

Special Relativistic Hydrodynamics with Gravitation

J. Hwang & H. Noh
CosPA 2016
December 1, 2016
Sydney

Perturbation theory:

- ❖ Perturbation expansion
- ❖ All perturbation variables are small
- ❖ Weakly nonlinear
- ❖ Strong gravity; fully relativistic
- ❖ Fully nonlinear and exact formulation (neglecting TT part)

Post-Newtonian approximation:

- ❖ Abandon geometric spirit of GR: recover the good old absolute space and absolute time
- ❖ Newtonian equations of motion with GR corrections
- ❖ Expansion in strength of gravity $\frac{\delta\Phi}{c^2} \sim \frac{GM}{Rc^2} \sim \frac{v^2}{c^2} \ll 1$
- ❖ Fully nonlinear
- ❖ No strong gravity; weakly relativistic

Special relativity with gravity:

- ❖ Assume weak gravity and action-at-a-distance
- ❖ Fully relativistic velocity, pressure and anisotropic stress
- ❖ Newtonian equations of motion with GR corrections

Fully NL & Exact Pert. Theory

JH, Noh MNRAS **433** (2013) 3472

Noh JCAP **07** (2014) 037

JH, Noh, Park MNRAS **461** (2016) 3239

JH, Noh ApJ (2016) to appear [arXiv:1610.05662](https://arxiv.org/abs/1610.05662)

Convention: (Bardeen 1988)

Decomposition, possible
to NL order (York 1973)

$$ds^2 = -\underline{a^2} (1 + 2\alpha) d\eta^2 - 2a^2 \left(\beta_{,i} + B_i^{(v)} \right) d\eta dx^i + \underline{a^2} \left[(1 + 2\varphi) g_{ij}^{(3)} + 2\gamma_{,i|j} + C_{i|j}^{(v)} + C_{j|i}^{(v)} + 2C_{ij}^{(t)} \right] dx^i dx^j,$$

$$\chi \equiv a\beta + a^2\dot{\gamma}, \quad \Psi_i^{(v)} \equiv B_i^{(v)} + a\dot{C}_i^{(v)},$$

No TT-pert!

Spatial gauge condition

$$\tilde{T}_{ab} = \tilde{\mu}\tilde{u}_a\tilde{u}_b + \tilde{p}(\tilde{u}_a\tilde{u}_b + \tilde{g}_{ab}) + \tilde{\pi}_{ab}, \quad \tilde{u}_i \equiv \frac{a}{c}\hat{\gamma}\hat{v}_i, \quad \hat{\gamma} \equiv \frac{1}{\sqrt{1 - \frac{\hat{v}^k\hat{v}_k}{c^2(1+2\varphi)}}},$$

$$\tilde{\mu} \equiv \mu + \delta\mu \equiv \mu(1 + \delta), \quad \tilde{p} \equiv p + \delta p, \quad \hat{v}_i \equiv -\hat{v}_{,i} + \hat{v}_i^{(v)}$$

Spatial gauge:

$$\gamma \equiv 0 \equiv C_i^{(v)},$$

$$\chi_i \equiv \chi_{,i} + a\Psi_i^{(v)} = a(\beta_{,i} + B_i^{(v)})$$

Temporal gauge still not taken yet!



Complete spatial gauge fixing.

Remaining variables are spatially gauge-invariant
to fully NL order! \therefore Lose no generality!

Metric convention:

$$\tilde{g}_{00} = -a^2 (1 + 2\alpha), \quad \tilde{g}_{0i} = -a\chi_i, \quad \tilde{g}_{ij} = a^2 (1 + 2\varphi) g_{ij}^{(3)}.$$

Inverse metric:

$$\tilde{g}^{00} = -\frac{1}{a^2} \frac{1 + 2\varphi}{(1 + 2\varphi)(1 + 2\alpha) + \chi^k \chi_k/a^2},$$

$$\tilde{g}^{0i} = -\frac{1}{a^2} \frac{\chi^i/a}{(1 + 2\varphi)(1 + 2\alpha) + \chi^k \chi_k/a^2},$$

$$\tilde{g}^{ij} = \frac{1}{a^2(1 + 2\varphi)} \left(g^{(3)ij} - \frac{\chi^i \chi^j/a^2}{(1 + 2\varphi)(1 + 2\alpha) + \chi^k \chi_k/a^2} \right). \quad \text{Exact!}$$

Using the ADM and the covariant
formalisms the rest are simple algebra.
We do not even need the connection!

Fully Nonlinear Perturbation Equations without taking temporal gauge condition:

Metric:

$$ds^2 = -(1 + 2\alpha) c^2 dt^2 - 2\chi_i c dt dx^i + (1 + 2\varphi) \delta_{ij} dx^i dx^j$$

$$\mathcal{N} \equiv \frac{1}{\sqrt{-g^{00}}} = \sqrt{1 + 2\alpha + \frac{\chi^k \chi_k}{1 + 2\varphi}}, \quad \gamma \equiv -u^c n_c = \frac{1}{\sqrt{1 - \frac{v^k v_k}{c^2(1+2\varphi)}}}$$

Temporal gauge (slicing, hypersurface):

synchronous gauge : $\alpha \equiv 0$,

zero-shear gauge : $\chi^i,_i \equiv 0$,

comoving gauge : $v^i,_i \equiv 0$,

uniform-expansion gauge/maximal slicing : $\kappa \equiv 0$.

Applicable to NL orders!

Except for synchronous gauge, complete gauge fixing

Remaining variables are gauge-invariant to fully NL order!

Definition of κ (the trace of extrinsic curvature, K_i^i):

$$\kappa \equiv -\frac{1}{\mathcal{N}(1+2\varphi)} \left[3\dot{\varphi} + c \left(\chi^k_{,k} + \frac{\chi^k \varphi_{,k}}{1+2\varphi} \right) \right]. \quad (\text{B3})$$

ADM energy constraint:

$$\frac{4\pi G}{c^2} \mu + \frac{c^2 \Delta \varphi}{(1+2\varphi)^2} = \frac{1}{6} \kappa^2 - \frac{4\pi G}{c^2} (\mu + p) (\gamma^2 - 1) + \frac{3}{2} \frac{c^2 \varphi^i \varphi_{,i}}{(1+2\varphi)^3} - \frac{c^2}{4} \overline{K}_j^i \overline{K}_i^j - \frac{1}{(1+2\varphi)^2} \frac{4\pi G}{c^4} \Pi_{ij} v^i v^j. \quad (\text{B4})$$

ADM momentum constraint:

$$\begin{aligned} & \frac{2}{3} \kappa_{,i} + \frac{c}{\mathcal{N}(1+2\varphi)} \left(\frac{1}{2} \Delta \chi_i + \frac{1}{6} \chi^k_{,ki} \right) + \frac{8\pi G}{c^4} (\mu + p) \gamma^2 v_i \\ &= \frac{c}{\mathcal{N}(1+2\varphi)} \left\{ \left(\frac{\mathcal{N}_{,j}}{\mathcal{N}} - \frac{\varphi_{,j}}{1+2\varphi} \right) \left[\frac{1}{2} (\chi^j_{,i} + \chi^j_{,i}) - \frac{1}{3} \delta_i^j \chi^k_{,k} \right] - \frac{\varphi^j}{(1+2\varphi)^2} \left(\chi_i \varphi_{,j} + \frac{1}{3} \chi_j \varphi_{,i} \right) \right. \\ & \quad \left. + \frac{\mathcal{N}}{1+2\varphi} \nabla_j \left[\frac{1}{\mathcal{N}} \left(\chi^j \varphi_{,i} + \chi_i \varphi_{,j} - \frac{2}{3} \delta_i^j \chi^k \varphi_{,k} \right) \right] \right\} - \frac{1}{1+2\varphi} \frac{8\pi G}{c^4} \Pi_{ij} v^j. \end{aligned} \quad (\text{B5})$$

Trace of ADM propagation:

$$\begin{aligned} & -\frac{4\pi G}{c^2} (\mu + 3p) + \frac{1}{\mathcal{N}} \dot{\kappa} + \frac{c^2 \Delta \mathcal{N}}{\mathcal{N}(1+2\varphi)} = \frac{1}{3} \kappa^2 + \frac{8\pi G}{c^2} (\mu + p) (\gamma^2 - 1) \\ & - \frac{c}{\mathcal{N}(1+2\varphi)} \left(\chi^i \kappa_{,i} + c \frac{\varphi^i \mathcal{N}_{,i}}{1+2\varphi} \right) + c^2 \overline{K}_j^i \overline{K}_i^j + \frac{1}{1+2\varphi} \frac{4\pi G}{c^2} \left(\Pi_i^i + \frac{1}{1+2\varphi} \Pi_{ij} \frac{v^i v^j}{c^2} \right). \end{aligned} \quad (\text{B6})$$

Tracefree ADM propagation:

$$\begin{aligned} & \left(\frac{1}{\mathcal{N}} \frac{\partial}{\partial t} - \kappa + \frac{c \chi^k}{\mathcal{N}(1+2\varphi)} \nabla_k \right) \left\{ \frac{c}{\mathcal{N}(1+2\varphi)} \left[\frac{1}{2} (\chi^i_{,j} + \chi^i_{,j}) - \frac{1}{3} \delta_j^i \chi^k_{,k} - \frac{1}{1+2\varphi} \left(\chi^i \varphi_{,j} + \chi_j \varphi_{,i} - \frac{2}{3} \delta_j^i \chi^k \varphi_{,k} \right) \right] \right\} \\ & - \frac{c^2}{(1+2\varphi)} \left[\frac{1}{1+2\varphi} \left(\nabla^i \nabla_j - \frac{1}{3} \delta_j^i \Delta \right) \varphi + \frac{1}{\mathcal{N}} \left(\nabla^i \nabla_j - \frac{1}{3} \delta_j^i \Delta \right) \mathcal{N} \right] \\ &= \frac{8\pi G}{c^2} (\mu + p) \left[\frac{\gamma^2 v^i v_j}{c^2(1+2\varphi)} - \frac{1}{3} \delta_j^i (\gamma^2 - 1) \right] + \frac{c^2}{\mathcal{N}^2(1+2\varphi)^2} \left[\frac{1}{2} (\chi^{i,k} \chi_{j,k} - \chi^{k,i} \chi_{j,k}) \right. \\ & \quad \left. + \frac{1}{1+2\varphi} (\chi^{k,i} \chi_{k,j} - \chi^{i,k} \chi_{j,k} + \chi_{k,j} \chi^k \varphi_{,i} - \chi_{j,k} \chi^i \varphi_{,k}) + \frac{2}{(1+2\varphi)^2} (\chi^i \chi_j \varphi^{,k} \varphi_{,k} - \chi^k \chi_k \varphi^{,i} \varphi_{,j}) \right] \\ & - \frac{c^2}{(1+2\varphi)^2} \left[\frac{3}{1+2\varphi} \left(\varphi^i \varphi_{,j} - \frac{1}{3} \delta_j^i \varphi^{,k} \varphi_{,k} \right) + \frac{1}{\mathcal{N}} \left(\varphi^i \mathcal{N}_{,j} + \varphi_{,j} \mathcal{N}^{,i} - \frac{2}{3} \delta_j^i \varphi^{,k} \mathcal{N}_{,k} \right) \right] \\ & + \frac{1}{1+2\varphi} \frac{8\pi G}{c^2} \left(\Pi_j^i - \frac{1}{3} \delta_j^i \Pi_k^k \right), \end{aligned} \quad (\text{B7})$$

ADM energy conservation:

$$\begin{aligned} & \frac{1}{\mathcal{N}} [\mu + (\mu + p) (\gamma^2 - 1)]_{\cdot} + \frac{c}{\mathcal{N}} \frac{\chi^i}{1+2\varphi} [\mu + (\mu + p) (\gamma^2 - 1)]_{,i} \\ & - (\mu + p) \frac{1}{3} (4\gamma^2 - 1) \kappa + \left(\frac{\mu + p}{1+2\varphi} \gamma^2 v^i \right)_{,i} + \left(\frac{3\varphi_{,i}}{1+2\varphi} + 2 \frac{\mathcal{N}_{,i}}{\mathcal{N}} \right) \frac{\mu + p}{1+2\varphi} \gamma^2 v^i \\ & + \frac{\gamma^2(\mu + p)}{c\mathcal{N}(1+2\varphi)^2} \left[\chi^{i,j} v_i v_j - \frac{1}{3} \chi^j_{,j} v^i v_i - \frac{2}{1+2\varphi} \left(v^i v^j \chi_i \varphi_{,j} - \frac{1}{3} v^i v_i \chi^j \varphi_{,j} \right) \right] = -\Pi^{(\text{ADM})}. \end{aligned} \quad (\text{B8})$$

ADM momentum conservation:

$$\begin{aligned} & \left(\frac{1}{\mathcal{N}} \frac{\partial}{\partial t} - \kappa \right) [(\mu + p) \gamma^2 v_i] + \frac{c}{\mathcal{N}} \frac{\chi^j}{1+2\varphi} [(\mu + p) \gamma^2 v_i]_{,j} + c^2 p_{,i} + c^2 (\mu + p) \frac{\mathcal{N}_{,i}}{\mathcal{N}} + \left(\frac{\mu + p}{1+2\varphi} \gamma^2 v^j v_i \right)_{,j} \\ & + \frac{c}{\mathcal{N}} \left(\frac{\chi^j}{1+2\varphi} \right)_{,i} (\mu + p) \gamma^2 v_j + \frac{\mu + p}{1+2\varphi} \gamma^2 v^j \left[\frac{1}{1+2\varphi} (3v_i \varphi_{,j} - v_j \varphi_{,i}) + \frac{1}{\mathcal{N}} (v_i \mathcal{N}_{,j} + v_j \mathcal{N}_{,i}) \right] = -c\Pi_i^{(\text{ADM})}. \end{aligned} \quad (\text{B9})$$

Covariant energy conservation:

$$\begin{aligned} & \left[\frac{\partial}{\partial t} + \frac{1}{1+2\varphi} (\mathcal{N} v^i + c \chi^i) \nabla_i \right] \mu + (\mu + p) \left\{ -\mathcal{N} \kappa + \frac{(\mathcal{N} v^i)_{,i}}{1+2\varphi} + \frac{\mathcal{N} v^i \varphi_{,i}}{(1+2\varphi)^2} \right. \\ & \left. + \frac{1}{\gamma} \left[\frac{\partial}{\partial t} + \frac{1}{1+2\varphi} (\mathcal{N} v^i + c \chi^i) \nabla_i \right] \gamma \right\} = -\Pi^{(\text{ADM})} + \frac{v^i}{c(1+2\varphi)} \Pi_i^{(\text{ADM})}. \end{aligned} \quad (\text{B10})$$

Covariant momentum conservation:

$$\begin{aligned} & \frac{\partial}{\partial t} (\gamma v_i) + \frac{1}{1+2\varphi} (\mathcal{N} v^k + c \chi^k) \nabla_k (\gamma v_i) + c^2 \gamma \mathcal{N}_{,i} + \frac{1-\gamma^2}{\gamma} \frac{c^2 \mathcal{N} \varphi_{,i}}{1+2\varphi} + c \gamma v^k \nabla_i \left(\frac{\chi_k}{1+2\varphi} \right) \\ & + \frac{1}{\mu+p} \left\{ c^2 \frac{\mathcal{N}}{\gamma} p_{,i} + \gamma v_i \left[\frac{\partial}{\partial t} + \frac{1}{1+2\varphi} (\mathcal{N} v^k + c \chi^k) \nabla_k \right] p \right\} \\ & = \frac{c \mathcal{N}}{(\mu+p)\gamma} \left[\frac{\gamma^2 v_i}{c} \Pi^{(\text{ADM})} - \left(\delta_i^j + \frac{\gamma^2 v_i v^j}{c^2(1+2\varphi)} \right) \Pi_j^{(\text{ADM})} \right]. \end{aligned} \quad (\text{B11})$$

We have

$$\begin{aligned} \overline{K}_j^i \overline{K}_i^j &= \frac{1}{\mathcal{N}^2(1+2\varphi)^2} \left\{ \frac{1}{2} \chi^{i,j} (\chi_{i,j} + \chi_{j,i}) - \frac{1}{3} \chi^i_{,i} \chi^j_{,j} - \frac{4}{1+2\varphi} \left[\frac{1}{2} \chi^i \varphi^{,j} (\chi_{i,j} + \chi_{j,i}) - \frac{1}{3} \chi^i_{,i} \chi^j \varphi_{,j} \right] \right. \\ &\quad \left. + \frac{2}{(1+2\varphi)^2} \left(\chi^i \chi_i \varphi^{,j} \varphi_{,j} + \frac{1}{3} \chi^i \chi^j \varphi_{,i} \varphi_{,j} \right) \right\}, \end{aligned} \quad (\text{B12})$$

for the tracefree part of extrinsic curvature \overline{K}_j^i , and

$$\begin{aligned} \Pi^{(\text{ADM})} &\equiv \left(\frac{1}{\mathcal{N}} \frac{\partial}{\partial t} + \frac{c\chi^k}{\mathcal{N}(1+2\varphi)} \nabla_k - \frac{4}{3} \kappa \right) \left(\frac{v^i v^j}{c^2(1+2\varphi)^2} \Pi_{ij} \right) \\ &\quad + \frac{c}{\mathcal{N}(1+2\varphi)^2} \left(\chi^i_{,j} - \frac{2\varphi^{,i} \chi_j}{1+2\varphi} \right) \left(\Pi_i^j - \frac{1}{3} \delta_i^j \Pi_k^k \right) + \left(\nabla_i + 2 \frac{\mathcal{N}_{,i}}{\mathcal{N}} + \frac{3\varphi_{,i}}{1+2\varphi} \right) \left(\frac{v^j}{(1+2\varphi)^2} \Pi_j^i \right), \end{aligned} \quad (\text{B13})$$

$$\begin{aligned} \Pi_i^{(\text{ADM})} &\equiv \left(\frac{1}{\mathcal{N}} \frac{\partial}{\partial t} + \frac{c\chi^k}{\mathcal{N}(1+2\varphi)} \nabla_k - \kappa \right) \left(\frac{v^j}{c(1+2\varphi)} \Pi_{ij} \right) + \frac{1}{\mathcal{N}(1+2\varphi)} \left(\frac{\chi^j}{1+2\varphi} \right)_{,i} v^k \Pi_{jk} \\ &\quad + \frac{c\mathcal{N}_{,i}}{\mathcal{N}(1+2\varphi)^2} \frac{v^j v^k}{c^2} \Pi_{jk} + \left(\nabla_j + 3 \frac{\varphi_{,j}}{1+2\varphi} + \frac{\mathcal{N}_{,j}}{\mathcal{N}} \right) \left(\frac{c}{1+2\varphi} \Pi_i^j \right) - \frac{c\varphi_{,i}}{(1+2\varphi)^2} \Pi_j^i. \end{aligned} \quad (\text{B14})$$

Here we add the continuity equation, $(\bar{\varrho} u^c)_{;c} = 0$,

$$\left[\frac{\partial}{\partial t} + \frac{1}{1+2\varphi} (\mathcal{N} v^i + c\chi^i) \nabla_i \right] \bar{\varrho} + \bar{\varrho} \left\{ -\mathcal{N} \kappa + \frac{(\mathcal{N} v^i)_{,i}}{1+2\varphi} + \frac{\mathcal{N} v^i \varphi_{,i}}{(1+2\varphi)^2} + \frac{1}{\gamma} \left[\frac{\partial}{\partial t} + \frac{1}{1+2\varphi} (\mathcal{N} v^i + c\chi^i) \nabla_i \right] \gamma \right\} = 0. \quad (\text{B15})$$

Special Relativistic Hydrodynamics with Gravity

**Weak gravity and Action-at-a-distance
With fully relativistic velocity, pressure
and anisotropic stress**

JH, Noh, Fabris, Piattella, Zimdahl, JCAP **07** (2016) 046
JH, Noh, ApJ (2016) to appear [arXiv:1610.05662](https://arxiv.org/abs/1610.05662)

Special Relativistic Hydrodynamics

Metric, Energy-momentum tensor:

$$g_{ab} = \eta_{ab}$$

$$T_{ab} = (\varrho c^2 + p) u_a u_b + p g_{ab} + \pi_{ab}$$

$$\mu = \varrho c^2, \quad \varrho \equiv \bar{\varrho} \left(1 + \frac{\Pi}{c^2} \right), \quad \pi_{ij} \equiv \Pi_{ij}, \quad u_i \equiv \gamma \frac{v_i}{c}, \quad \gamma \equiv \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}.$$

Continuity, Energy conservation, Momentum conservation:

$$(\bar{\varrho} u^c)_{;c} \equiv 0, T_{0;b}^b \equiv 0, T_{i;b}^b \equiv 0:$$

$$\begin{aligned} & (\bar{\varrho} \gamma)^. + \nabla \cdot (\bar{\varrho} \gamma \mathbf{v}) = 0, \\ & \left[\left(\varrho + \frac{p}{c^2} \right) \gamma^2 \right]^. + \nabla \cdot \left[\left(\varrho + \frac{p}{c^2} \right) \gamma^2 \mathbf{v} \right] = \frac{\dot{p}}{c^2} - \frac{1}{c^2} \nabla_i (\Pi_j^i v^j) - \frac{1}{c^4} (\Pi_{ij} v^i v^j)^., \\ & \left[\left(\varrho + \frac{p}{c^2} \right) \gamma^2 v_i \right]^. + \nabla \cdot \left[\left(\varrho + \frac{p}{c^2} \right) \gamma^2 v_i \mathbf{v} \right] = -\nabla_i p - \Pi_{i,j}^j - \frac{1}{c^2} (\Pi_{ij} v^j)^.. \end{aligned}$$

SR Hydrodynamics with Gravity

$$\frac{d\bar{\varrho}}{dt} + \bar{\varrho} \left(\nabla \cdot \mathbf{v} + \frac{d}{dt} \ln \gamma \right) = 0, \quad \frac{d}{dt} \equiv \frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla$$

$$\frac{d\varrho}{dt} + \left(\varrho + \frac{p}{c^2} \right) \left(\nabla \cdot \mathbf{v} + \frac{d}{dt} \ln \gamma \right) = -\frac{1}{c^2} \Pi_i^j \nabla_j v^i - \frac{1}{c^4} \Pi_{ij} v^i \dot{v}^j,$$

$$\begin{aligned} \frac{d\mathbf{v}}{dt} &= \boxed{\nabla \Phi} - \frac{1}{\varrho + p/c^2} \frac{1}{\gamma^2} \left(\nabla p + \frac{1}{c^2} \mathbf{v} \dot{p} \right) \\ &\quad + \frac{1}{\varrho + p/c^2} \frac{1}{\gamma^2} \left\{ -\Pi_{i,j}^j + \frac{1}{c^2} \left[\mathbf{v} (\Pi_j^k v^j)_{,k} - \frac{1}{\gamma^2} (\Pi_{ij} v^j)^\cdot \right] + \frac{1}{c^4} \mathbf{v} (\Pi_{jk} v^j v^k)^\cdot \right\}, \end{aligned}$$

$$\bar{\varrho} \frac{d\Pi}{dt} + p \left(\nabla \cdot \mathbf{v} + \frac{d}{dt} \ln \gamma \right) = -\Pi_i^j \nabla_j v^i - \frac{1}{c^2} \Pi_{ij} v^i \dot{v}^j$$

$$\Delta \Phi + 4\pi G \left(\varrho + 3 \frac{p}{c^2} \right) = -\frac{8\pi G}{c^2} \left[\left(\varrho + \frac{p}{c^2} \right) \gamma^2 v^2 + \Pi_i^i \right] \text{ Maximal Slicing: } K^i_i \equiv 0$$

$$\Delta \Phi + 4\pi G \varrho = -\frac{8\pi G}{c^2} \left[\left(\varrho + \frac{p}{c^2} \right) \gamma^2 v^2 + \Pi_i^i \right] \quad \text{Zero-shear Slicing: } \chi \equiv 0$$

$$+ \frac{12\pi G}{c^2} \Delta^{-1} \nabla_i \nabla_j \left[\left(\varrho + \frac{p}{c^2} \right) \gamma^2 v^i v^j + \Pi^{ij} \right]$$

$$\Delta \Psi + 4\pi G \varrho = -\frac{4\pi G}{c^2} \left[\left(\varrho + \frac{p}{c^2} \right) \gamma^2 v^2 + \Pi_i^i \right]$$

Metric:

$$ds^2 = - \left(1 - \frac{2\Phi}{c^2} \right) c^2 dt^2 - 2\chi_i c dt dx^i + \left(1 + \frac{2\Psi}{c^2} \right) \delta_{ij} dx^i dx^j$$

$$\chi_i \equiv c\chi_{,i} + \chi_i^{(v)} \text{ with } \chi^{(v)i}_{,i} \equiv 0$$

Assumptions:

$$\boxed{\frac{\Phi}{c^2} \ll 1, \quad \frac{\Psi}{c^2} \ll 1, \quad \gamma^2 \frac{t_\ell^2}{t_g^2} \ll 1}$$

Weak Gravity

Action-at-a-distance

$$t_g \sim 1/\sqrt{G\varrho} \quad t_\ell \sim \ell/c \sim 2\pi/(kc)$$

$$\gamma^2 \frac{t_\ell^2}{t_g^2} \sim \gamma^2 \frac{G\varrho\ell^2}{c^2} \sim \gamma^2 \frac{GM}{\ell c^2} \sim \frac{\Phi}{c^2} \sim \frac{\Psi}{c^2}$$

Gauge:

Maximal Slicing: $K^i_i \equiv 0$ (trace of extrinsic curvature)

Zero-shear Slicing: $\chi \equiv 0$

Newtonian Limit

Infinite speed-of-light limit :

$$\frac{\partial \bar{\varrho}}{\partial t} + \nabla \cdot (\bar{\varrho} \mathbf{v}) = 0,$$

$$\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} = \nabla \Phi - \frac{1}{\varrho} \left(\nabla p + \nabla_j \Pi_i^j \right),$$

$$\Delta \Phi + 4\pi G \varrho = 0. \quad \text{Both gauges}$$

$$\bar{\varrho} \frac{d\Pi}{dt} + p \nabla \cdot \mathbf{v} = -\Pi_i^j \nabla_j v^i.$$

$$\frac{d\varrho}{dt} + \left(\varrho + \frac{p}{c^2} \right) \nabla \cdot \mathbf{v} = -\frac{1}{c^2} \Pi_i^j \nabla_j v^i.$$

Slow-motion, Newtonian Stress

$$\frac{d\bar{\varrho}}{dt} + \bar{\varrho} \nabla \cdot \mathbf{v} = \frac{\bar{\varrho}}{\varrho + p/c^2} \frac{1}{c^2} \mathbf{v} \cdot \nabla p,$$

$$\frac{d\varrho}{dt} + \left(\varrho + \frac{p}{c^2} \right) \nabla \cdot \mathbf{v} = \frac{1}{c^2} \mathbf{v} \cdot \nabla p - \frac{1}{c^2} \Pi_i^j \nabla_j v^i,$$

$$\frac{d\mathbf{v}}{dt} = \nabla \Phi - \frac{1}{\varrho + p/c^2} \left(\nabla p + \frac{1}{c^2} \mathbf{v} \dot{p} + \nabla_j \Pi_i^j \right),$$

$$\Delta \Phi + 4\pi G \left(\varrho + 3 \frac{p}{c^2} \right) = 0, \quad \text{Maximal Slicing}$$

$$\Delta \Phi + 4\pi G \varrho = 0. \quad \text{Zero-shear Slicing}$$

$$\Delta \Psi + 4\pi G \varrho = 0.$$

Tolman-Oppenheimer-Volkoff

Static limit:

$$\nabla_i \Phi = \frac{1}{\varrho + p/c^2} \left(\nabla_i p + \nabla_j \Pi_i^j \right),$$
$$\Delta \Phi = -4\pi G \left(\varrho + 3 \frac{p}{c^2} \right). \quad \text{← Maximal Slicing}$$

Spherical symmetry:

$$\frac{dp}{dr} = -\frac{4\pi G}{r^2} \left(\varrho + \frac{p}{c^2} \right) \int_0^r \left(\varrho + 3 \frac{p}{c^2} \right) r^2 dr$$

Absence of 3p-term in ZSG:

$$\Delta \Phi = -4\pi G \left(\varrho + \frac{3}{c^2} \Delta^{-1} \nabla_i \nabla_j \Pi^{ij} \right)$$

← Zero-shear Slicing

Post-Newtonian Approximation

**Weakly relativistic gravity, velocity, pressure
and anisotropic stress**

Chandrasekhar ApJ (1965): **1PN, Minkowsky**

JH, Noh & Puetzfeld JCAP (2008): **Cosmological**

JH, Noh ApJ (2016) to appear [arXiv:1610.05662](https://arxiv.org/abs/1610.05662)

1PN Hydrodynamics:

$$\begin{aligned}
\dot{\bar{\varrho}} + \nabla \cdot (\bar{\varrho} \bar{\mathbf{v}}) &= -\frac{1}{c^2} \bar{\varrho} \left(\frac{\partial}{\partial t} + \bar{\mathbf{v}} \cdot \nabla \right) \left(\frac{1}{2} \bar{v}^2 + 3U \right), \\
\dot{\bar{\varrho}} + \nabla \cdot (\bar{\varrho} \bar{\mathbf{v}}) &= -\frac{1}{c^2} \left[\bar{\varrho} \left(\frac{\partial}{\partial t} + \bar{\mathbf{v}} \cdot \nabla \right) \left(\frac{1}{2} \bar{v}^2 + 3U + \Pi \right) + p \nabla \cdot \bar{\mathbf{v}} + \Pi_i^j \nabla_j v^i \right], \\
\dot{\bar{\mathbf{v}}} + \bar{\mathbf{v}} \cdot \nabla \bar{\mathbf{v}} - \nabla U + \frac{1}{\bar{\varrho}} \left(\nabla p + \nabla_j \Pi_i^j \right) \\
&= \frac{1}{c^2} \left\{ -2\nabla \left(U^2 - \tilde{\Phi} \right) + \dot{P}_i + \bar{v}^j (P_{i,j} - P_{j,i}) - \bar{\mathbf{v}} \left(\frac{\partial}{\partial t} + \bar{\mathbf{v}} \cdot \nabla \right) \left(\frac{1}{2} \bar{v}^2 + 3U \right) + \bar{v}^2 \nabla U \right. \\
&\quad \left. + \frac{1}{\bar{\varrho}} \left[\left(\bar{v}^2 + 4U + \Pi + \frac{p}{\bar{\varrho}} \right) (\nabla p + \nabla_j \Pi_i^j) - \bar{\mathbf{v}} \left(\frac{\partial}{\partial t} + \bar{\mathbf{v}} \cdot \nabla \right) p - (\Pi_i^j \bar{v}^j) \dot{} - \bar{\mathbf{v}} \Pi_k^j \nabla_j v^k + 2U \nabla_j \Pi_i^j \right] \right\}, \\
\Delta U + 4\pi G \bar{\varrho} &= -\frac{1}{c^2} \left[3\ddot{U} - 2U \Delta U + 2\Delta \tilde{\Phi} + \dot{P}_i^i + 8\pi G \left(\bar{\varrho} \bar{v}^2 + \frac{1}{2} \bar{\varrho} \Pi + \frac{3}{2} p \right) \right], \\
0 &= \frac{1}{4} (P_{,ji}^j - \Delta P_i) + \nabla \dot{U} - 4\pi G \bar{\varrho} \bar{\mathbf{v}}, \\
0 &= U - V.
\end{aligned}$$

General gauge: $P^i_{,i} + n \dot{U} = 0.$

Harmonic gauge: $n \equiv 4$

(Weinberg 1972)

Maximal Slicing: $n \equiv 3$

(Chandrasekhar 1965)

Zero-shear Slicing: $n \equiv 0$

$$g_{00} = - \left[1 - \frac{1}{c^2} 2U + \frac{1}{c^4} (2U^2 - 4\tilde{\Phi}) \right], \quad g_{0i} = -\frac{1}{c^3} P_i, \quad g_{ij} = \left(1 + \frac{1}{c^2} 2V \right) \delta_{ij}.$$

$$u^i \equiv u^0 \frac{\bar{v}^i}{c} \quad v_i = \bar{v}_i + \frac{1}{c^2} [(U + 2V) \bar{v}_i - P_i]$$

Gauge dependence of Propagation speed & Pressure

Under general gauge condition:

$$\Delta U + 4\pi G \varrho + \frac{1}{c^2} [- (n-3) \ddot{U} + 12\pi G p + \dots] = 0.$$

Propagation speed

$$= \frac{c}{\sqrt{n-3}}$$

= **c** for field strength, the Weyl tensor

$$\begin{cases} - \ddot{U} & \text{for Harmonic gauge } (n \equiv 4) \sim \text{Lorentz} \\ 0 & \text{for Maximal Slicing } (n \equiv 3) \sim \text{Coulomb} \\ + 3 \ddot{U} & \text{for Zero-shear Slicing } (n \equiv 0) \end{cases}$$

Pressure:

$$\Delta U + 4\pi G \varrho + \frac{1}{c^2} [4\pi G n p + \dots] = 0.$$

$$\begin{cases} 16\pi G p & \text{for Harmonic gauge} \\ 12\pi G p & \text{for Maximal Slicing} \\ 0 & \text{for Zero-shear Slicing} \end{cases}$$

Gauge transformation

$$\hat{x}^c = x^c + \xi^t$$

↑
MS ↑
ZSG ↑
ZSG to MS

$$\begin{aligned}\hat{\Phi} &= \Phi + c\dot{\xi}^t, & \hat{\Psi} &= \Psi, & \hat{\chi} &= \chi - \xi^t, & \hat{\kappa} &= \kappa + c\Delta\xi^t, \\ \hat{\varrho} &= \varrho, & \hat{\bar{\varrho}} &= \bar{\varrho}, & \hat{p} &= p, & \hat{v} &= v - c\xi^t.\end{aligned}$$

$$\Phi_{\text{MS}} = \Phi_{\text{ZSG}} + c\dot{\xi}^t, \quad \chi_{\text{MS}} = -\xi^t, \quad \kappa_{\text{ZSG}} = -c\Delta\xi^t, \quad v_{\text{MS}} = v_{\text{ZSG}} - c\xi^t.$$

$$\Delta\Phi_{\text{MS}} - \Delta\Phi_{\text{ZSG}} = c\Delta\dot{\xi}^t = -c\Delta\dot{\chi}_{\text{MS}} = -\dot{\kappa}_{\text{ZSG}} = 3\frac{\ddot{\Psi}}{c^2} = -12\pi G\frac{p}{c^2}.$$

Einstein's eqs.

Summary

- ❖ Fully nonlinear and exact perturbation formulation
- ❖ 1PN hydrodynamic equations in gauge-ready form
- ❖ Special relativistic hydrodynamics with gravity formulation
- ❖ Absence of pressure term in Poisson's equation in the zero-shear gauge is **real** but **troublesome**.
- ❖ The Maximal Slicing (Uniform-expansion gauge) is better than the Zero-shear Slicing (Longitudinal or Conformal-Newtonian gauge).
- ❖ Anticipate numerical applications in Astrophysical situations; need **B**, working on it ...