delta_5(\tilde{m}))\right) \delta(\Delta_5(\tilde{m}))\, d\{4,5\} = \prod_{\text{events}} \frac{\Theta\left(\text{disc}(\Delta_5(\tilde{m}))\right)}{\sqrt{\text{disc}(\Delta_5(\tilde{m}))}}$$

Question: If two particles from one vertex are both visible, couldn't I just combine them and think of this as a 4-body event?

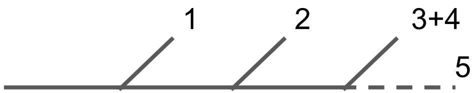
Yes.

There is an interesting factorization relation:

$\text{disc}(\Delta_5)$ = Δ_4 (



) x Δ_4 (

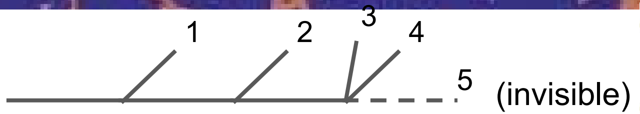


)

The boundary of the 5-body event coincides with the boundary of the corresponding 4-body event.

5-body results

Topology:



Histogram of winning mass spectra for many samples of Monte Carlo data, each with 100 events.

True value indicated by red.
Phase space boundary winners in blue.
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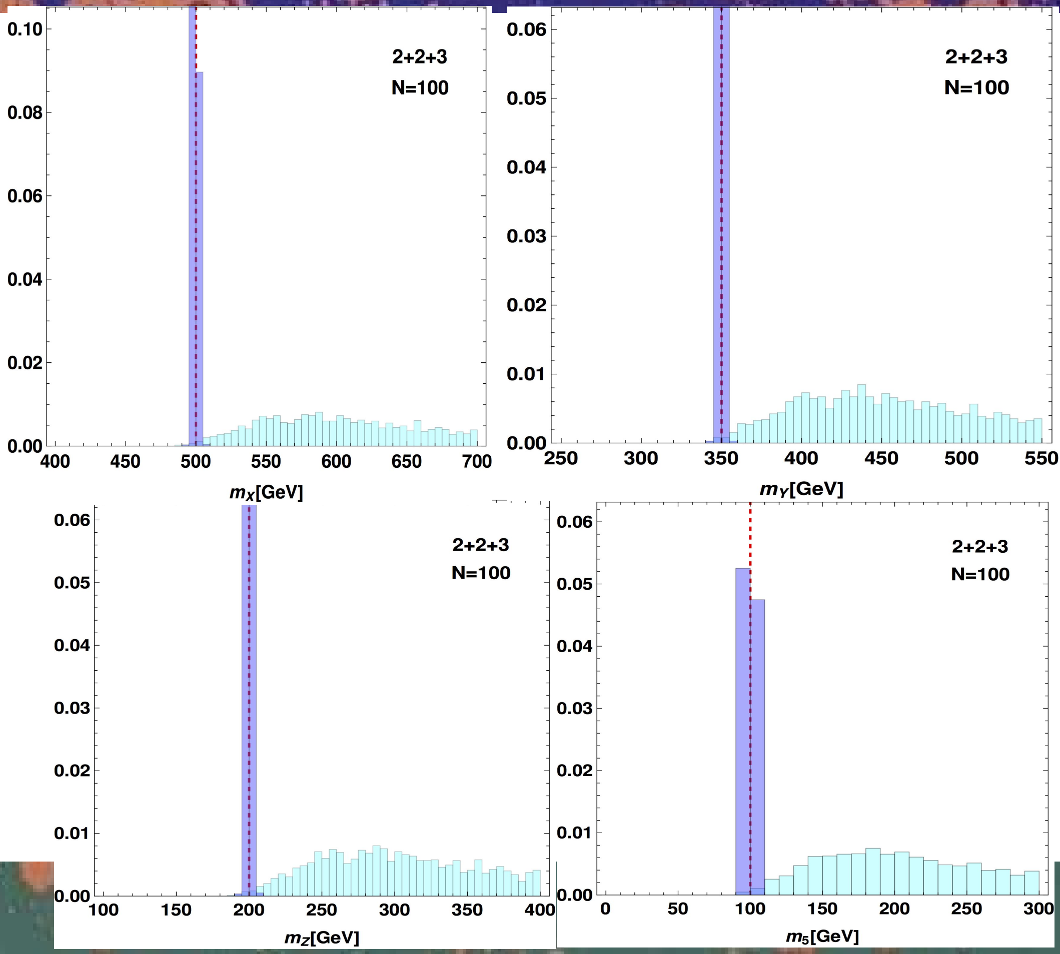
Note insensitivity of endpoints to overall scale again.

The boundary method rapidly converges on the correct masses, even for this complicated and high-multiplicity final state.



Conclusions

- **A likelihood technique that considers the full dimensionality of phase space can then rapidly reconstruct the involved masses with few events**
  - The mass content of particle decays is encoded in the shape of the phase space boundary
  - For $n > 3$ final states, events preferentially cluster on the boundary, which becomes *stronger* as the n increases!
  - This powerful technique easily outperforms traditional techniques which are based on 1-d projections of phase space
  - Interesting structure in phase space that hasn't been explored yet!