Axion as a CDM Candidate: Proof to Fully NL Perturbation

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Axion as a CDM

Scalar field:

$$\tilde{\phi}^{;c}{}_{c} = \tilde{V}_{,\tilde{\phi}},$$

$$\tilde{T}^{a}_{b} = \tilde{\phi}^{,a}\tilde{\phi}_{,b} - \left(\frac{1}{2}\tilde{\phi}^{,c}\tilde{\phi}_{,c} + \tilde{V}\right)\delta^{a}_{b}.$$

axion [6] is the pseudo-Goldstone boson of the global Peccei–Quinn [7] U(1) symmetry introduced to solve the strong *CP* problem. The symmetry is spontaneously broken at a scale f_a giving rise to a massless Goldstone boson. QCD instanton effects generate explicit symmetry breaking terms below 1 GeV which give the axion a small mass. The only viable realization of this idea is in invisible axion models [8] in which both the axion mass and its couplings to other fields are suppressed by f_a^{-1}.

PLB **126** 179 (1991) Axenides et al

Axion: $V = \frac{1}{2} m^2 \phi^2$

Axion, background order:

Strictly ignore: $\frac{H}{m} = 2.133 \times 10^{-28} h \left(\frac{10^{-5}\mathrm{eV}}{m}\right)\left(\frac{H}{H_0}\right)$

Ansatz: $\phi(t) = \phi_+(t)\sin(mt) + \phi_-(t)\cos(mt)$

Ratra (1991)

EOM: $\ddot{\phi} + 3H\dot{\phi} + m^2\phi = 0,$

$$\Longrightarrow \phi(t) = a^{-3/2}[\phi_{+0}\sin(mt) + \phi_{-0}\cos(mt)]$$

Fluid quantities:

$$\mu = \frac{1}{2}\langle\dot{\phi}^2 + m^2\phi^2\rangle = \frac{1}{2}m^2 a^{-3}(\phi_{+0}^2 + \phi_{-0}^2),$$

$$p = \frac{1}{2}\langle\dot{\phi}^2 - m^2\phi^2\rangle = 0,$$

$$\langle f(t)\rangle \equiv \frac{m}{2\pi}\int_0^{2\pi/m} f(t')dt'$$

$\therefore$ **Pressureless fluid!**

Ratra, PRD **44** 352 (1991) **Synchronous Gauge**

Zero-pressure Irrotational Fluid

Linear-order:

Axion

$$\ddot{\delta} + 2H\dot{\delta} - 4\pi G\varrho\delta = -\frac{\hbar^2\Delta^2}{4m^2a^4}\delta.$$

Relativistic/Newtonian correspondence to second order.
This equation is valid to fully nonlinear order in Newtonian theory.

Second-order:

Axion

$$\ddot{\delta} + 2\frac{\dot{a}}{a}\dot{\delta} - 4\pi G\mu\delta = -\frac{1}{a^2}\frac{\partial}{\partial t}\left[a\nabla\cdot(\delta\mathbf{u})\right] + \frac{1}{a^2}\nabla\cdot(\mathbf{u}\cdot\nabla\mathbf{u}) - \frac{\hbar^2\Delta}{2m^2a^4}\frac{\Delta\sqrt{1+\delta}}{\sqrt{1+\delta}}$$

Pure relativistic correction appearing from third order.
All involving φ.

Third-order:

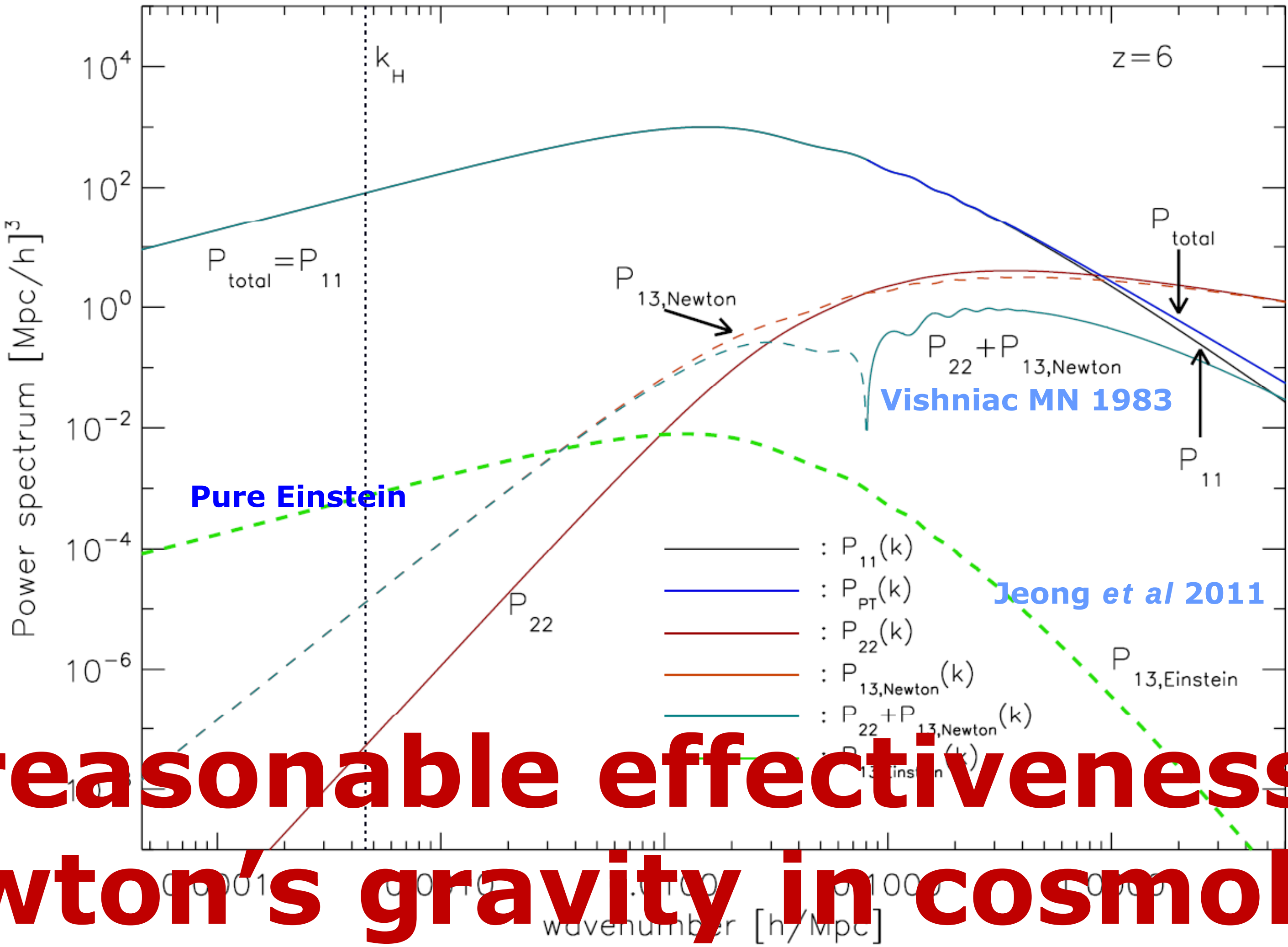
Axion

$$\ddot{\delta} + 2\frac{\dot{a}}{a}\dot{\delta} - 4\pi G\mu\delta = -\frac{1}{a^2}\frac{\partial}{\partial t}\left[a\nabla\cdot(\delta\mathbf{u})\right] + \frac{1}{a^2}\nabla\cdot(\mathbf{u}\cdot\nabla\mathbf{u}) - \frac{\hbar^2\Delta}{2m^2a^4}\frac{\Delta\sqrt{1+\delta}}{\sqrt{1+\delta}}$$

$$\boxed{\begin{aligned} &+\frac{1}{a^2}\frac{\partial}{\partial t}\{a[2\varphi\mathbf{u} - \nabla(\Delta^{-1}X)]\cdot\nabla\delta\} - \frac{4}{a^2}\nabla\cdot\left[\varphi\left(\mathbf{u}\cdot\nabla\mathbf{u} - \frac{1}{3}\mathbf{u}\nabla\cdot\mathbf{u}\right)\right] \\ &+\frac{2}{3a^2}\varphi\mathbf{u}\cdot\nabla(\nabla\cdot\mathbf{u}) + \frac{\Delta}{a^2}[\mathbf{u}\cdot\nabla(\Delta^{-1}X)] - \frac{1}{a^2}\mathbf{u}\cdot\nabla X - \frac{2}{3a^2}X\nabla\cdot\mathbf{u}, \\ &X \equiv 2\varphi\nabla\cdot\mathbf{u} - \mathbf{u}\cdot\nabla\varphi + \frac{3}{2}\Delta^{-1}\nabla\cdot\left[\mathbf{u}\cdot\nabla\left(\nabla\varphi\right) + \mathbf{u}\Delta\varphi\right].\end{aligned}}$$

Valid in ALL cosmological scales

Leading Nonlinear Density Power-spectrum in the Comoving gauge:



Unreasonable effectiveness of Newton's gravity in cosmology!

Jeong, et al., ApJ **722**, 1 (2011)

Fully NL & Exact Pert. Theory

JH, Noh, MNRAS **433** (2013) 3472
Noh, JCAP **07** (2014) 037
JH, Noh, Park, MNRAS **461** (2016) 3239

Convention:

(Bardeen 1988)

Decomposition, possible to NL order (York 1973)

No TT-pert (GW)

$$ds^2 = -a^2\left(1+2\alpha\right)d\eta^2 - 2a^2\left(\beta_{,i} + B_i^{(v)}\right)d\eta dx^i$$
$$+a^2\left[\left(1+2\varphi\right)g^{(3)}_{ij} + 2\cancel{\gamma_{,i|j}} + \cancel{C^{(v)}_{i|j}} + \cancel{C^{(v)}_{j|i}} + 2\cancel{C^{(t)}_{ij}}\right]dx^i dx^j,$$

$$\chi \equiv a\beta + a^2\cancel{\dot{\gamma}}, \quad \Psi_i^{(v)} \equiv B_i^{(v)} + a\cancel{\dot{C}_i^{(v)}},$$

Spatial gauge condition

$$\widetilde{T}_{ab} = \widetilde{\mu}\widetilde{u}_a\widetilde{u}_b + \widetilde{p}\left(\widetilde{u}_a\widetilde{u}_b + \widetilde{g}_{ab}\right) + \widetilde{\pi}_{ab}, \quad \widetilde{u}_i \equiv \frac{a}{c}\widehat{\gamma}\widehat{v}_i, \quad \widehat{\gamma} \equiv \frac{1}{\sqrt{1-\frac{\widehat{v}^k\widehat{v}_k}{c^2(1+2\varphi)}}},$$

$$\widetilde{\mu} \equiv \mu + \delta\mu \equiv \mu\left(1+\delta\right), \quad \widetilde{p} \equiv p + \delta p, \quad \widetilde{\pi}_{ij} \equiv a^2\Pi_{ij} \quad \widehat{v}_i \equiv -\widehat{v}_{,i} + \widehat{v}_i^{(v)}$$

Spatial gauge:

$$\gamma \equiv 0 \equiv C_i^{(v)},$$
$$\chi_i \equiv \chi_{,i} + a\Psi_i^{(v)} = a\left(\beta_{,i} + B_i^{(v)}\right)$$

Temporal gauge still not taken yet!

➡ Complete spatial gauge fixing.
Remaining variables are spatially gauge-invariant to fully NL order! ∴ Lose no generality!

HJ, Noh MNRAS (2013); Noh JCAP (2014); HN, Noh, Park MNRAS (2016)

Metric convention:

$$\boxed{\widetilde{g}_{00} = -a^2(1+2\alpha), \qquad \widetilde{g}_{0i} = -a\chi_i, \qquad \widetilde{g}_{ij} = a^2(1+2\varphi)\,g^{(3)}_{ij}.}$$

Inverse metric:

$$\widetilde{g}^{00} = -\frac{1}{a^2}\frac{1+2\varphi}{(1+2\varphi)(1+2\alpha)+\chi^k\chi_k/a^2},$$

$$\widetilde{g}^{0i} = -\frac{1}{a^2}\frac{\chi^i/a}{(1+2\varphi)(1+2\alpha)+\chi^k\chi_k/a^2},$$

$$\widetilde{g}^{ij} = \frac{1}{a^2(1+2\varphi)}\left(g^{(3)ij} - \frac{\chi^i\chi^j/a^2}{(1+2\varphi)(1+2\alpha)+\chi^k\chi_k/a^2}\right).$$

Exact!

Using the ADM and the covariant formalisms the rest are simple algebra. We do not even need the connection!

HJ, Noh MNRAS (2013); Noh JCAP (2014)

Axion, perturbation:

Ansatz: $\delta\phi(k,t) = \delta\phi_+(k,t)\sin(mt) + \delta\phi_-(k,t)\cos(mt).$

Ratra (1991)

$$\widetilde{T}_{ab} = \langle \widetilde{T}_{ab} \rangle$$

$$\widetilde{T}_{ab} = \langle \left[\widetilde{\phi}_{,a}\widetilde{\phi}_{,b} - \frac{1}{2}\left(\widetilde{\phi}^{;c}\widetilde{\phi}_{,c} + m^2\widetilde{\phi}^2 \right) \widetilde{g}_{ab} \right] \rangle$$

$$\widetilde{T}_{ab} = \widetilde{\mu}\widetilde{u}_a\widetilde{u}_b + \widetilde{p}\left(\widetilde{g}_{ab} + \widetilde{u}_a\widetilde{u}_b\right) + \widetilde{\pi}_{ab}$$

Axion-comoving gauge: $\widetilde{T}^{0,i}_{i} \equiv 0$

Zero-shear gauge: $\chi^{i}_{\ ,i} \equiv 0$

JH, Noh PLB **680** 1 (2009) **Axion-CG**; Nambu-Sasaki (1990), Sikivie-Yang (2009) **ZSG**

Lapse & Fluid quantities:

Perturbed Lapse function

$$\delta\mathcal{N} = \frac{\hbar^2}{2m^2c^2a^2(1+2\varphi)}\frac{\Delta\sqrt{1+\delta}}{\sqrt{1+\delta}},$$

$$\frac{\delta p}{\mu} = -\frac{\hbar^2\Delta\delta}{4m^2c^2a^2(1+2\varphi)},$$

$$\Pi_{ij} = \mu\frac{\hbar^2}{4m^2c^2a^2(1+\delta)}\left(\delta_{,i}\delta_{,j} - \frac{1}{3}\delta_{ij}\delta^{,k}\delta_{,k}\right)$$

Even non-vanishing anisotropic stress!

Zero-pressure fluid in the comoving gauge

Axion-

Exact equations (flat background):

Covariant energy-conservation, Trace of ADM propagation, ADM momentum & energy constraint:

$$\dot{\delta} - \kappa + \frac{c}{a^2(1+2\varphi)}\delta_{,i}\chi^i - \delta\kappa = 0,$$

$$\dot{\kappa} + 2H\kappa - 4\pi G\delta\varrho + \frac{c}{a^2(1+2\varphi)}\kappa_{,i}\chi^i - \frac{1}{3}\kappa^2 - \frac{c^2}{a^4(1+2\varphi)^2}\left\{\frac{1}{2}\chi^{i,j}\left(\chi_{i,j}+\chi_{j,i}\right) - \frac{1}{3}\chi^i{}_{,i}\chi^j{}_{,j}\right.$$
$$\left. - \frac{4}{1+2\varphi}\left[\frac{1}{2}\chi^i\varphi^{,j}\left(\chi_{i,j}+\chi_{j,i}\right) - \frac{1}{3}\chi^i{}_{,i}\chi^j\varphi_{,j}\right] + \frac{2}{(1+2\varphi)^2}\left(\chi^i\chi_i\varphi^{,j}\varphi_{,j} + \frac{1}{3}\chi^i\chi^j\varphi_{,i}\varphi_{,j}\right)\right\}$$
$$= -\frac{c^2\Delta\mathcal{N}}{a^2(1+2\varphi)} = -\frac{\hbar^2\Delta}{2m^2a^4(1+2\varphi)}\frac{\Delta\sqrt{1+\delta}}{\sqrt{1+\delta}}. \longleftarrow \textbf{Axion}$$

$$\frac{2}{3}\kappa_{,i} + \frac{c}{2a^2(1+2\varphi)}\left(\Delta\chi_i + \frac{1}{3}\chi^k{}_{,ik}\right) = \frac{c}{a^2(1+2\varphi)}\left\{-\frac{\varphi_{,j}}{1+2\varphi}\left[\frac{1}{2}\left(\chi^j{}_{,i} + \chi_i{}^{,j}\right) - \frac{1}{3}\delta^j_i\chi^k{}_{,k}\right]\right.$$
$$\left. - \frac{\varphi^{,j}}{(1+2\varphi)^2}\left(\chi_i\varphi_{,j} + \frac{1}{3}\chi_j\varphi_{,i}\right) + \frac{1}{1+2\varphi}\nabla_j\left(\chi^j\varphi_{,i} + \chi_i\varphi^{,j} - \frac{2}{3}\delta^j_i\chi^k\varphi_{,k}\right)\right\},$$

$$\frac{c^2\Delta\varphi}{a^2(1+2\varphi)^2} + 4\pi G\delta\varrho + H\kappa = \frac{1}{6}\kappa^2 + \frac{3}{2}\frac{c^2\varphi^{,i}\varphi_{,i}}{a^2(1+2\varphi)^3} - \frac{c^2}{4a^4(1+2\varphi)^2}\left\{\frac{1}{2}\chi^{i,j}\left(\chi_{i,j}+\chi_{j,i}\right) - \frac{1}{3}\chi^i{}_{,i}\chi^j{}_{,j}\right.$$
$$\left. - \frac{4}{1+2\varphi}\left[\frac{1}{2}\chi^i\varphi^{,j}\left(\chi_{i,j}+\chi_{j,i}\right) - \frac{1}{3}\chi^i{}_{,i}\chi^j\varphi_{,j}\right] + \frac{2}{(1+2\varphi)^2}\left(\chi^i\chi_i\varphi^{,j}\varphi_{,j} + \frac{1}{3}\chi^i\chi^j\varphi_{,i}\varphi_{,j}\right)\right\}.$$

Identify: $\kappa \equiv -\frac{1}{a}\nabla\cdot\mathbf{u}$

$$K^i_i \equiv -3H + \kappa = -\widetilde{\theta}^{(n)}$$

Extrinsic curvature   Expansion scalar of the normal frame

Weak gravity limit: $\varphi \ll 1$

Axion

$$\ddot{\delta} + 2H\dot{\delta} - 4\pi G\varrho\delta + \frac{1}{a^2}\left[a\nabla \cdot (\delta \mathbf{u})\right]^{\cdot} - \frac{1}{a^2}\nabla \cdot (\mathbf{u} \cdot \nabla \mathbf{u}) = -\frac{\hbar^2 \Delta}{2m^2 a^4} \frac{\Delta\sqrt{1+\delta}}{\sqrt{1+\delta}}$$

LHS = Exactly Newtonian valid to fully NL order.
In the axion-comoving gauge

$$c^2 \frac{\Delta}{a^2}\varphi + 4\pi G \delta\varrho = H\frac{1}{a}\nabla \cdot \mathbf{u} + \frac{1}{4a^2}\left[(\nabla \cdot \mathbf{u})^2 - u^{,ij}u_{,ij}\right]$$

Curvature perturbation
in the comoving gauge
≠ Newonian potential

Solution & Jeans scale:

$$\ddot{\delta} + 2H\dot{\delta} - 4\pi G\varrho\delta + \frac{\hbar^2 k^4}{4m^2 a^4}\delta = 0$$

Exact solution in mde with K=0=Lambda:

$$\delta_{\pm} \propto t^{-1/6} J_{\mp 5/2}\left(3At^{-1/3}\right), \quad A \equiv \frac{k^2}{2m}\left(\frac{t^{2/3}}{a}\right)^2.$$

Sound speed: $c_s \equiv \sqrt{\frac{\delta p}{\delta \varrho}} = \frac{1}{2}\frac{\hbar k}{ma}$

Jeans scale: Khlopov, Malomed, Zel'dovich, MN (1985), **Newtonian**
Nambu, Sasaki, PRD (1990), **Zero-shear gauge**

$$\lambda_{J_a} \equiv \frac{2\pi a}{k_{J_a}} = \sqrt{\frac{\pi\hbar}{m}\sqrt{\frac{\pi}{G\varrho}}} \sim 5.4 \times 10^{14}\mathrm{cm}\sqrt{\frac{10^{-5}\mathrm{eV}}{mh}}$$

$\therefore$ **CDM in ALL cosmological scales**

JH, Noh PLB (2009) **Axion-CG, in all scales**;
Sikivie, Yang PRL **103** 111301 (2009) **Zero-shear gauge, in sub-horizon only**

In Bose-Einstein condensate literature:

$$c^2 \frac{\Delta}{a^2} \delta\mathcal{N} = \frac{\hbar^2 \Delta}{2m^2 a^4} \frac{\Delta\sqrt{1+\delta}}{\sqrt{1+\delta}} = \frac{\hbar^2 \Delta}{2m^2 a^4} \frac{\Delta\sqrt{\widetilde{\varrho}}}{\sqrt{\widetilde{\varrho}}}$$

M.Yu. Khlopov, B.A. Malomed and Ya.B. Zeldovich, Mon. Not. R. Astron. Soc. **215**, 575 (1985); M. Bianchi, D. Grasso and R. Ruffini, Astron. Astrophys. **231**, 301 (1990).

F. Dalfovo, S. Giorgini, L. Pitaevskii, S. Stringari, Rev. Mod. Phys. **71**, 463 (1999); C.J. Pethick, H. Smith, *Bose-Einstein condensation in dilute gases*, (Cambridge Univ. Press, 2002), Chapter 7; L. Pitaevskii, S. Stringari, *Bose-Einstein Condensation*, (Oxford Univ. Press, 2003), Chapter 5; C. Barceló, S. Liberati and M. Visser, Living Rev. Relat. **8**, 12 (2005).

Interpretation as de Broglie wavelength:

$$\lambda \sim \frac{\hbar}{m v_g} \sim \frac{\hbar}{m\lambda/t_g} \sim \frac{\hbar}{m\lambda\sqrt{G\varrho}}$$

W. Hu, R. Barkana and A. Gruzinov, Phys. Rev. Lett. **85**, 1158 (2000)

Low-mass Axion

Conventional mass:

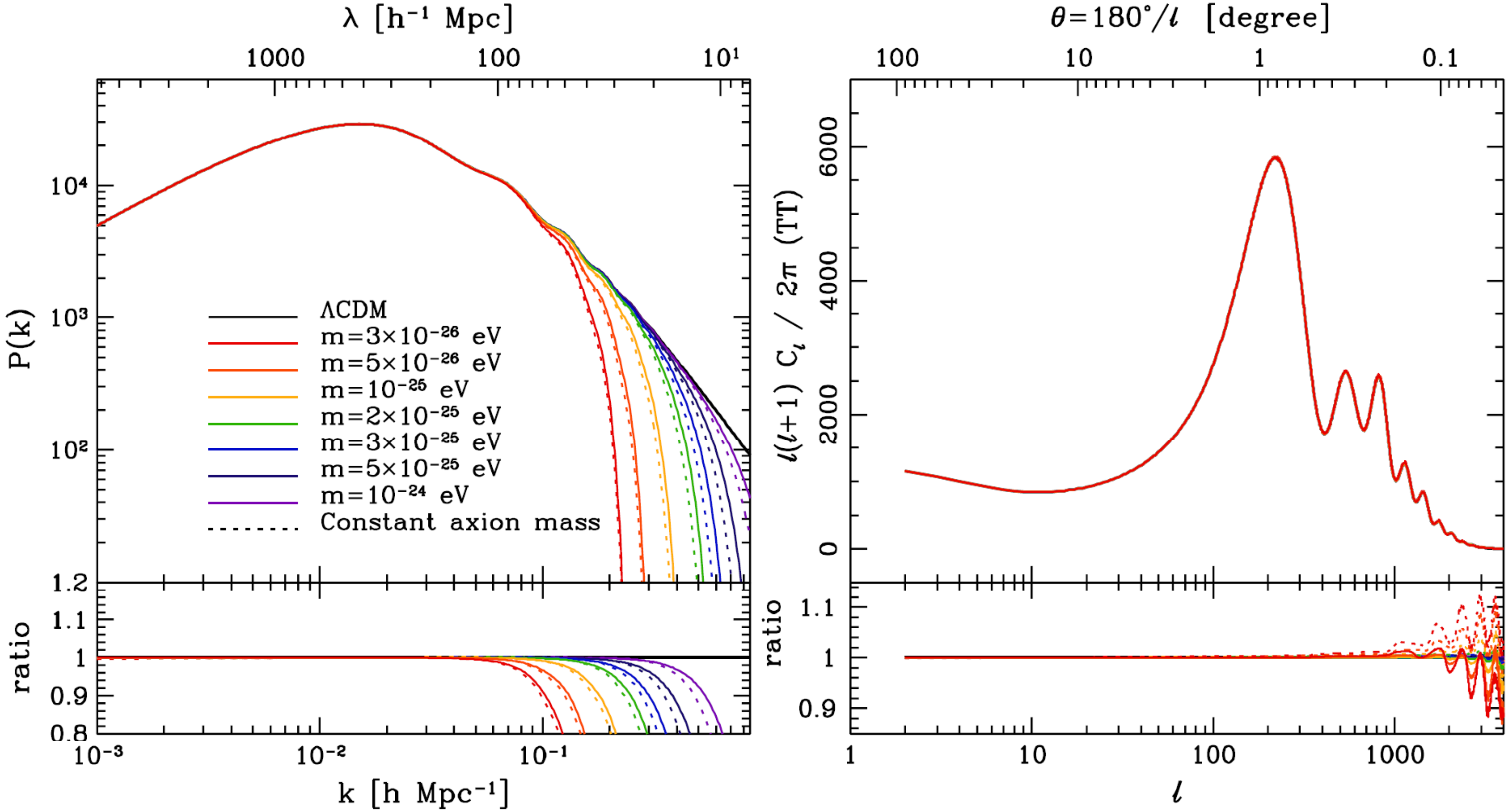
$$\lambda_J = \left(\frac{\pi^3}{G\mu_{a0}m^2}\right)^{1/4} = 50h^{-1/2}\left(\frac{m}{10^{-5}\ \mathrm{eV}}\right)^{-1/2}\ \mathrm{AU}$$

Extreme low mass:

$$\lambda_J = 2.4h^{-1/2}\left(\frac{m}{10^{-25}\ \mathrm{eV}}\right)^{-1/2}\ \mathrm{Mpc}$$

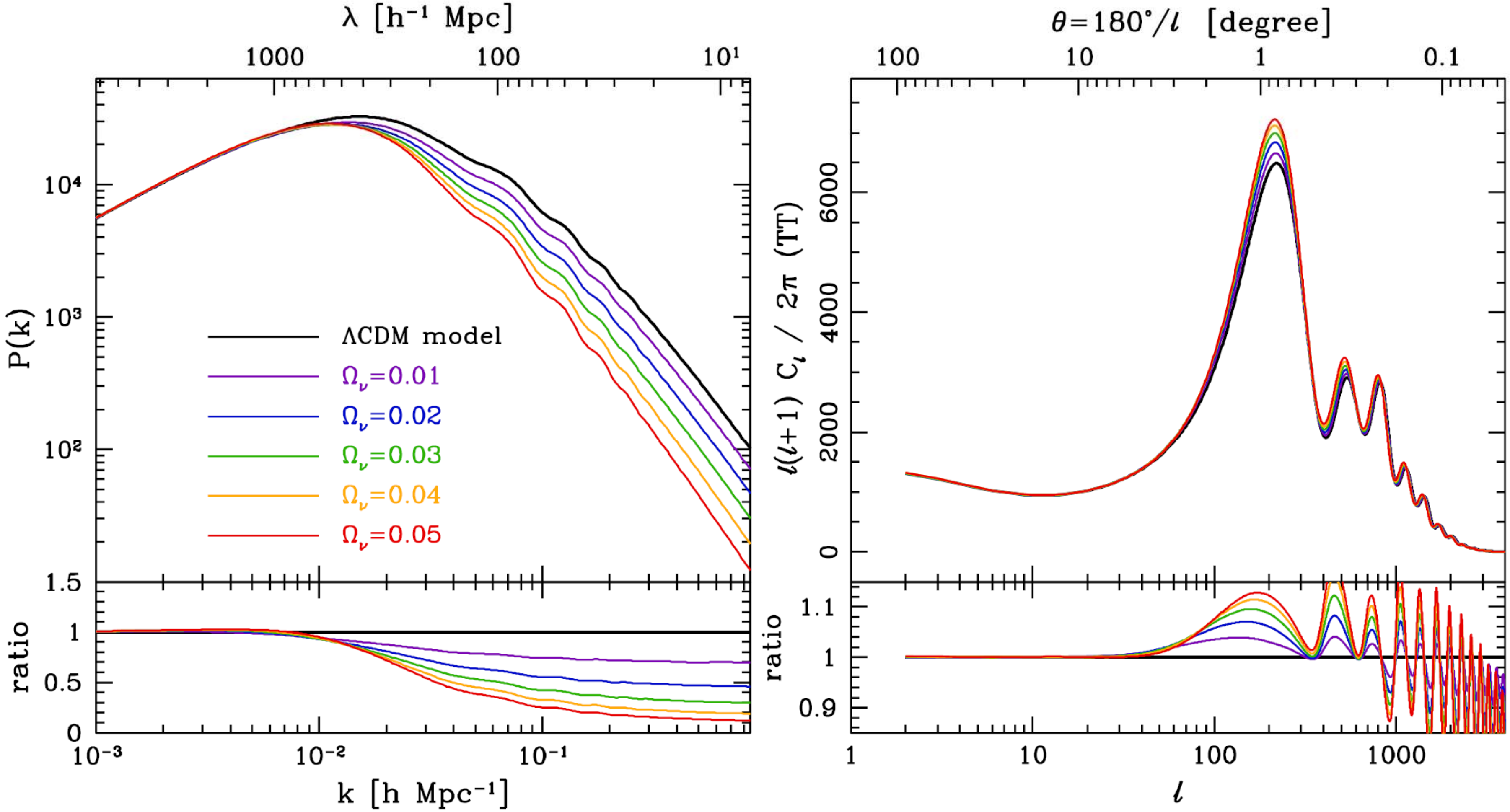
Park, JH, Noh PRD **86** 083535 (2012)

Low-mass axion as a CDM with SS cut:



Park, JH, Noh PRD (2012)

Neutrino as a HDM with SS damping:



mass ranges from $m_\nu = 0.154$ eV ($\Omega_{\nu 0} = 0.01$; red) to 0.769 eV ($\Omega_{\nu 0} = 0.05$; violet curves), with a relation $(\Omega_{\nu 0} + \Omega_{c0})h^2 =$ 0.1123. Black curves represent the power spectrum of the fiducial ΛCDM model with massless neutrinos. The curves in the bottom

Park, JH, Noh PRD (2012)

Summary

- Axion ($m \sim 10^{-5}$eV) as a CDM, shown to fully nonlinear order perturbation
- Axion has characteristic pressure and anisotropic stress.
- Extreme low-mass axion like particle as a WDM with small-scale cut off in the power spectrum