Basics of QCD

Lecture II

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e+ e- collisions : QCD in the final state

1. Infrared safety
2. Towards realistic final states
3. Jets

New set of questions

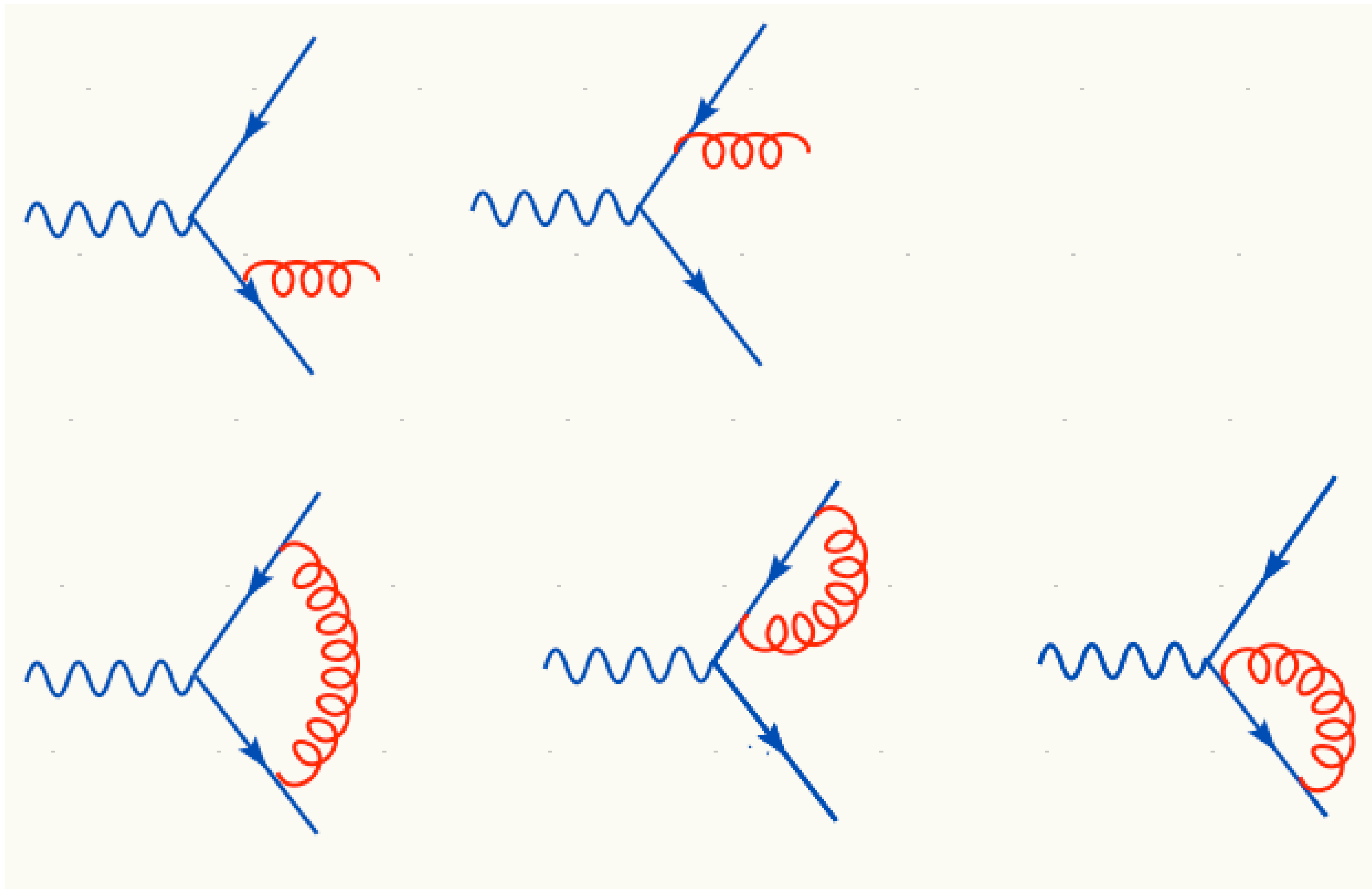
The “infrared” behaviour of QCD

1. How can we identify a cross sections for producing quarks and gluons with a cross section for producing hadrons?

2. Given the fact that free quarks are not observed, why is the computed Born cross section so good?

3. Are there other calculable, i.e., that do not depend on the non-perturbative dynamics (like hadronization), quantities besides the total cross section?

Anatomy of a NLO calculation



Real

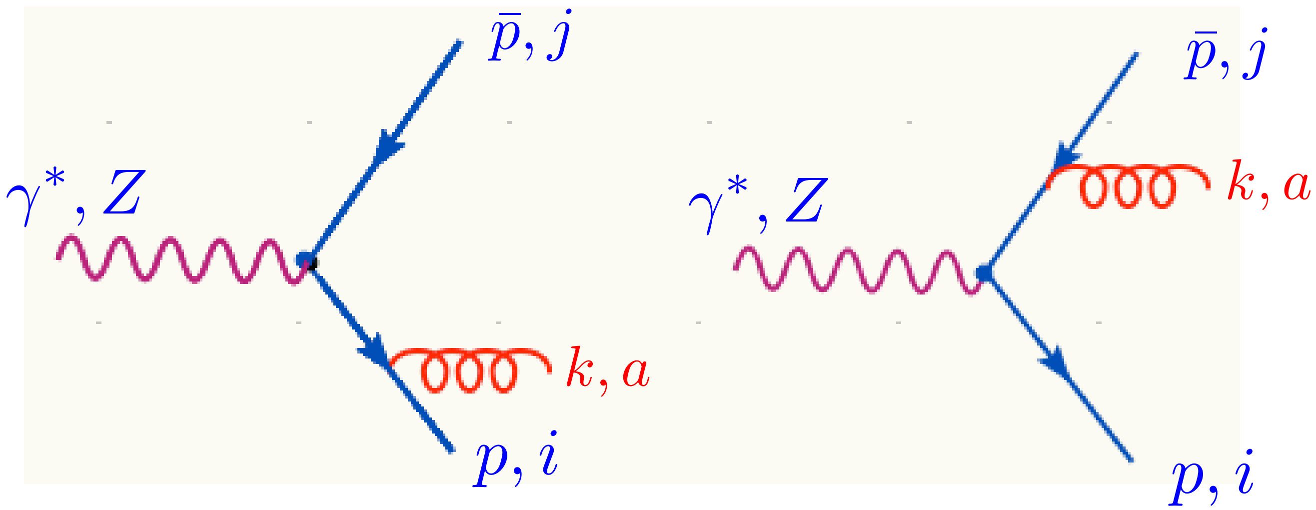
Virtual

The KLN theorem states that divergences appear because some of the final state are physically degenerate but we treated them as different. A final state with a soft gluon is nearly degenerate with a final state with no gluon at all (virtual).

$$\int \frac{d^d k}{(2\pi)^d} \cdots$$

$$\sigma^{\mathrm{NLO}} = \int_R |M_{real}|^2 d\Phi_3 + \int_V 2Re\left(M_0 M^*_{virt}\right) d\Phi_2 = \text{finite!}$$

Anatomy of a NLO calculation



Let's consider the real gluon emission corrections to the process e+e- →qq. The full calculation is a little bit tedious, but since we in any case interested in the issues arising in the infra-red, we already start in that approximation.

$$A = \bar{u}(p)\not{\epsilon}(-ig_s)\frac{-i}{\not{p}+\not{k}}\Gamma^\mu v(\bar{p})t^a + \bar{u}(p)\Gamma^\mu \frac{i}{\not{\bar{p}}+\not{k}}(-ig_s)\not{\epsilon}v(\bar{p})t^a$$

$$= -g_s\left[\frac{\bar{u}(p)\not{\epsilon}(\not{p}+\not{k})\Gamma^\mu v(\bar{p})}{2p\cdot k} - \frac{\bar{u}(p)\Gamma^\mu(\not{\bar{p}}+\not{k})\not{\epsilon}v(\bar{p})}{2\bar{p}\cdot k}\right]t^a$$

The denominators $2p\cdot k = p_0 k_0(1-\cos\theta)$ give singularities for collinear ($\cos\theta \to 1$) or soft ($k_0 \to 0$) emission. By neglecting k in the numerators and using the Dirac equation, the amplitude simplifies and factorizes over the Born amplitude:

$$A_{soft} = -g_s t^a\left(\frac{p\cdot\epsilon}{p\cdot k} - \frac{\bar{p}\cdot\epsilon}{\bar{p}\cdot k}\right)A_{Born} \qquad A_{Born} = \bar{u}(p)\Gamma^\mu v(\bar{p})$$

Factorization: Independence of long-wavelength (soft) emission form the hard (short-distance) process. Soft emission is universal!!

Anatomy of a NLO calculation

By squaring the amplitude we obtain:

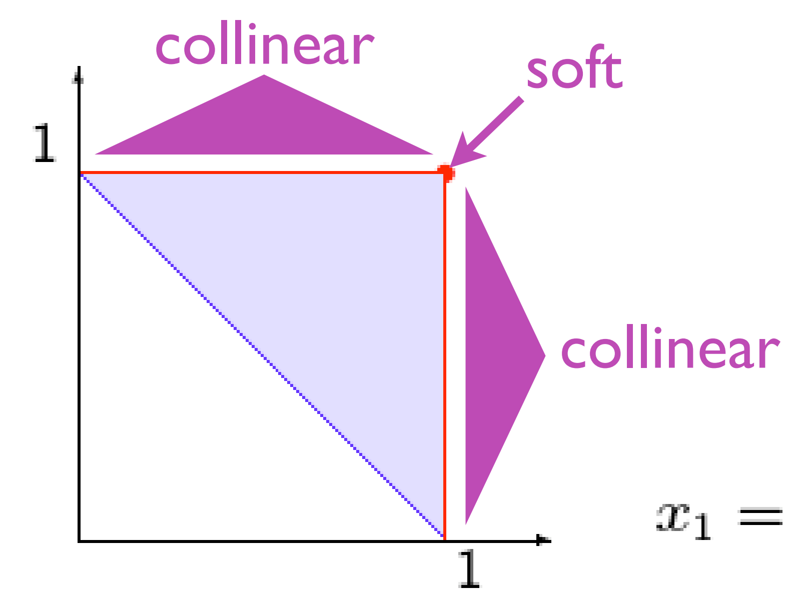
$$\sigma^{\rm REAL}_{q\bar q g} = C_F g_s^2 \sigma^{\rm Born}_{q\bar q} \int \frac{d^3k}{2k^0(2\pi)^3} 2 \frac{p\cdot\bar p}{(p\cdot k)(\bar p\cdot k)}$$

$$= C_F \frac{\alpha_S}{2\pi}\sigma^{\rm Born}_{q\bar q} \int d\cos\theta \frac{dk^0}{k^0}\frac{4}{(1-\cos\theta)(1+\cos\theta)}$$

Two collinear divergences and a soft one. Very often you find the integration over phase space expressed in terms of x1 and x2, the fraction of energies of the quark and anti-quark:

$$x_1 = 1 - x_2 x_3(1-\cos\theta_{23})/2$$
$$x_2 = 1 - x_1 x_3(1-\cos\theta_{13})/2$$
$$x_1 + x_2 + x_3 = 2$$
$$0 \le x_1, x_2 \le 1, \quad \text{and} \quad x_1 + x_2 \ge 1$$

$$x_2 = \frac{2E_{\bar q}}{\sqrt{s}}$$



$$x_1 = \frac{2E_q}{\sqrt{s}}$$

So we can now predict the divergent part of the virtual contribution, while for the finite part an explicit calculation is necessary:

$$\sigma^{\rm VIRT}_{q\bar q} = -\sigma^{\rm Born}_{q\bar q} C_F \frac{\alpha_S}{2\pi}\int d\cos\theta' \frac{dk_0'}{k_0'}\frac{1}{1-\cos^2\theta} 2\delta(k_0')[\delta(1-\cos\theta')+\delta(1+\cos\theta')] + \ldots$$

Anatomy of a NLO calculation

Summary:

$$\sigma^{\mathrm{REAL}} + \sigma^{\mathrm{VIRT}} = \infty - \infty = ?$$

Solution: regularize the "intermediate" divergences, by giving a gluon a mass (see later) or going to d=4-2ε dimensions.

$$\int^1 \frac{1}{1-x}dx = -\log 0 \xrightarrow{\text{regularization}} \int^1 \frac{(1-x)^{-2\epsilon}}{1-x}dx = -\frac{1}{2\epsilon}$$

This gives:

$$\sigma^{\mathrm{REAL}} = \sigma^{\mathrm{Born}} C_F \frac{\alpha_S}{2\pi}\left(\frac{2}{\epsilon^2} + \frac{3}{\epsilon} + \frac{19}{2} - \pi^2\right)$$

$$\sigma^{\mathrm{VIRT}} = \sigma^{\mathrm{Born}} C_F \frac{\alpha_S}{2\pi}\left(-\frac{2}{\epsilon^2} - \frac{3}{\epsilon} - 8 + \pi^2\right)$$

$$\lim_{\epsilon\to 0}(\sigma^{\mathrm{REAL}} + \sigma^{\mathrm{VIRT}}) = C_F \frac{3}{4}\frac{\alpha_S}{\pi}\sigma^{\mathrm{Born}}$$

$$R_1 = R_0\left(1 + \frac{\alpha_S}{\pi}\right)$$ as presented before

New set of questions

1. How can we identify a cross sections for producing (few) quarks and gluons with a cross section for producing (many) hadrons?

2. Given the fact that free quarks are not observed, why is the computed Born cross section so good?

Answers:

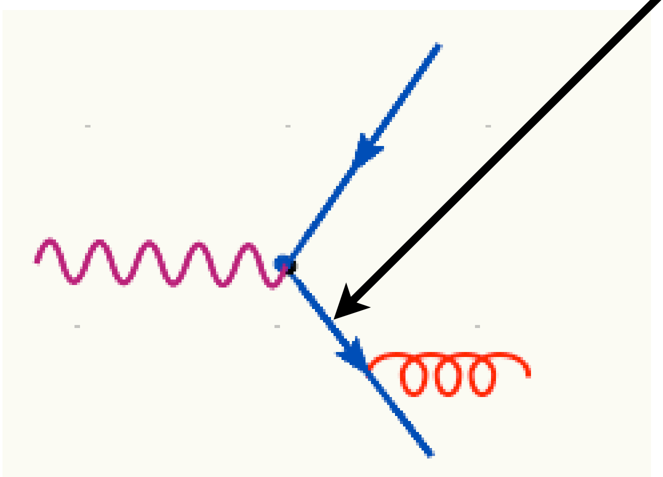
The Born cross section IS NOT the cross section for producing q qbar, since the coefficients of the perturbative expansion are infinite! But this is not a problem since we don't observe q qbar and nothing else. So there is no contradiction here.

On the other hand the cross section for producing hadrons is finite order by order and its lowest order approximation IS the Born.

A further insight can be gained by thinking of what happens in QED and what is different there. For instance soft and collinear divergence are also there. In QED one can prove that cross section for producing "only two muons" is zero...

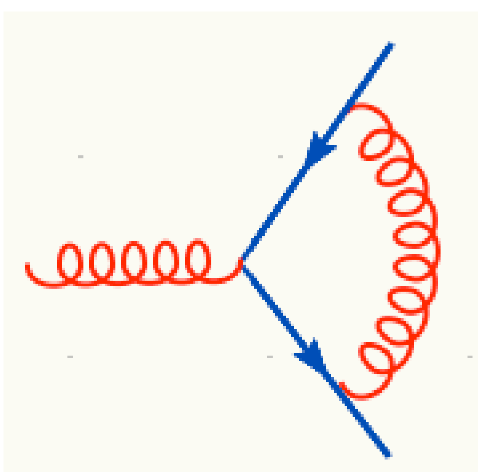
Infrared divergences

$$A_{soft} = -g_s t^a \left(\frac{p \cdot \epsilon}{p \cdot k} - \frac{\bar{p} \cdot \epsilon}{\bar{p} \cdot k} \right) A_{Born}$$



Even in high-energy, short-distance regime, long-distance aspects of QCD cannot be ignored.

This is because there are configurations in phase space for gluons and quarks, i.e. when gluons are soft and/or when are pairs of partons are collinear.

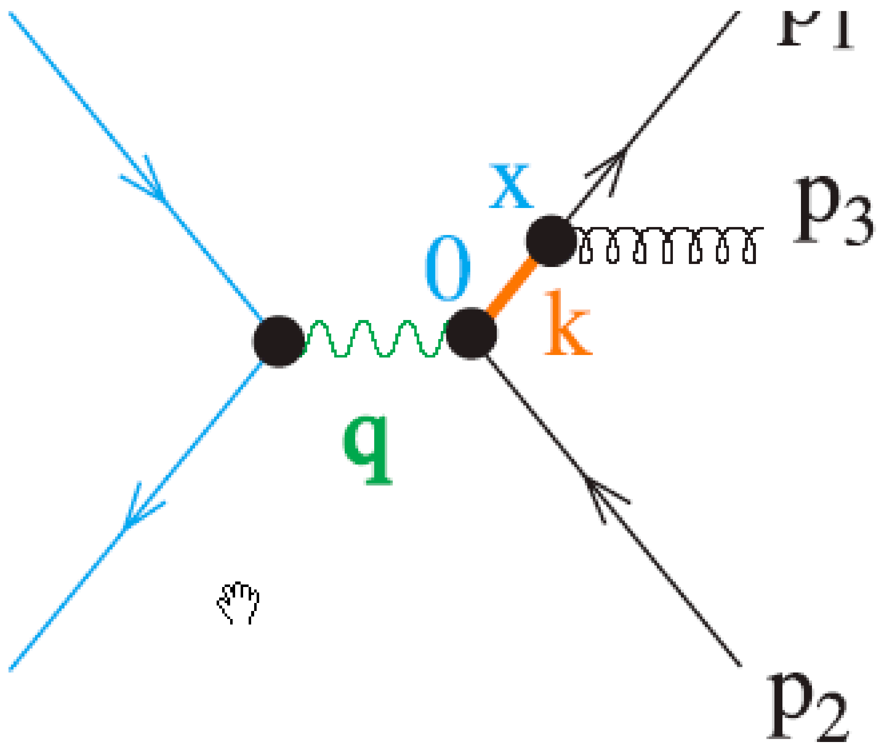


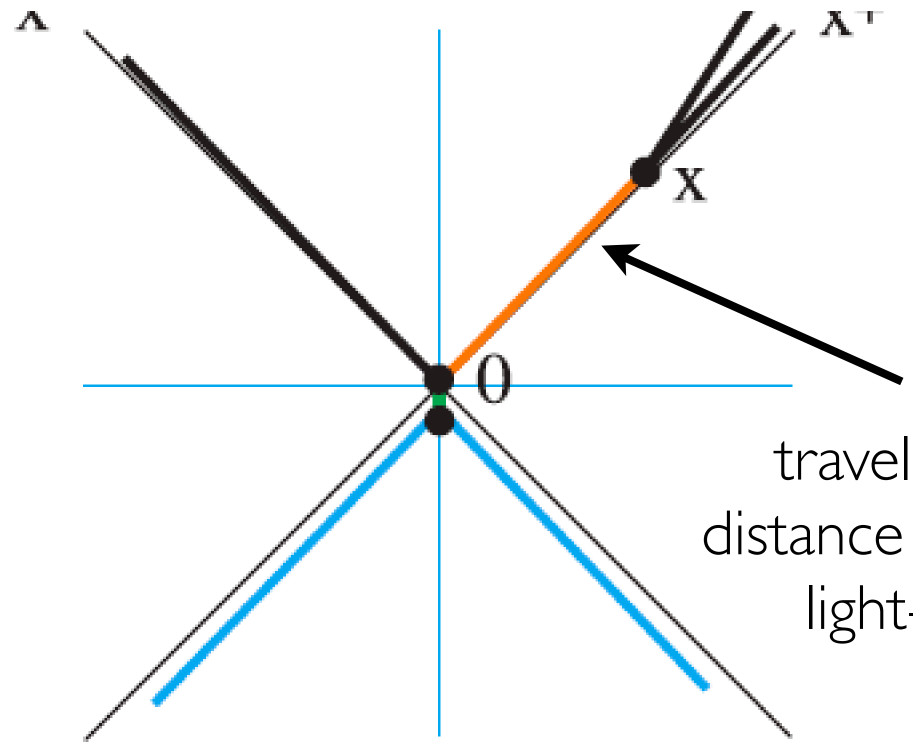
$$\Rightarrow \int \frac{d^d k}{(2\pi)^d} \frac{1}{k^2 (k+p)^2 (k-\bar{p})^2}$$

also for soft and collinear or collinear configurations associated to the virtual partons with the region of integration of the loop momenta.

Space-time picture of IR singularities

The singularities can be understood in terms of light-cone coordinates. Take $p^{\mu}=(p^0, p^1, p^2, p^3)$ and define $p^{\pm}=(p^0 \pm p^3)/\sqrt{2}$. Then choose the direction of the + axis as the one of the largest between + and - . A particle with small virtuality travels for a long time along the x^+ direction.





$$k^+ \simeq \sqrt{s}/2 \qquad \text{large}$$
$$k^- \simeq (k^T + 2k^+k^-)\sqrt{s}/2 \qquad \text{small}$$

$$x^+ \simeq 1/k^- \qquad \text{large}$$
$$x^- \simeq 1/k^+ \qquad \text{small}$$

Infrared divergences

Infrared divergences arise from interactions that happen a long time after the creation of the quark/antiquark pair.

When distances become comparable to the hadron size of ~1 Fermi, quasi-free partons of the perturbative calculation are confined/hadronized non-perturbatively.

We have seen that in total cross sections such divergences cancel. But what about for other quantities?

Obviously, the only possibility is to try to use the pQCD calculations for quantities that are not sensitive to the to the long-distance physics.

Can we formulate a criterium that is valid in general?

YES! It is called INFRARED SAFETY

Infrared-safe quantities

DEFINITION: quantities are that are insensitive to soft and collinear branching.

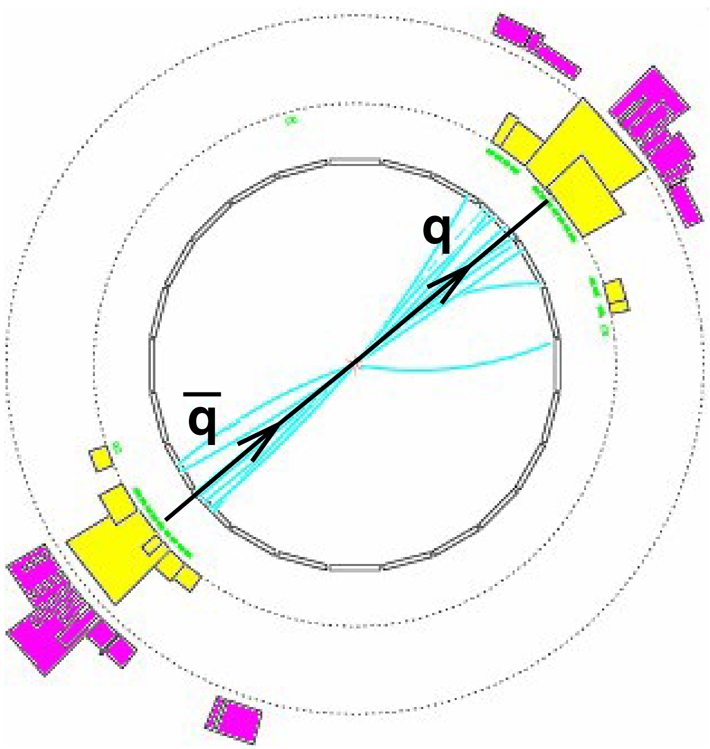
For these quantities, an extension of the general theorem (KLN) exists which proves that infrared divergences cancel between real and virtual or are simply removed by kinematic factors.

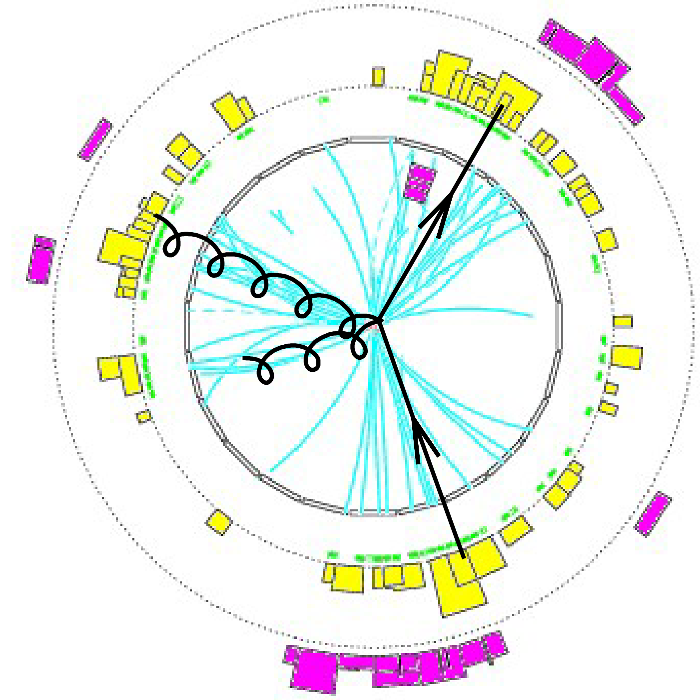
Such quantities are determined primarily by hard, short-distance physics. Long-distance effects give power corrections, suppressed by the inverse power of a large momentum scale (which must be present in the first place to justify the use of PT).

Examples:

1. Multiplicity of gluons is not IRC safe
2. Energy of hardest particle is not IRC safe
3. Energy flow into a cone is IRC safe

Event shape variables





pencil-like → spherical

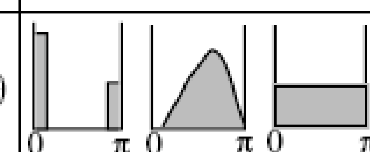
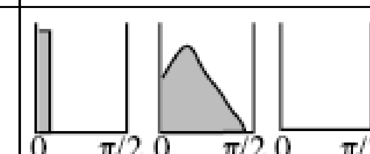
Event shape variables

The idea is to give more information than just total cross section by defining “shapes” of an hadronic event (pencil-like, planar, spherical, etc..)

In order to be comparable with theory it MUST be IR-safe, that means that the quantity should not change if one of the parton “branches” $p_k \to p_i + p_j$

Examples are: Thrust, Spherocity, C-parameters,...

Similar quantities exist for hadron collider too, but they much less used.

Name of Observable	Definition	Typical Value for:			QCD calculation
Thrust	$T = \max_{\vec{n}}\left(\frac{\sum_i \lvert\vec{p}_i\vec{n}\rvert}{\sum_i \lvert\vec{p}_i\rvert}\right)$	1	$\geq 2/3$	$\geq 1/2$	(resummed) $O(\alpha_s^2)$
Thrust major	Like T, however T_{maj} and $\vec{n}_{maj}$ in plane $\perp \vec{n}_T$	0	$\leq 1/3$	$\leq 1/\sqrt{2}$	$O(\alpha_s^2)$
Thrust minor	Like T, however T_{min} and $\vec{n}_{min}$ in direction $\perp$ to $\vec{n}_T$ and $\vec{n}_{maj}$	0	0	$\leq 1/2$	$O(\alpha_s^2)$
Oblateness	$O = T_{maj} - T_{min}$	0	$\leq 1/3$	0	$O(\alpha_s^2)$
Sphericity	$S = 1.5\,(Q_1 + Q_2)$; $Q_1 \leq \ldots \leq Q_3$ are Eigenvalues of $S^{\alpha\beta} = \frac{\sum_i p_i^\alpha p_i^\beta}{\sum_i p_i^2}$	0	$\leq 3/4$	≤ 1	none (not infrared safe)
Aplanarity	$A = 1.5\, Q_1$	0	0	$\leq 1/2$	none (not infrared safe)
Jet (Hemisphere) masses	$M_\pm^2 = (\sum_i E_i^2 - \sum_i \vec{p}_i^{\,2})_{i \in S_\pm}$ ($S_\pm$: Hemispheres $\perp$ to $\vec{n}_T$) $M_H^2 = \max(M_+^2, M_-^2)$ $M_D^2 = \lvert M_+^2 - M_-^2 \rvert$	0 0	$\leq 1/3$ $\leq 1/3$	$\leq 1/2$ 0	(resummed) $O(\alpha_s^2)$
Jet broadening	$B_\pm = \frac{\sum_{i\in S_\pm} \lvert\vec{p}_i \times \vec{n}_T\rvert}{2\sum_i \lvert\vec{p}_i\rvert}$; $B_T = B_+ + B_-$ $B_w = \max(B_+, B_-)$	0 0	$\leq 1/(2\sqrt{3})$ $\leq 1/(2\sqrt{3})$	$\leq 1/(2\sqrt{2})$ $\leq 1/(2\sqrt{3})$	(resummed) $O(\alpha_s^2)$
Energy-Energy Correlations	$EEC(\chi) = \sum_{events} \sum_{i,j} \frac{E_i E_j}{E_{vis}^2} \int_{\chi-\frac{\Delta\chi}{2}}^{\chi+\frac{\Delta\chi}{2}} \delta(\chi - \chi_{ij})$				(resummed) $O(\alpha_s^2)$
Asymmetry of EEC	$AEEC(\chi) = EEC(\pi - \chi) - EEC(\chi)$				$O(\alpha_s^2)$
Differential 2-jet rate	$D_2(y) = \frac{R_2(y - \Delta y) - R_2(y)}{\Delta y}$				(resummed) $O(\alpha_s^2)$

Is the thrust IR safe?

$$T = \max_{\vec{n}} \frac{\sum_i \vec{p}_i \cdot \vec{n}}{\sum_i \vec{p}_i}$$

$$|(1-\lambda)\vec{p}_k \cdot \vec{u}| + |\lambda \vec{p}_k \cdot \vec{u}| = |\vec{p}_k \cdot \vec{u}|$$

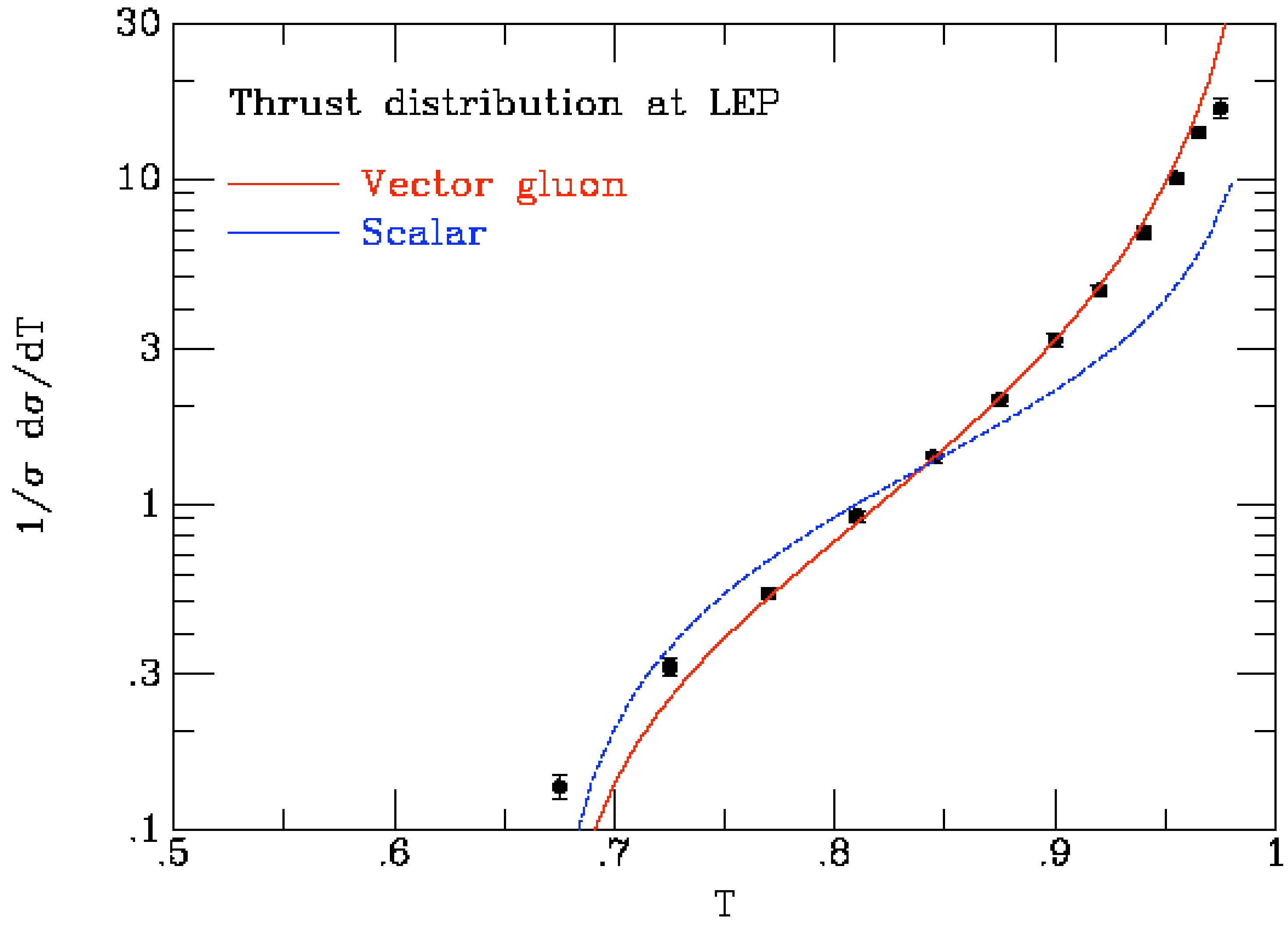
and

$$|(1-\lambda)\vec{p}_k| + |\lambda \vec{p}_k| = |\vec{p}_k|$$

Calculation of event shape variables: Thrust

The values of the different event-shape variables for different topologies are

$$\frac{1}{\sigma}\frac{d\sigma}{dT} = C_F \frac{\alpha_S}{2\pi}\left[\frac{2(3T^2-3T+2)}{T(1-T)}\log\left(\frac{2T-1}{1-T}\right) - \frac{3(3T-2)(2-T)}{1-T}\right].$$



$O(\alpha_S^2)$ corrections (NLO) are also known. Comparison with data provide test of QCD matrix elements, through shape distribution and measurement of α_S from overall rate. Care must be taken around T=1 where

(a) hadronization effects become large and

(b) large higher order terms of the form $\alpha_S^N [\log^{2N-1}(1-T)]/(1-T)$ need to be resummed.

At lower T multi-jet matrix element become important.

QCD in the final state

1. Infrared safety
2. Towards realistic final states
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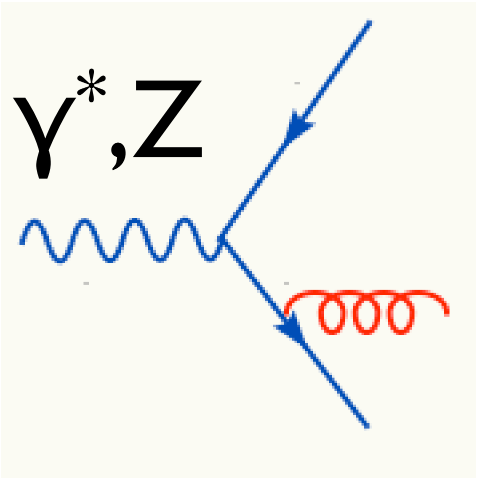
Towards a realistic predictions



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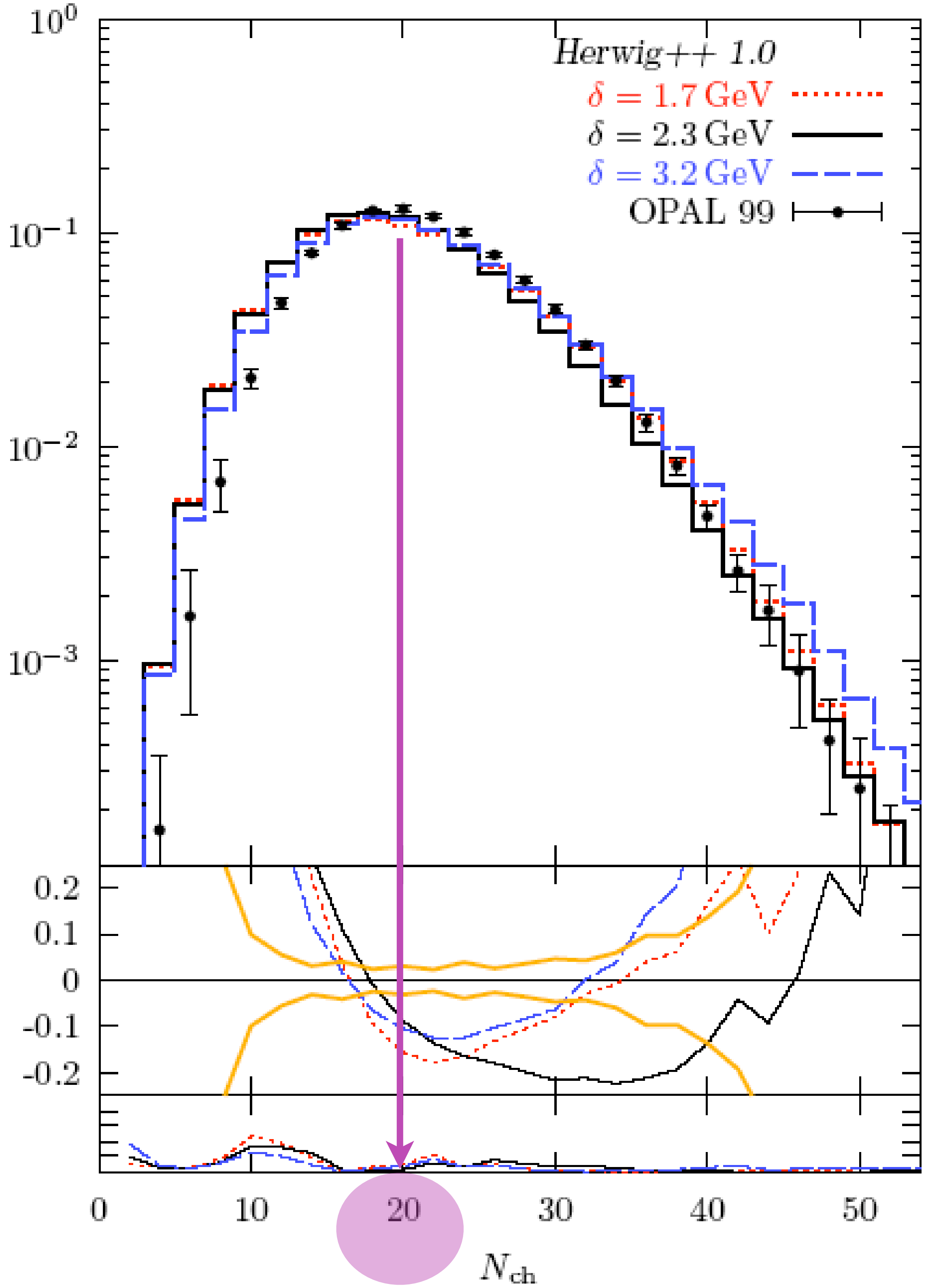


More exclusive quantities (AKA, the power of exponentiation)

Assuming "abelian" gluons one finds that something magic happens at higher orders:

$$\sigma_{2j} = \sigma^{\rm Born}\left[1 - \frac{\alpha_S C_F}{\pi}\log^2 y + \frac{1}{2!}\left(\frac{\alpha_S C_F}{\pi}\log^2 y\right)^2 + \ldots\right] = \sigma^{\rm Born} e^{-\frac{\alpha_S C_F}{\pi}\log^2 y}$$

$$\sigma_{3j} = \sigma^{\rm Born}\frac{\alpha_S C_F}{\pi}\log^2 y\, e^{-\frac{\alpha_S C_F}{\pi}\log^2 y}$$

$$\vdots$$

$$\sigma_{nj} = \sigma^{\rm Born}\frac{1}{n!}\left(\frac{\alpha_S C_F}{\pi}\log^2 y\right)^n e^{-\frac{\alpha_S C_F}{\pi}\log^2 y}$$

The number of jets is distributed as a Poisson with average (and the full QCD result):

$$< n_j >= 2 + \frac{\alpha_S C_F}{\pi}\log^2 y \qquad < n_j >_{\rm QCD} = \frac{C_F}{C_A}\exp\sqrt{\frac{\alpha_S C_A}{2\pi}\log^2\frac{1}{y}}$$

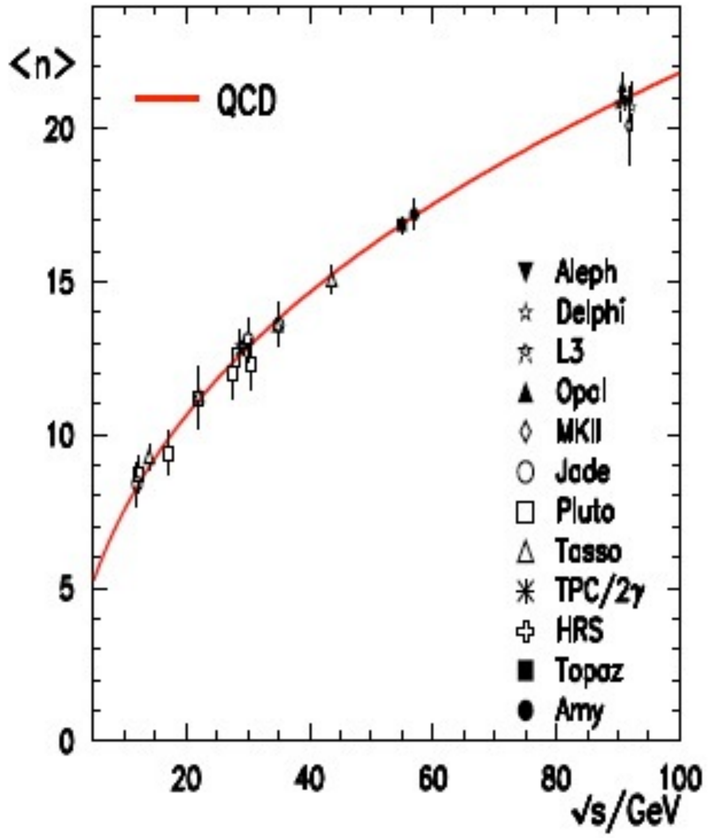
More exclusive quantities (AKA, the power of exponentiation)

Identifying one particle with one jet at resolution scale of Λs one obtains an estimate for the average number of particles in an event (multiplicity):

$$< n_p >= \frac{\alpha_S C_F}{\pi} \log^2 \frac{s}{\Lambda_s^2} = \frac{C_F}{\pi b_0} \log \frac{s}{\Lambda_s^2}$$

$$< n_p >_{\rm QCD} = \exp \sqrt{\frac{2 C_A}{\pi b_0} \log \frac{s}{\Lambda_s}}$$

ie. the multiplicity grows with the log of the com energy.



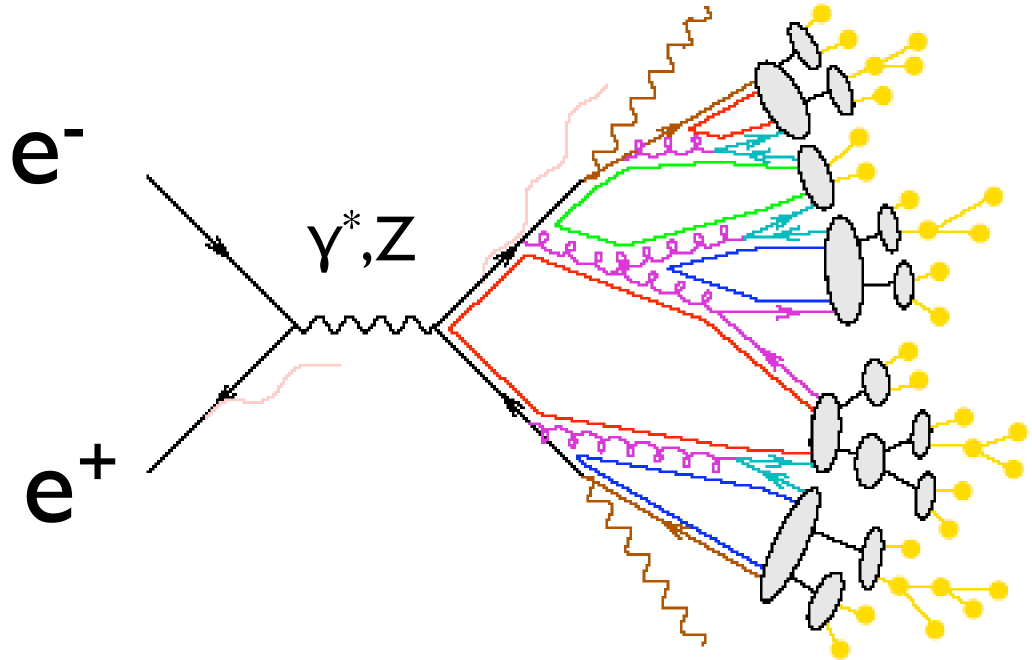
Finally the jet mass can also be easily estimated by integrating the cross sections over two emispheres identified by the thrust axis:

$$< m_j^2 >= \frac{1}{2\sigma_{\rm Born}} \left[\int_{(I)} (q+k)^2 d\sigma_g + \int_{(II)} (q+k)^2 d\sigma_g \right] = \frac{\alpha_S C_F}{\pi} s$$

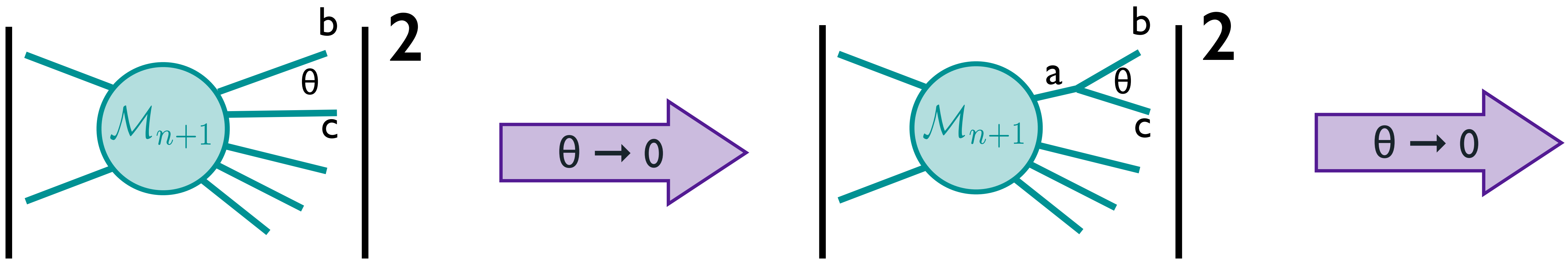
This result gives the correct scaling of the jet mass, mj ~√αs Ej , which is also valid at hadron colliders (replacing E with pt)!

Parton showers

- We need to be able to describe an arbitrarily number of parton branchings, i.e. we need to 'dress' partons with radiation
- This effect should be unitary: the inclusive cross section shouldn't change when extra radiation is added
- And finally we want to turn partons into hadrons (hadronization)....

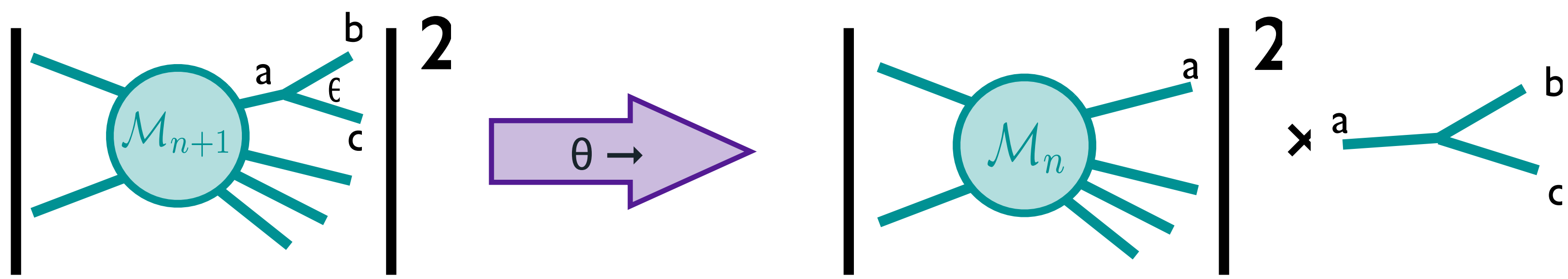


Collinear factorization



- Consider a process for which two particles are separated by a small angle θ.
- In the limit of θ → 0 the contribution is coming from a single parent particle going on shell: therefore its branching is related to time scales which are very long with respect to the hard subprocess.
- The inclusion of such a branching cannot change the picture set up by the hard process: the whole emission process must be writable in this limit as the simpler one times a branching probability.
- The first task of Monte Carlo physics is to make this statement quantitative.

Collinear factorization



- The process factorizes in the collinear limit. This procedure it universal!

$$|\mathcal{M}_{n+1}|^2 d\Phi_{n+1} \;\simeq\; |\mathcal{M}_n|^2 d\Phi_n \frac{dt}{t} dz \frac{d\phi}{2\pi} \frac{\alpha_S}{2\pi} P_{a\to bc}(z)$$

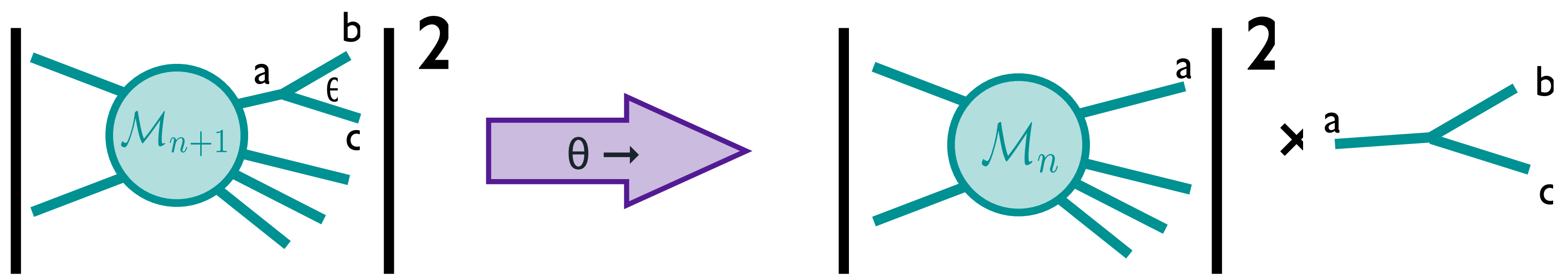
- Notice that what has been roughly called ‘branching probability’ is actually a singular factor, so one will need to make sense precisely of this definition.

- At the leading contribution to the (n+1)-body cross section the Altarelli-Parisi splitting kernels are defined as:

$$P_{g\to qq}(z) = T_R\left[z^2 + (1-z)^2\right], \qquad P_{g\to gg}(z) = C_A\left[z(1-z) + \frac{z}{1-z} + \frac{1-z}{z}\right],$$

$$P_{q\to qg}(z) = C_F\left[\frac{1+z^2}{1-z}\right], \qquad P_{q\to gq}(z) = C_F\left[\frac{1+(1-z)^2}{z}\right].$$

Collinear factorization



- The process factorizes in the collinear limit. This procedure it universal!

$$|\mathcal{M}_{n+1}|^2 d\Phi_{n+1} \;\simeq\; |\mathcal{M}_n|^2 d\Phi_n \frac{dt}{t} dz \frac{d\phi}{2\pi} \frac{\alpha_S}{2\pi} P_{a\to bc}(z)$$

- t can be called the ‘evolution variable’ (will become clearer later): it can be the virtuality m^2 of particle a or its p_T^2 or $E^2\theta^2$ …

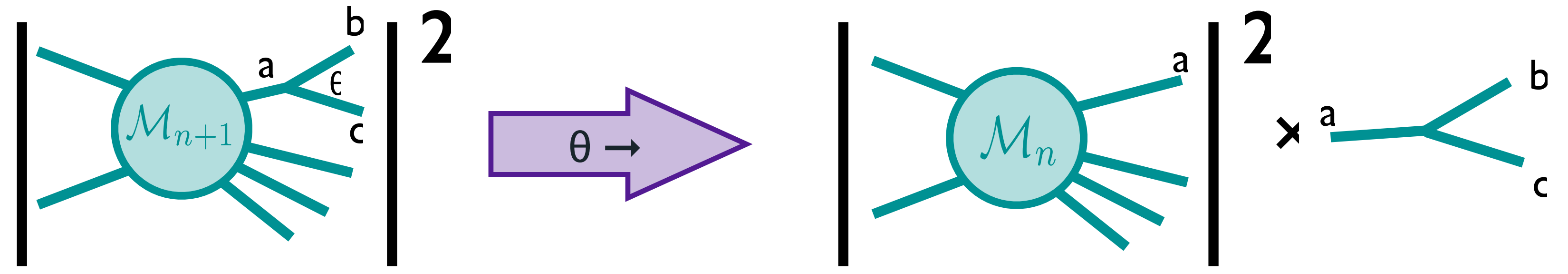
  - It represents the hardness of the branching and tends to 0 in the collinear limit.

- Indeed in the collinear limit one has:
  so that the factorization takes place
  for all these definitions:

$$m^2 \simeq z(1-z)\theta^2 E_a^2$$
$$p_T^2 \simeq z m^2$$

$$d\theta^2/\theta^2 \;=\; dm^2/m^2 \;=\; dp_T^2/p_T^2$$

Collinear factorization

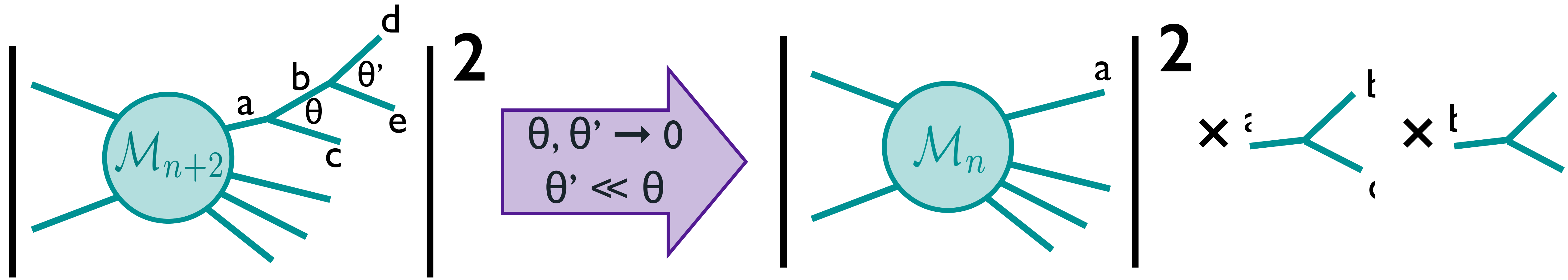


- The process factorizes in the collinear limit. This procedure it universal!

$$|\mathcal{M}_{n+1}|^2 d\Phi_{n+1} \simeq |\mathcal{M}_n|^2 d\Phi_n \frac{dt}{t} dz \frac{d\phi}{2\pi} \frac{\alpha_S}{2\pi} P_{a\to bc}(z)$$

- z is the "energy variable": it is defined to be the energy fraction taken by parton b from parton a. It represents the energy sharing between b and c and tends to 1 in the soft limit (parton c going soft)
- Φ is the azimuthal angle. It can be chosen to be the angle between the polarization of a and the plane of the branching.

Multiple emission

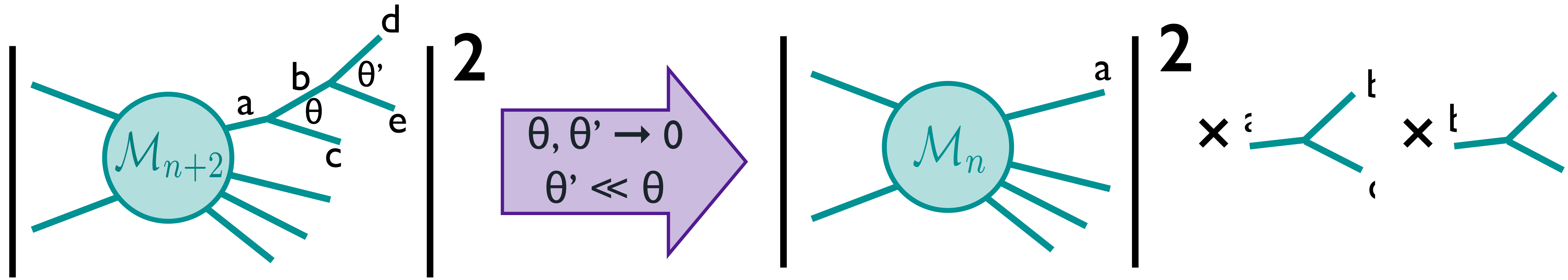


- Now consider $\mathbf{M}_{n+1}$ as the new core process and use the recipe we used for the first emission in order to get the dominant contribution to the (n+2)-body cross section: add a new branching at angle much smaller than the previous one:

$$|\mathcal{M}_{n+2}|^2 d\Phi_{n+2} \simeq |\mathcal{M}_n|^2 d\Phi_n \frac{dt}{t} dz \frac{d\phi}{2\pi} \frac{\alpha_S}{2\pi} P_{a\to bc}(z) \times \frac{dt'}{t'} dz' \frac{d\phi'}{2\pi} \frac{\alpha_S}{2\pi} P_{b\to de}(z')$$

- This can be done for an arbitrary number of emissions. The recipe to get the leading collinear singularity is thus cast in the form of an iterative sequence of emissions whose probability does not depend on the past history of the system: a 'Markov chain'. No interference!!!

Multiple emission



- The dominant contribution comes from the region where the subsequently emitted partons satisfy the strong ordering requirement: θ » θ' » θ''...
  For the rate for multiple emission we get

$$\sigma_{n+k} \propto \alpha_S^k \int_{Q_0^2}^{Q^2} \frac{dt}{t} \int_{Q_0^2}^{t} \frac{dt'}{t'} \ldots \int_{Q_0^2}^{t^{(k-2)}} \frac{dt^{(k-1)}}{t^{(k-1)}} \propto \sigma_n \left(\frac{\alpha_S}{2\pi}\right)^k \log^k(Q^2/Q_0^2)$$

  where Q is a typical hard scale and Q_0 is a small infrared cutoff that separates perturbative from non perturbative regimes.

- Each power of α_s comes with a logarithm. The logarithm can be easily large, and therefore it can lead to a breakdown of perturbation theory.

Absence of interference

- The collinear factorization picture gives a branching sequence for a given leg starting from the hard subprocess all the way down to the non-perturbative region.
- Suppose you want to describe two such histories from two different legs: these two legs are treated in a completely uncorrelated way. And even within the same history, subsequent emissions are uncorrelated.
- The collinear picture completely misses the possible interference effects between the various legs. The extreme simplicity comes at the price of quantum inaccuracy.
- Nevertheless, the collinear picture captures the leading contributions: it gives an excellent description of an arbitrary number of (collinear) emissions:
    - it is a “resummed computation”
    - it bridges the gap between fixed-order perturbation theory and the non-perturbative hadronization.

Sudakov form factor

The differential probability for the branching **a ⟶ bc** between scales **t** and **t+dt** knowing that no emission occurred before:

$$dp(t) = \sum_{bc} \frac{dt}{t} \int dz \frac{d\phi}{2\pi} \frac{\alpha_S}{2\pi} P_{a\to bc}(z)$$

The probability that a parton does NOT split between the scales **t** and **t+dt** is given by **1-dp(t)**. Probability that particle **a** does not emit between scales **Q²** and **t**

$$\Delta(Q^2,t) = \prod_k \left[1 - \sum_{bc} \frac{dt_k}{t_k} \int dz \frac{d\phi}{2\pi} \frac{\alpha_S}{2\pi} P_{a\to bc}(z)\right] =$$

$$\exp\left[-\sum_{bc} \int_t^{Q^2} \frac{dt'}{t'} dz \frac{d\phi}{2\pi} \frac{\alpha_S}{2\pi} P_{a\to bc}(z)\right] = \exp\left[-\int_t^{Q^2} dp(t')\right]$$

$\Delta(Q^2,t)$ is the Sudakov form factor

Parton shower algorithm

- The Sudakov form factor is the heart of the parton shower. It gives the probability that a parton does not branch between two scales
- Using this no-emission probability the branching tree of a parton is generated.
- Define dP_k as the probability for k ordered splittings from leg a at given scales

$$
\begin{aligned}
dP_1(t_1) &= \Delta(Q^2, t_1)\; dp(t_1)\Delta(t_1, Q_0^2), \\
dP_2(t_1, t_2) &= \Delta(Q^2, t_1)\; dp(t_1)\; \Delta(t_1, t_2)\; dp(t_2)\; \Delta(t_2, Q_0^2)\Theta(t_1 - t_2), \\
\ldots &= \ldots \\
dP_k(t_1, \ldots, t_k) &= \Delta(Q^2, Q_0^2) \prod_{l=1}^{k} dp(t_l)\Theta(t_{l-1} - t_l)
\end{aligned}
$$

- Q_0^2 is the hadronization scale (~1 GeV). Below this scale we do not trust the perturbative description for parton splitting anymore.
- This is what is implemented in a parton shower, taking the scales for the splitting t_i randomly (but weighted according to the no-emission probability).

Unitarity

$$dP_k(t_1, ..., t_k) \quad = \quad \Delta(Q^2, Q_0^2) \prod_{l=1}^{k} dp(t_l) \Theta(t_{l-1} - t_l)$$

- The parton shower has to be unitary (the sum over all branching trees should be 1). We can explicitly show this by integrating the probability for k splittings:

$$P_k \equiv \int dP_k(t_1, ..., t_k) = \Delta(Q^2, Q_0^2) \frac{1}{k!} \left[\int_{Q_0^2}^{Q^2} dp(t) \right]^k, \quad \forall k = 0, 1, ...$$

- Summing over all number of emissions

$$\sum_{k=0}^{\infty} P_k = \Delta(Q^2, Q_0^2) \sum_{k=0}^{\infty} \frac{1}{k!} \left[\int_{Q_0^2}^{Q^2} dp(t) \right]^k = \Delta(Q^2, Q_0^2) \exp \left[\int_{Q_0^2}^{Q^2} dp(t) \right] = 1$$

- Hence, the total probability is conserved

Cancellation of singularities

- We have shown that the showers is unitary. However, how are the IR divergences cancelled explicitly? Let's show this for the first emission: Consider the contributions from (exactly) 0 and 1 emissions from leg a:

$$\frac{d\sigma}{\sigma_n} = \Delta(Q^2, Q_0^2) + \Delta(Q^2, Q_0^2) \sum_{bc} dz \frac{dt}{t} \frac{d\phi}{2\pi} \frac{\alpha_S}{2\pi} P_{a\to bc}(z)$$

- Expanding to first order in α_s gives

$$\frac{d\sigma}{\sigma_n} \simeq 1 - \sum_{bc} \int_{Q_0^2}^{Q^2} \frac{dt'}{t'} dz \frac{d\phi}{2\pi} \frac{\alpha_S}{2\pi} P_{a\to bc}(z) + \sum_{bc} dz \frac{dt}{t} \frac{d\phi}{2\pi} \frac{\alpha_S}{2\pi} P_{a\to bc}(z)$$

- Same structure of the two latter terms, with opposite signs: cancellation of divergences between the approximate virtual and approximate real emission cross sections.

- The probabilistic interpretation of the shower ensures that infrared divergences will cancel for each emission.

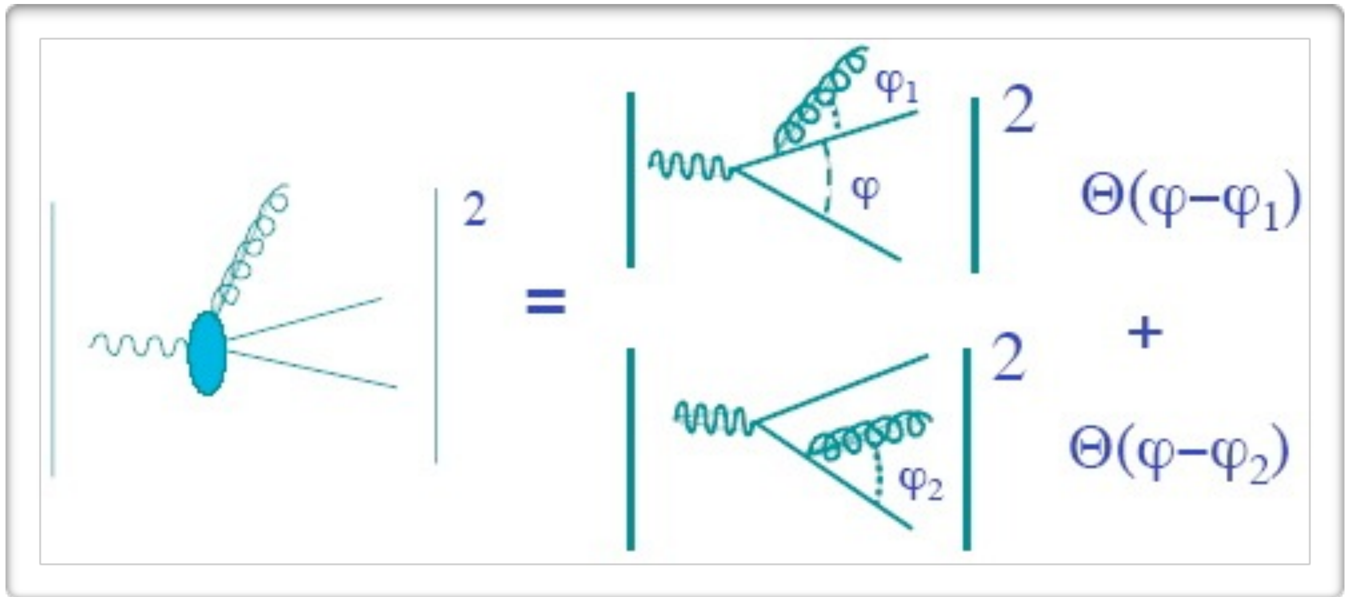
Choice of evolution parameter

$$\Delta(Q^2,t) = \exp\left[-\sum_{bc}\int_t^{Q^2}\frac{dt'}{t'}dz\frac{d\phi}{2\pi}\frac{\alpha_S}{2\pi}P_{a\to bc}(z)\right]$$

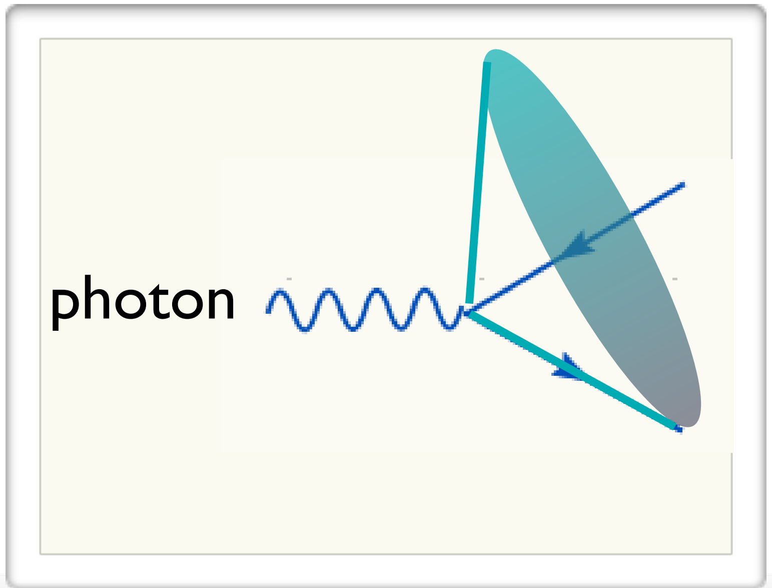
- There is a lot of freedom in the choice of evolution parameter t. It can be the virtuality m^2 of particle a or its p_T^2 or $E^2\theta^2$ ... For the collinear limit they are all equivalent
- However, in the soft limit ($z \longrightarrow 1$) they behave differently
- Can we chose it such that we get the correct soft limit?

YES! It should be (proportional to) the angle θ

Angular ordering



Radiation inside cones around the orginal partons is allowed (and described by the eikonal approximation), outside the cones it is zero (after averaging over the azimuthal angle)



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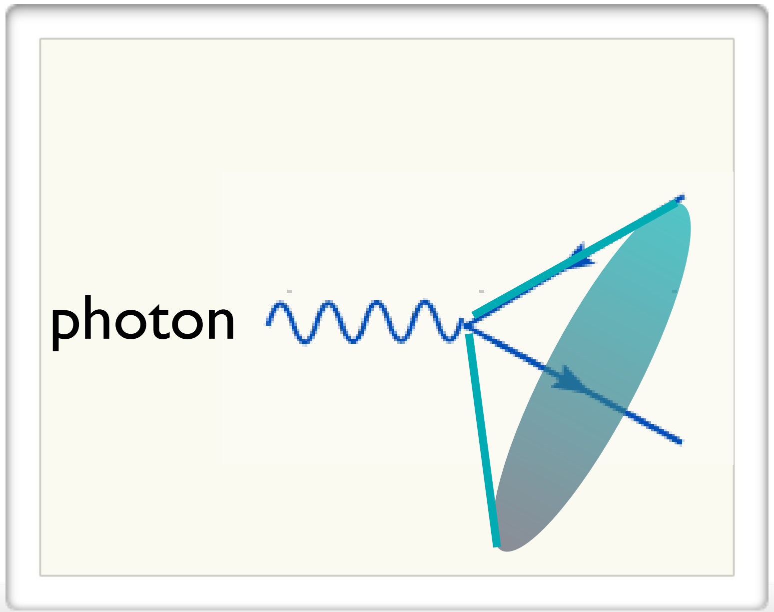
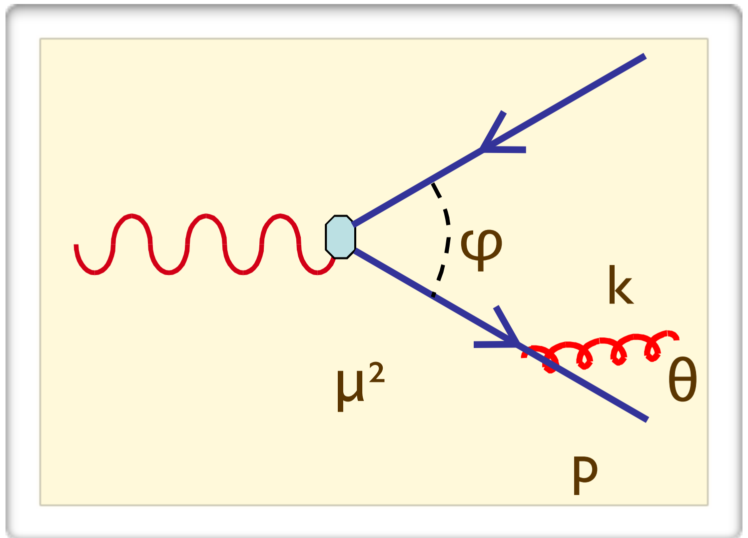


Intuitive explanation



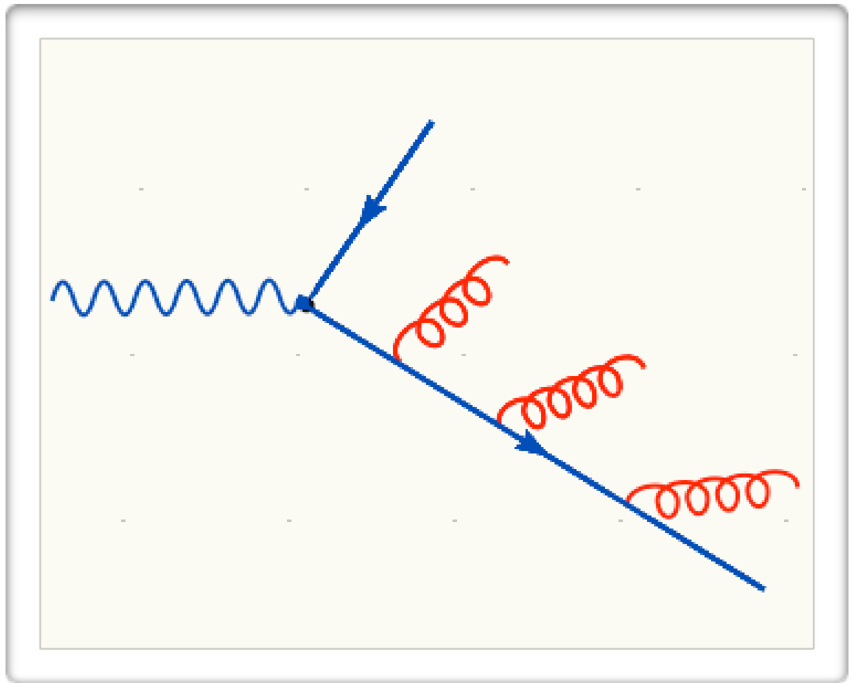
- Lifetime of the virtual intermediate state: $\tau < \gamma/\mu = E/\mu^2 = 1/(k_0\theta^2) = 1/(k_\perp\theta)$
- Distance between q and qbar after τ: $d = \varphi\tau = (\varphi/\theta)\ 1/k_\perp$

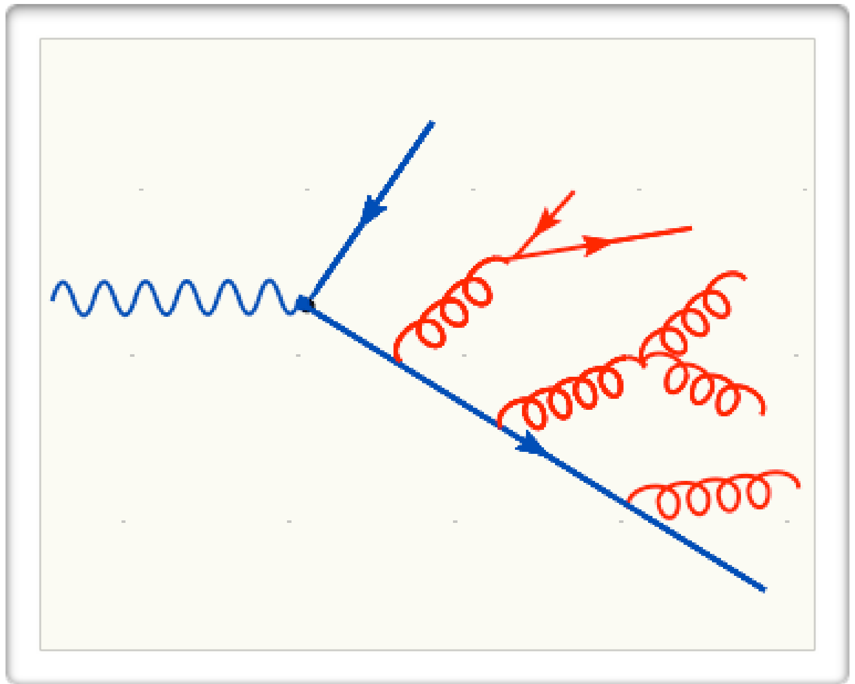
$\mu^2 = (p+k)^2 = 2E\, k_0\, (1\text{-}\cos\theta)$
$\sim E\, k_0\, \theta^2 \sim E\, k_\perp\, \theta$

If the transverse wavelength of the emitted gluon is longer than the separation between q and qbar, the gluon emission is suppressed, because the q qbar system will appear as colour neutral (i.e. dipole-like emission, suppressed)

Therefore $d>1/k_\perp$, which implies $\theta < \varphi$.

Angular ordering

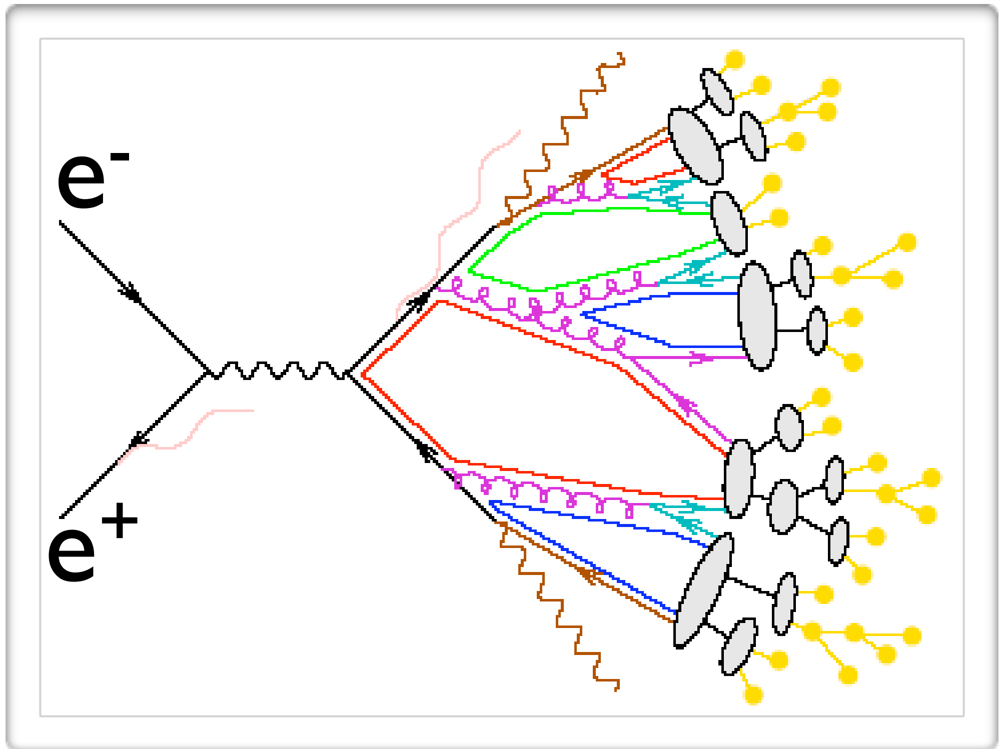


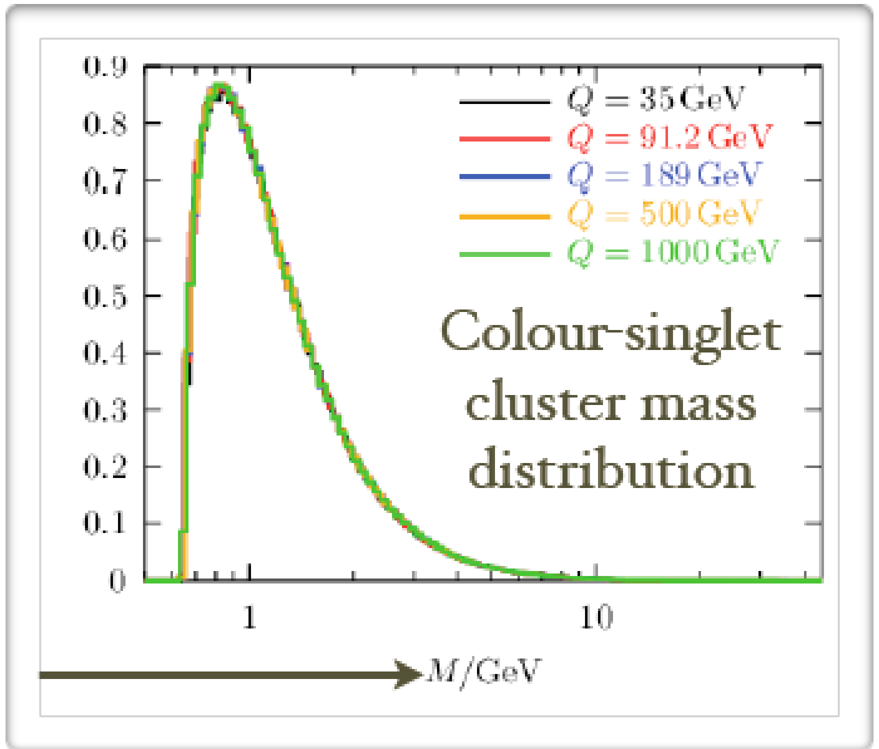


- The construction can be iterated to the next emission, with the result that the emission angles keep getting smaller and smaller.
- One can generalize it to a generic parton of color charge Q_k splitting into two partons i and j, $Q_k=Q_i+Q_j$. The result is that inside the cones i and j emit as independent charges, and outside their angular-ordered cones the emission is coherent and can be treated as if it was directly from color charge Q_k.
- KEY POINT FOR THE MC!
- Angular ordering is automatically satisfied in θ ordered showers! (and easy to account for in p_T ordered showers).

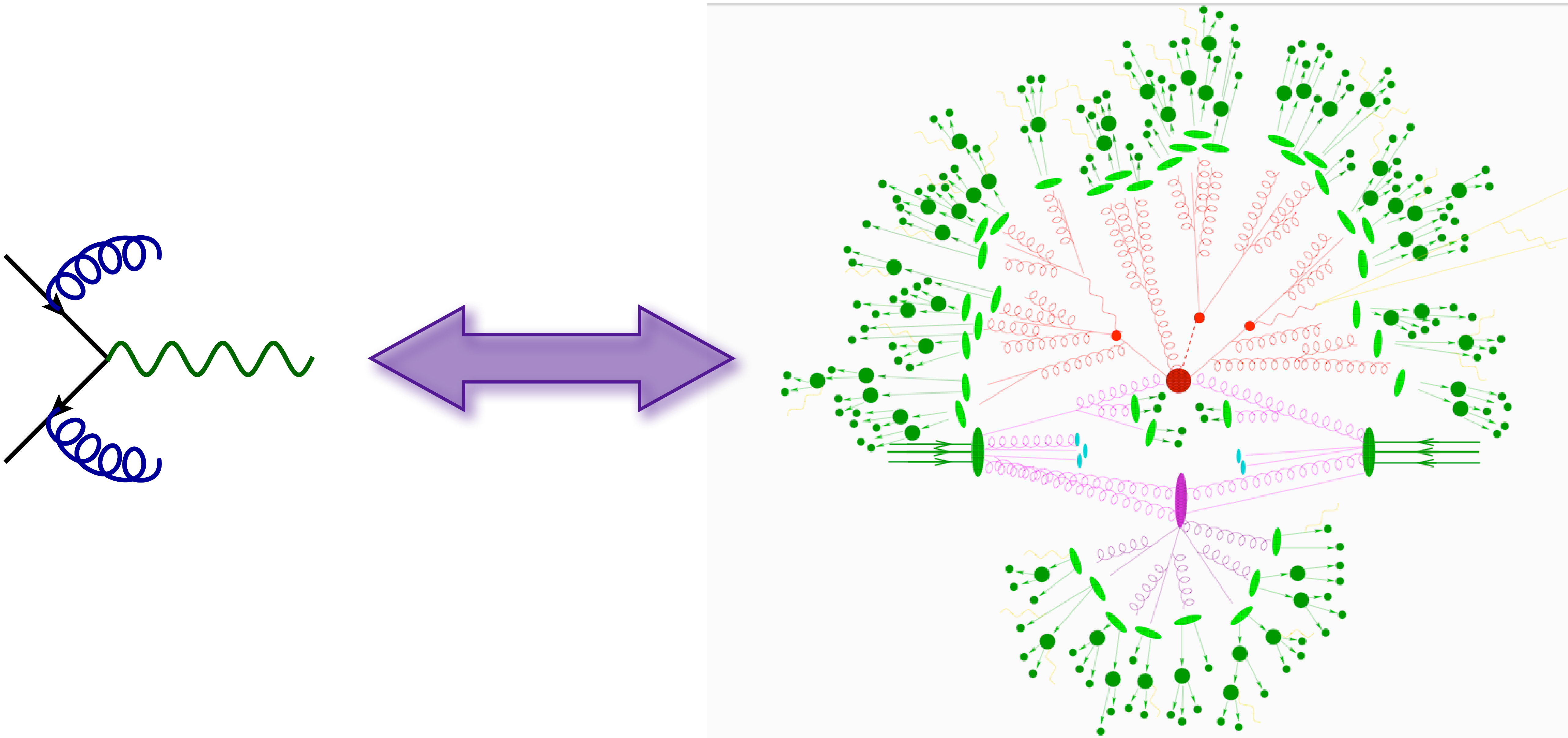
Cluster model

The structure of the perturbative evolution including angular ordering, leads naturally to the clustering in phase-space of color-singlet parton pairs (preconfinement). Long-range correlations are strongly suppressed. Hadronization will only act locally, on low-mass color singlet clusters.





Parton Shower MC



A parton shower program associates one of the possible histories (and pre-histories in case of pp collisions) of an hard event in an explicit and fully detailed way, such that the sum of the probabilities of all possible histories is unity.

QCD in the final state

1. Infrared safety
2. Towards realistic final states
3. Jets

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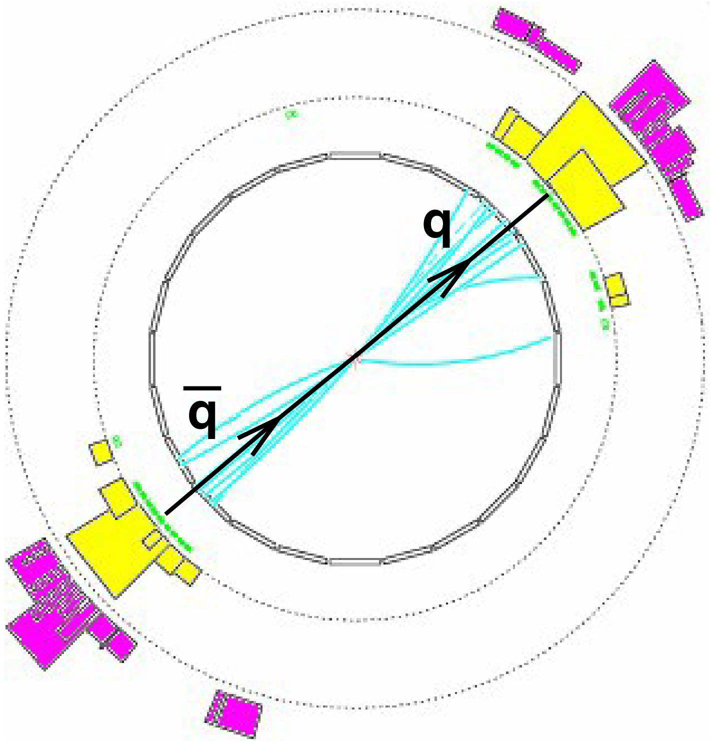
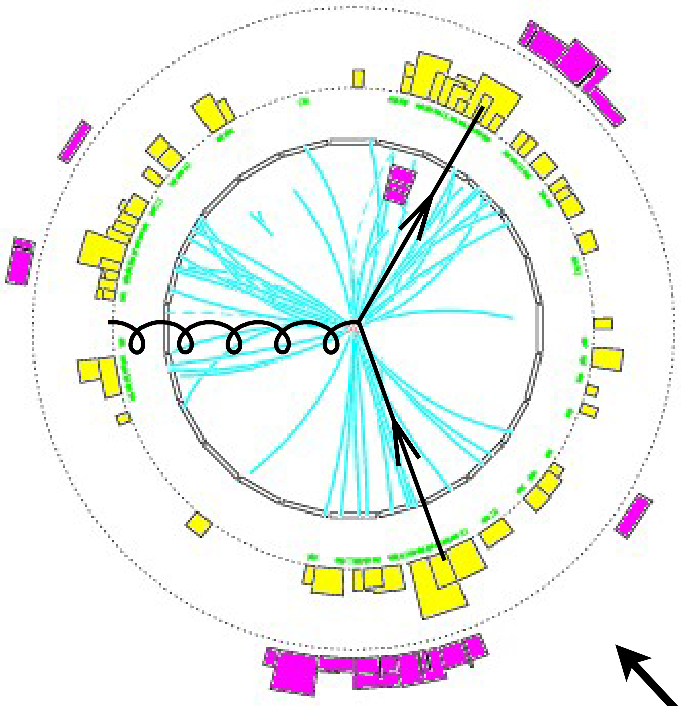
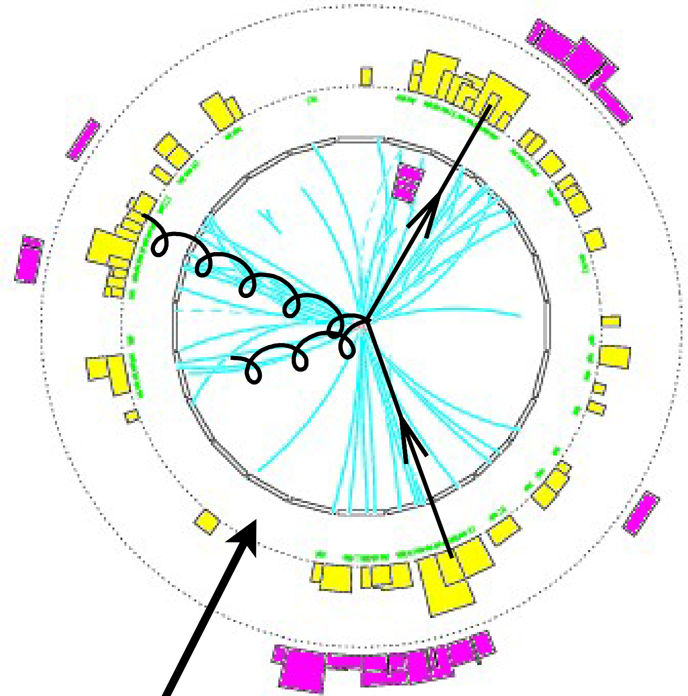
Jets







2-jets

3-jets

same event!!

4-jets

Jets are in the eye of the beholder!

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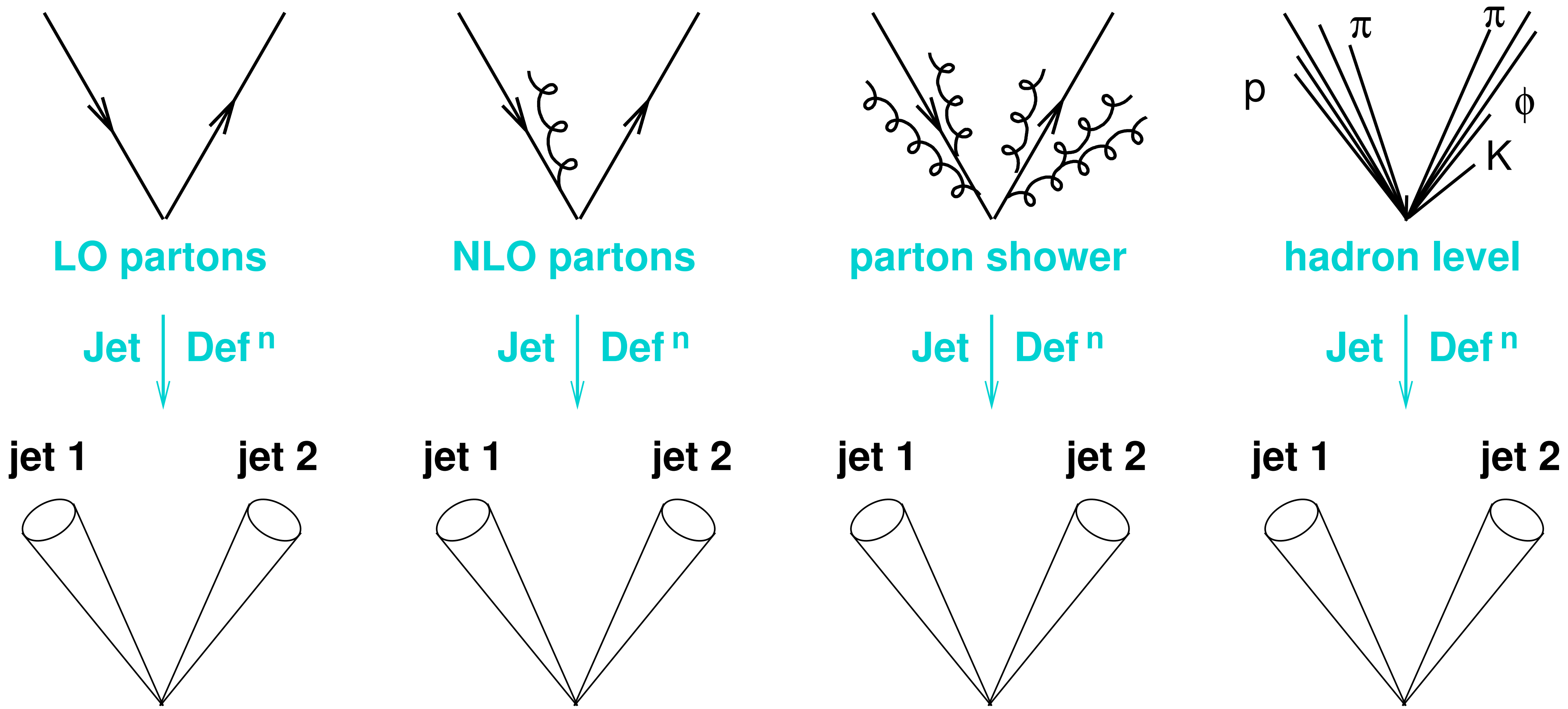
Jet algorithms

A jet definition is a fully specified set of rules for projecting information from hundreds of hadrons, onto a handful of parton-like objects.



In the projection a lot of information is lost.

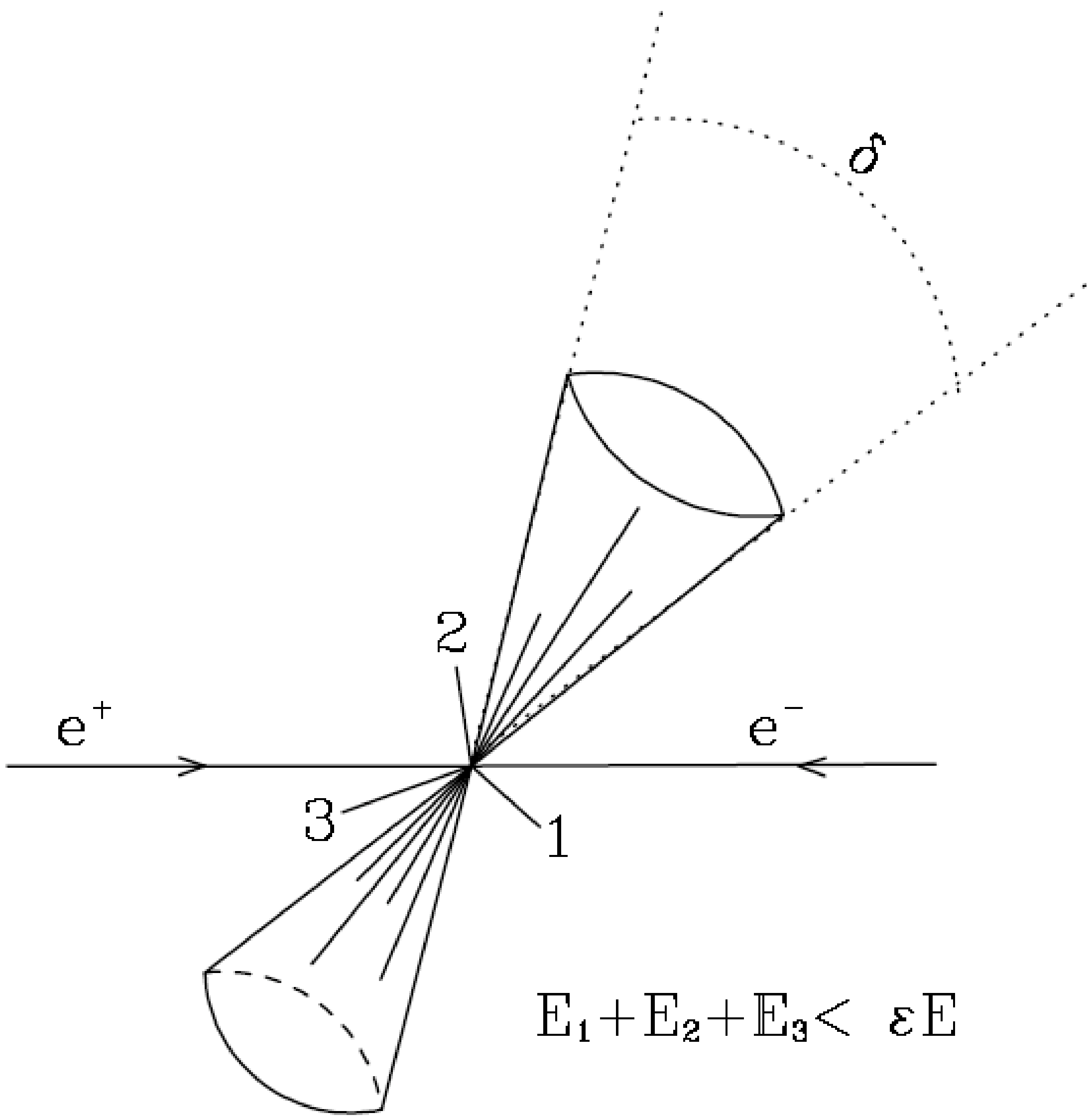
Projection to jets must be resilient to QCD effects

Jet algorithms

- The precise definition of a procedure how to cut be three-jet (and multi-jet) events is called "jet algorithm".

- Which jet algorithm to use for a given measurement/ experiment needs to be found out. Different algorithms have very different behaviors both experimentally and theoretically. Of course, it is important that a complete information is given on the jet algorithm when experimental data are to be compared with theory predictions!

- Weinberg-Sterman jets (intuitive definition): "An event is identified as a 2-jets if one can find 2 cones with opening angle δ that contain all but a small fraction εE of the total energy E".



Jets (top-down) at e-e+

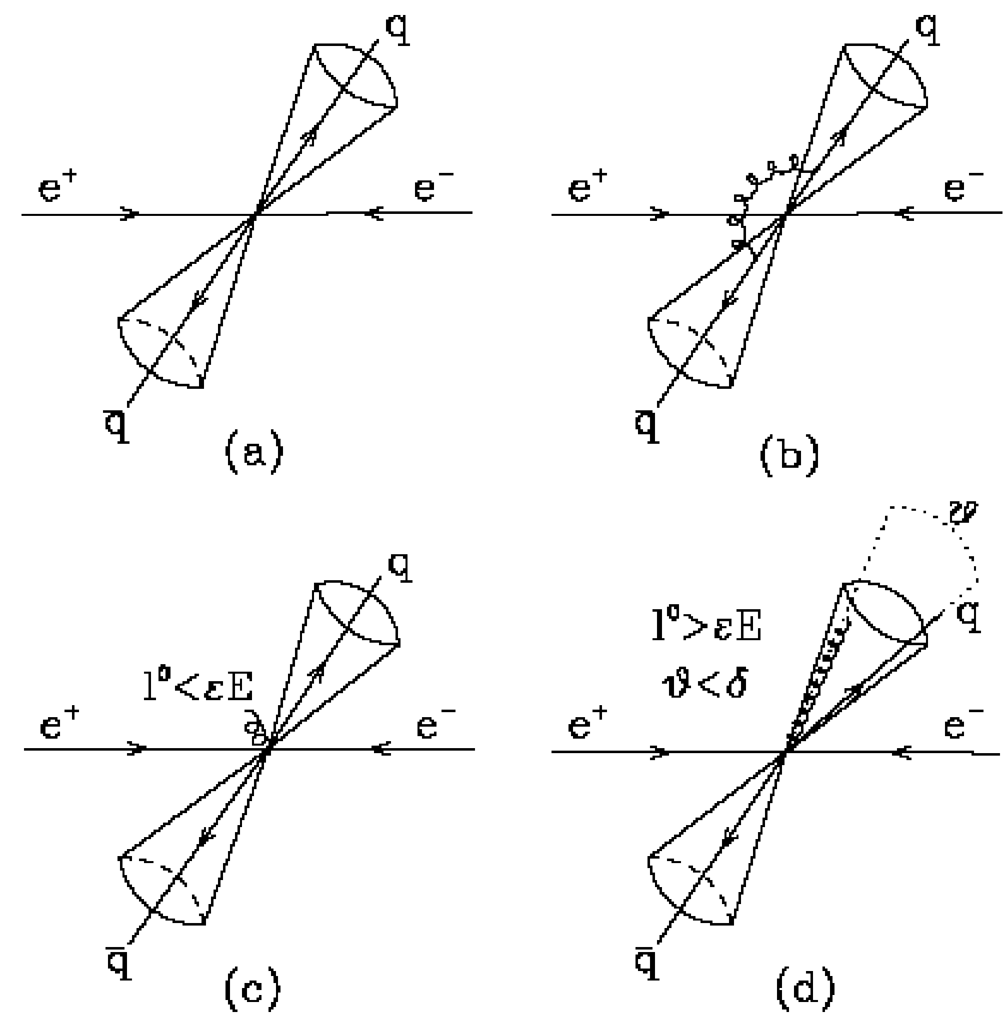
Let's see when the various contributions add up to the Sterman-Weinberg 2-jet cross section:

✷ The Born cross section contributes to the 2-jet cross section, INDEPENDENTLY of ε and δ.

✷The SAME as above for the virtual corrections.

✷The real corrections when $k^0<\varepsilon E$ (soft).

✷The real corrections when $k^0>\varepsilon E$ AND $\theta<\delta$ (collinear).



$$\text{Born} + \text{Virtual} + \text{Real (a)} + \text{Real (b)} = \sigma^{\text{Born}} - \sigma^{\text{Born}} \frac{4\alpha_S C_F}{2\pi} \int_{\epsilon E}^{E} \frac{dk^0}{k^0} \int_{\delta}^{\pi-\delta} \frac{d\cos\theta}{1-\cos^2\theta}$$

$$= \sigma^{\text{Born}} \left(1 - \frac{4\alpha_S C_F}{2\pi} \log\epsilon \log\delta \right)$$

As long as δ and ε are not too small, we find that the fraction of 2-jet cross section is almost 1! At high energy most of the events are two-jet events. As the energy increases the jets become thinner.

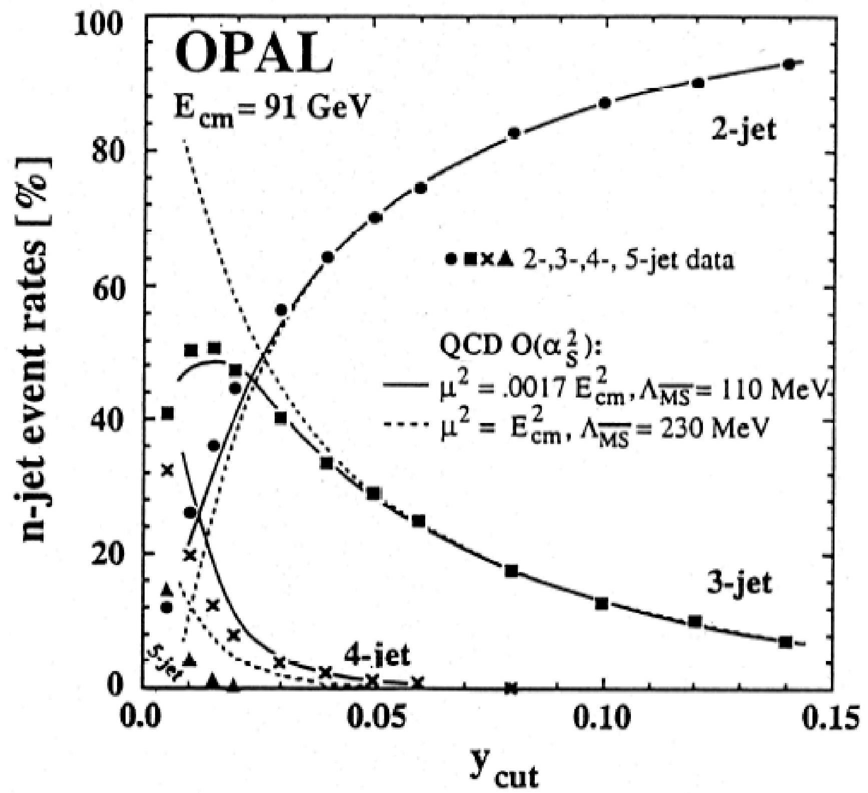
A very simple jet iterative algorithm (bottom-up)

1. Consider $e^+e^- \rightarrow$N partons
2. Consider all pairs i and j and calculate
   IF
   $\min\,(p_i + p_j)^2 < y_{cut}\, s$
   THEN
   replace the two partons i,j by $p_{ij} = p_i + p_j$ and decrease N → N-1
3. IF N=1 THEN stop ELSE goto 2.
4. N = number of jets in the event using the "scale" y.

The result of the algo can be calculated analytically at NLO:

$$\sigma_{2j} = \sigma^{\text{Born}}\left(1 - \frac{\alpha_S C_F}{\pi}\log^2 y + \ldots\right)$$

$$\sigma_{3j} = \sigma^{\text{Born}}\frac{\alpha_S C_F}{\pi}\log^2 y + \ldots$$



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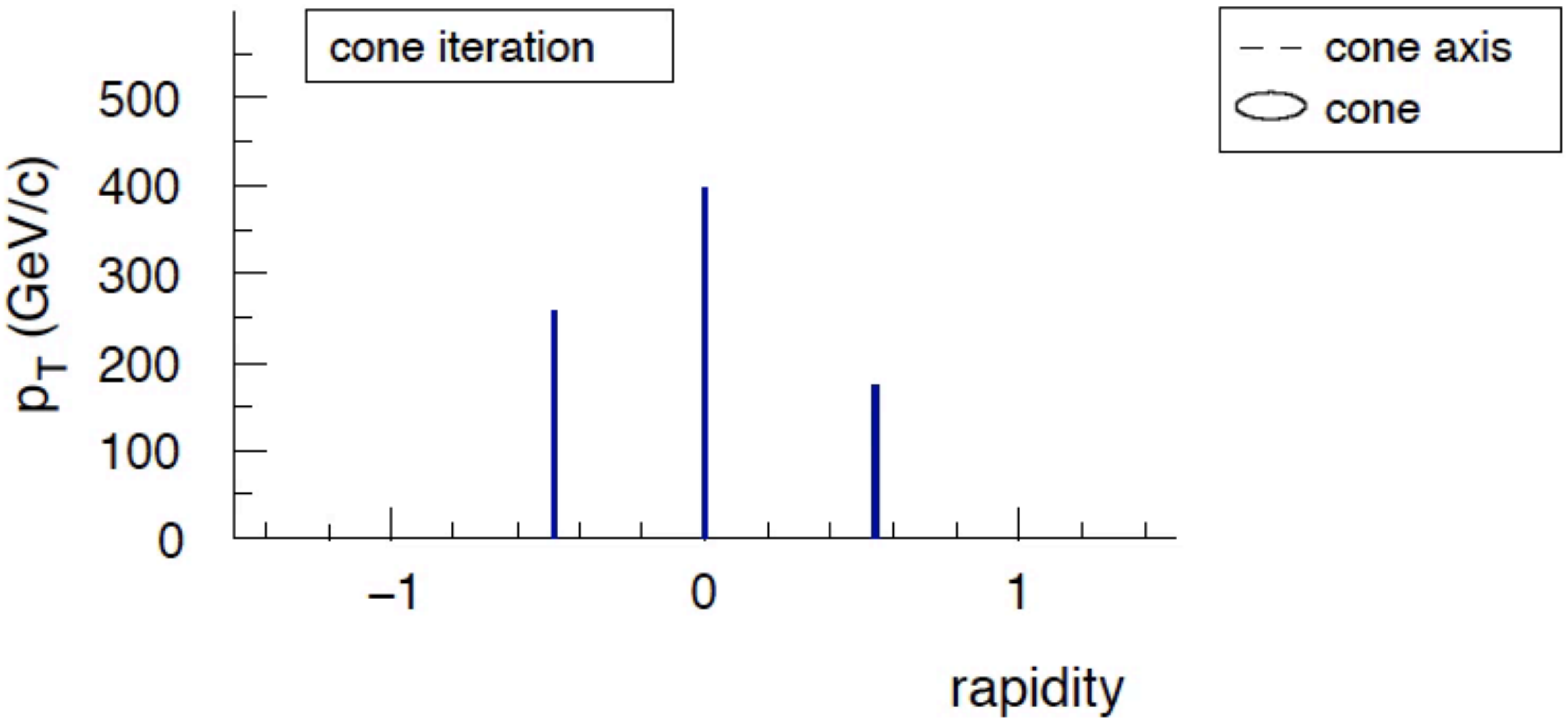
Infrared safety and jet algo’s



- Take hardest particle as seed for cone axis
- Draw cone around seed
- Sum the momenta use as new seed direction, iterate until stable
- Convert contents into a “jet” and remove from event

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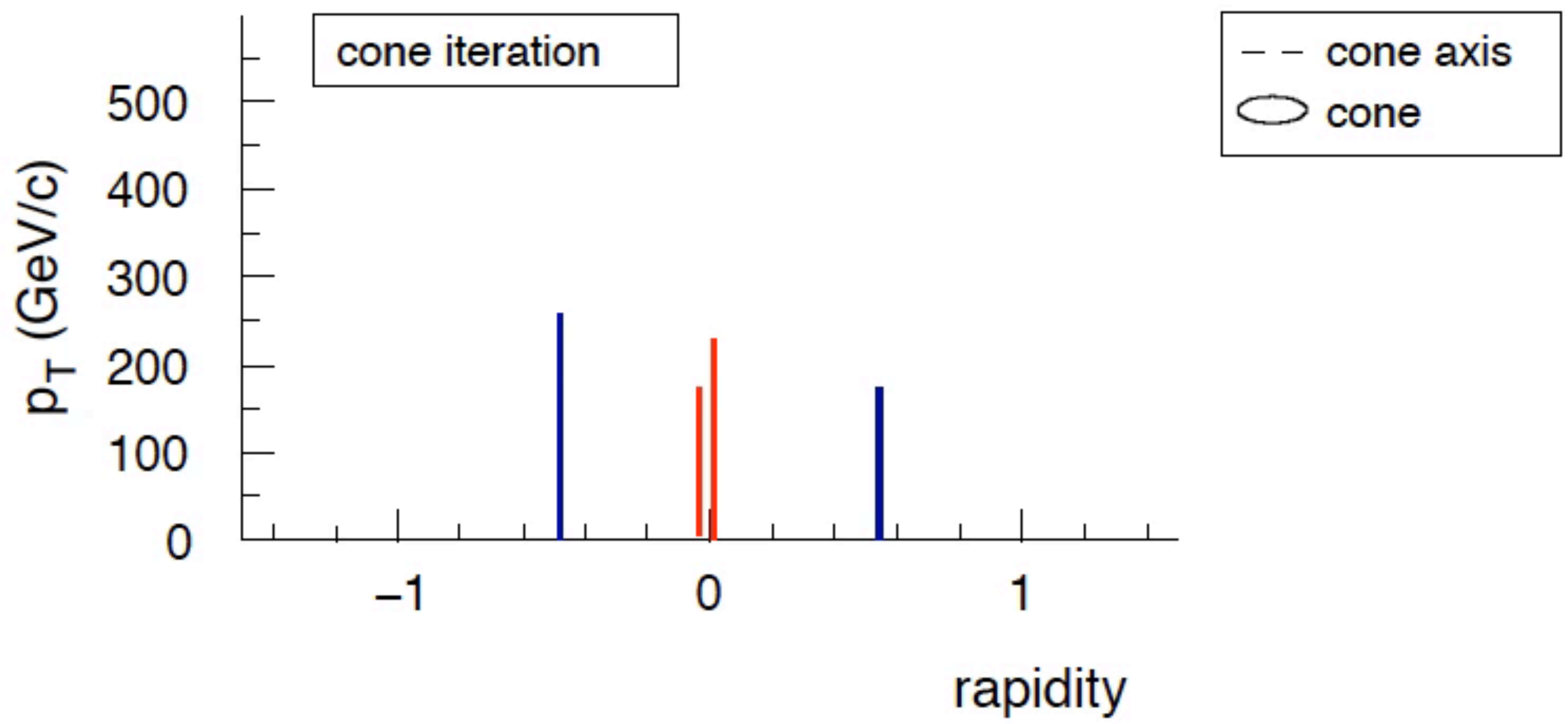
Infrared safety and jet algo's



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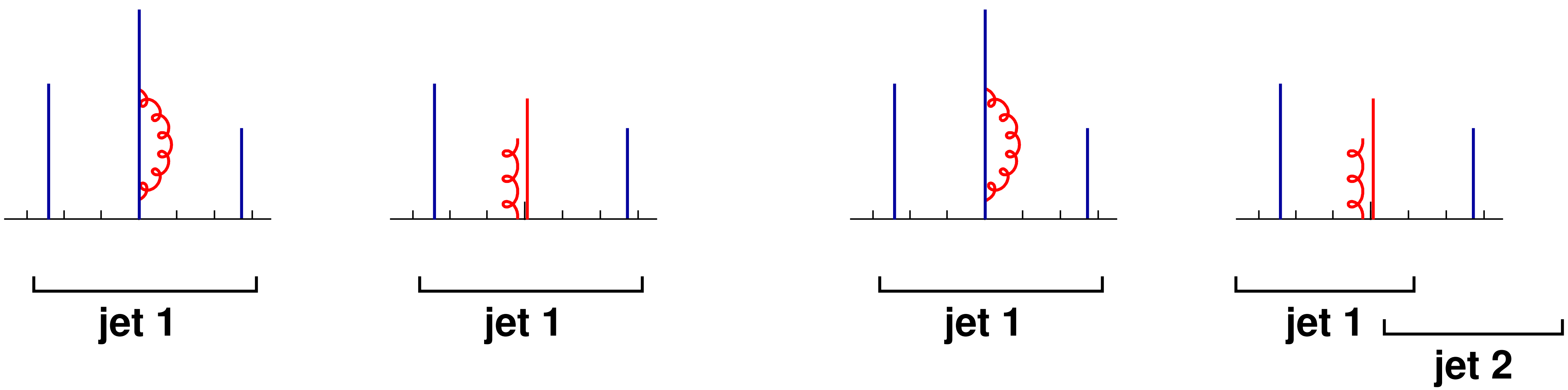
Infrared safety and jet algo's

Collinear Safe | **Collinear Unsafe**



α_s^n x $(-\infty)$ α_s^n x $(+\infty)$ α_s^n x $(-\infty)$ α_s^n x $(+\infty)$

Infinities cancel **Infinities do not cancel**

Invalidates comparison with perturbation theory results

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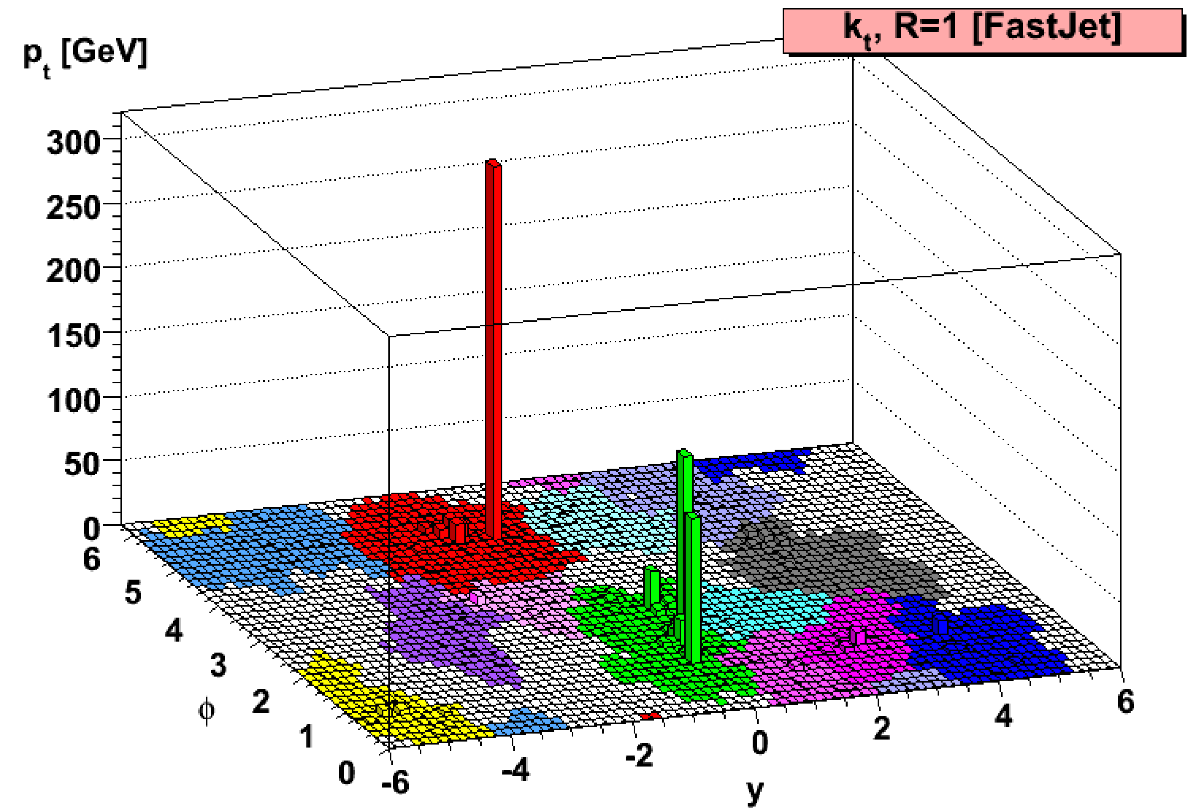
k_T algorithm at hadron colliders

Measure (dimensionful):

$$d_{ij} = \min(p_{ti}^2, p_{tj}^2)\frac{\Delta R_{ij}^2}{R^2}$$

$$d_{iB} = p_{ti}^2$$



The algorithm proceeds by searching for the smallest of the dij and the diB.
If it is a then dij particles i and j are recombined* into a single new particle.
If it is a diB then i is removed from the list of particles, and called a jet.

This is repeated until no particles remain.

kT algorigthm "undoes" the QCD shower

*a 4-momenta recombination scheme is needed (E-scheme)

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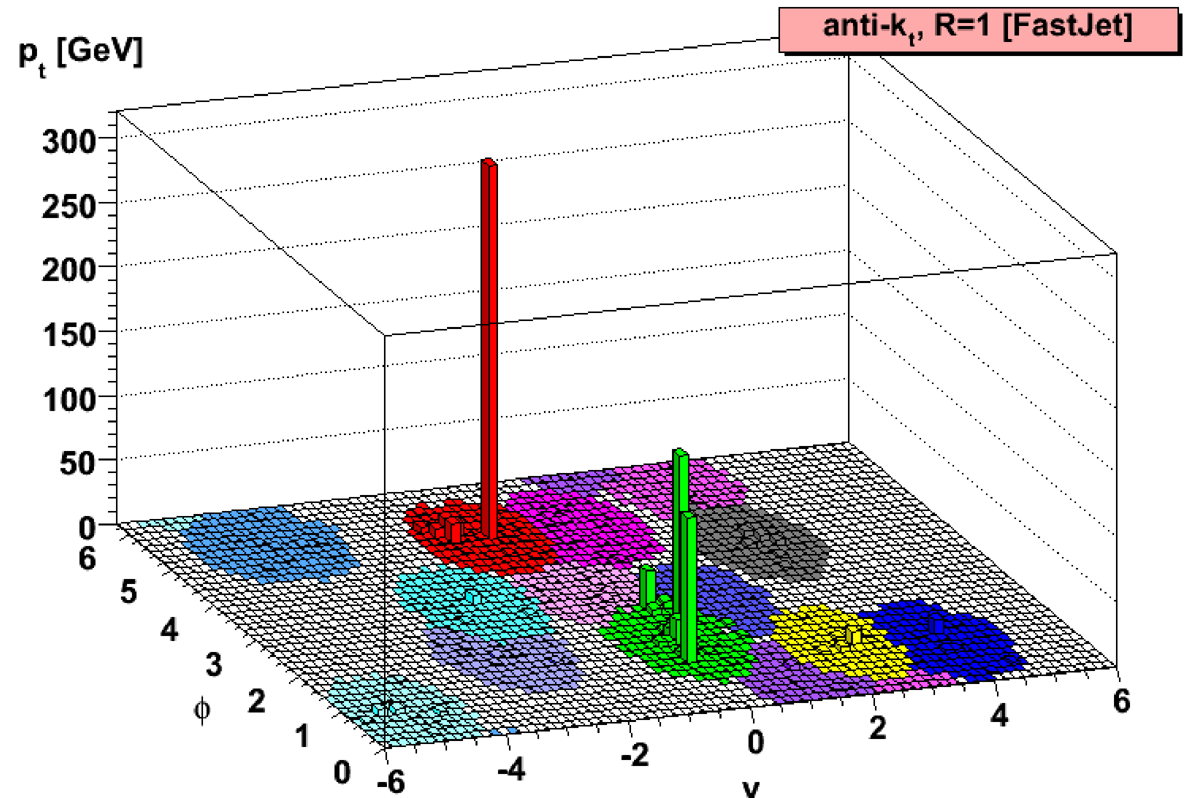
Anti-k_T algorithm

Measure (dimensionful):

$$d_{ij} = \frac{1}{\max(p_{ti}^2, p_{tj}^2)} \frac{\Delta R_{ij}^2}{R^2}$$

$$d_{iB} = \frac{1}{p_{ti}^2}$$



Objects that are close in angle prefer to cluster early, but that clustering tends to occur with a hard particle (rather than necessarily involving soft particles). This means that jets `grow' in concentric circles out from a hard core, until they reach a radius R, giving circular jets.

Unlike cone algorithms the `anti-kT' algorithm is collinear (and infrared) safe.

This, (and the fact that it has been implemented efficiently in FastJet), has led to be the default jet algorithm at the LHC.

It's a handy algorithm but it does not provide internal structure information.

Summary

1. We have studied the problem of infrared divergences in the calculation of the fully inclusive cross section, with the help of the soft limit.

2. We have introduced the concept of an Infrared Safe quantity, i.e. an observable which is both computable at all orders in pQCD and has a well defined counterpart at the experimental level.

3. We have discussed more exclusive quantities, from shape functions to fully exclusive quantities and compared them with e+ e- data.

3. We have explained the basic concept idea of a parton shower MC.

4. We have introduced the idea of jet algorithms (top-down and bottom-up) and discussed the most recent algorithms.