

## One loop further: 4 and 5 loop physics

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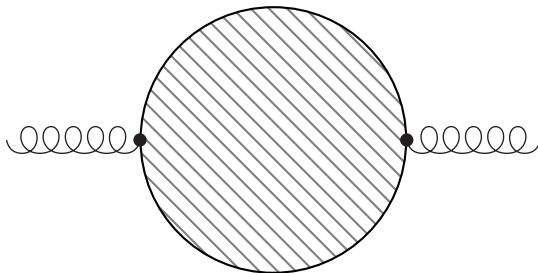


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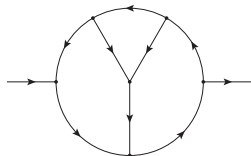
# Forcer package

- We have built a program that calculates four-loop massless propagator diagrams
- Express diagrams as linear combinations of diagrams with fewer propagators



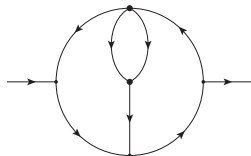
# IBP identities

Through integration by parts (IBP) identities we find rules to remove lines:



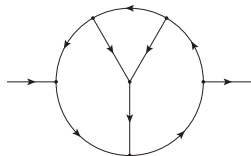
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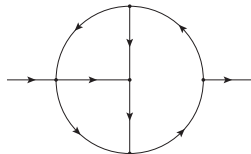
# IBP identities

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# IBP identities

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# Identify substructures

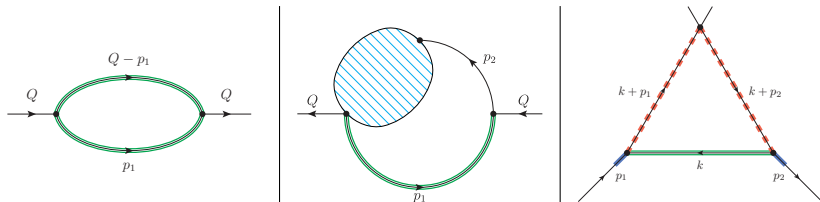


Figure: Guaranteed to remove a green or blue line

Triangle generalizations: diamonds [Ruijl,Ueda,Vermaseren '15]

# Example reduction

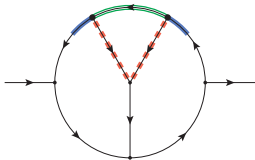


Figure: Reduction of the Benz diagram



# Example reduction

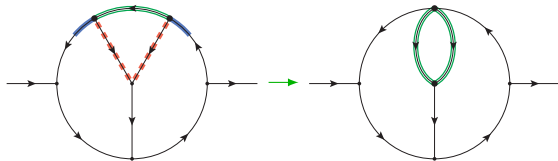


Figure: Reduction of the Benz diagram

# Example reduction

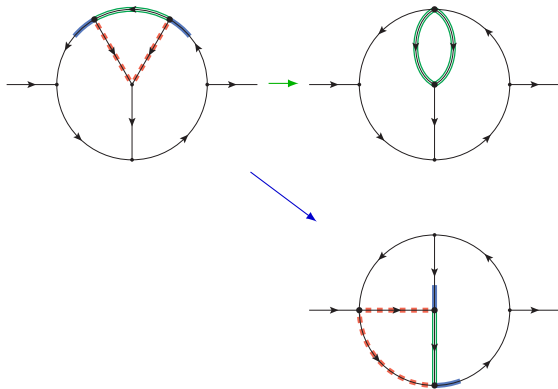


Figure: Reduction of the Benz diagram

# Example reduction

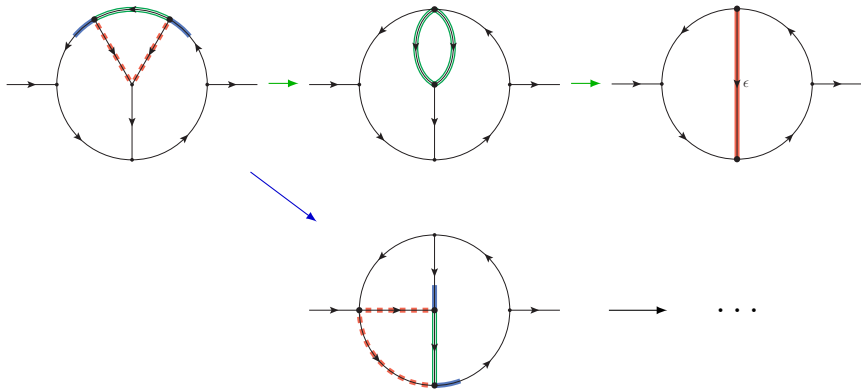


Figure: Reduction of the Benz diagram

# 3-loop reduction graph

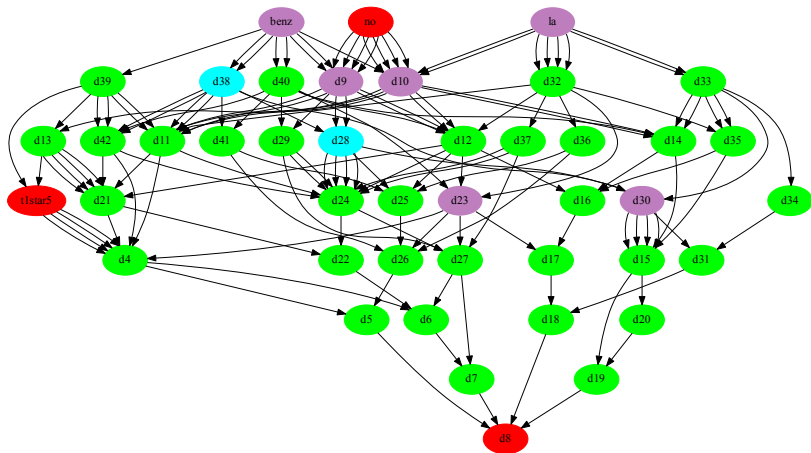


Figure: Reduction graph for 3 loop diagrams

# 4-loop reduction graph

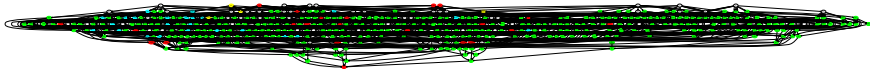


Figure: Reduction graph for 4 loop diagrams

# 4-loop reduction graph

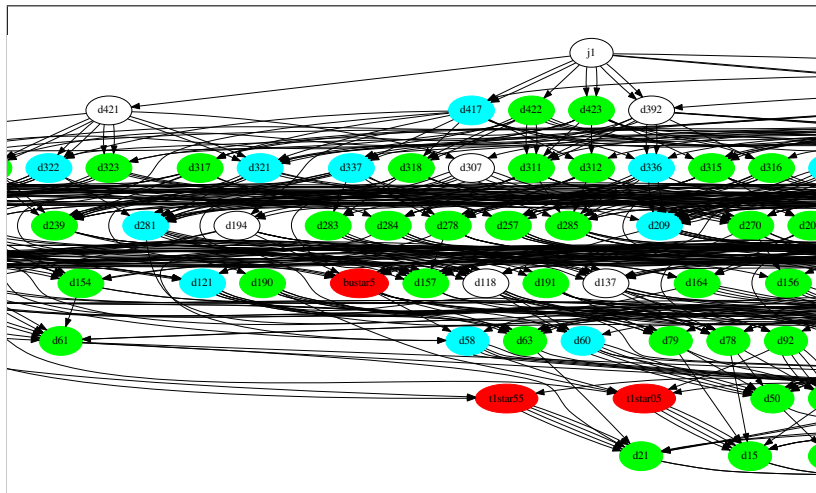
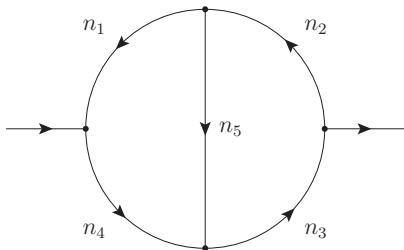


Figure: Part of reduction graph for 4 loops

# Manual solutions

- 21 topologies do not have substructures
- Solve recursion relations parametrically
- Much faster than Laporta methods



$$I(n_1, n_2, n_3, n_4, n_5) = n_1 I(n_1 - 1, n_2, n_3, n_4, n_5) + n_2 I(n_1, n_2 + 1, n_3, n_4, n_5) + \dots$$

$$I(n_1, n_2, n_3, n_4, n_5) = n_3 I(n_1, n_2, n_3 - 1, n_4, n_5) + n_4 I(n_1, n_2, n_3, n_4 + 1, n_5) + \dots$$

## Parametric solution vs. Laporta (I)

$$1) I(n_1, n_2, n_3, n_4, n_5) = + I(n_1 - 1, n_2, n_3, n_4, n_5) \\ + n_2 I(n_1, n_2 + 1, n_3, n_4, n_5)$$

$$2) I(n_1, n_2, n_3, n_4, n_5) = + I(n_1, n_2, n_3, n_4 - 1, n_5) \\ + I(n_1, n_2 + 1, n_3, n_4, n_5)$$

$$\Rightarrow I(n_1, n_2, n_3, n_4, n_5) = \frac{1}{n_2 - 1} \left( n_2 I(n_1, n_2, n_3, n_4 - 1, n_5) \right. \\ \left. - I(n_1 - 1, n_2, n_3, n_4, n_5) \right)$$

Next step in reduction

Solve  $I(n_1, 1, n_3, n_4, n_5)$  since  $n_2 > 2$  will have  $n_1 = 0$  or  $n_4 = 0$ !



# Parametric solution vs. Laporta (II)

Laporta for  $n_2 = 2$ :

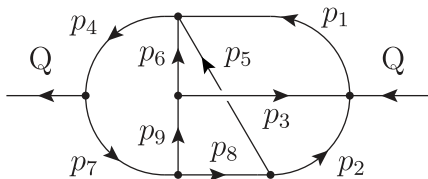
$$1) I(1, 2, 1, 1, 1) = I(0, 2, 1, 1, 1) + 2I(1, 3, 1, 1, 1)$$

$$2) I(1, 2, 1, 1, 1) = I(1, 2, 1, 0, 1) + I(1, 3, 1, 1, 1)$$

$$\Rightarrow I(1, 2, 1, 1, 1) = 2I(1, 2, 1, 0, 1) - I(0, 2, 1, 1, 1)$$

- System needs to be brute forced again for every  $n_2 > 2$
- A lot of equations are generated that drop out of the answer

# Public enemy #1



- 9 propagators that should go to 1
- 5 irreducible dot products that should go to 0
- Took 3 months to find solution!

## Reduction rule

```

id,ifmatch->bubul,
  Z(n1?pos_,n2?pos_,n3?pos_,n4?pos_,n5?pos_,n6?pos_,n7?pos_,
    n8?pos_,n9?pos_,n10?neg0_,n11?neg0_,n12?neg0_,n13?neg0_,n14?neg_)
    = -rat(1,-2*ep-2*n1-n3-n6-n12-n14+4)*(
+Z(-1+n1,-1+n2,n3,n4,1+n5,n6,n7,n8,n9,n10,n11,n12,n13,1+n14)*rat(-n5,1)
+Z(-1+n1,1+n2,n3,n4,-1+n5,n6,n7,n8,n9,n10,n11,n12,n13,1+n14)*rat(n2,1)
+Z(-1+n1,1+n2,n3,n4,n5,n6,n7,-1+n8,n9,n10,n11,n12,n13,1+n14)*rat(-n2,1)
+Z(-1+n1,n2,1+n3,n4,n5,n6,n7,n8,-1+n9,n10,n11,n12,n13,1+n14)*rat(-n3,1)
+Z(-1+n1,n2,n3,n4,-1+n5,n6,n7,n8,n9,n10,n11,1+n12,n13,1+n14)*rat(-n12,1)
+Z(-1+n1,n2,n3,n4,1+n5,n6,n7,-1+n8,n9,n10,n11,n12,n13,1+n14)*rat(n5,1)
+Z(-1+n1,n2,n3,n4,n5,n6,-1+n7,n8,n9,n10,n11,n12,1+n13,1+n14)*rat(-n13,1)
+Z(-1+n1,n2,n3,n4,n5,n6,n7,-1+n8,n9,n10,n11,1+n12,n13,1+n14)*rat(2*n12,1)
+Z(-1+n1,n2,n3,n4,n5,n6,n7,-1+n8,n9,n10,n11,n12,1+n13,1+n14)*rat(n13,1)
+Z(-1+n1,n2,n3,n4,n5,n6,n7,-1+n8,n9,n10,n11,n12,n13,2+n14)*rat(2*n14+2,1)
+Z(-1+n1,n2,n3,n4,n5,n6,n7,n8,-1+n9,n10,n11,n12,1+n13,1+n14)*rat(n13,1)
+Z(-1+n1,n2,n3,n4,n5,n6,n7,n8,n9,1+n10,-1+n11,n12,n13,1+n14)*rat(-n10,1)
+Z(-1+n1,n2,n3,n4,n5,n6,n7,n8,n9,n10,n11,1+n12,n13,n14)*rat(-n12,1)
+Z(-1+n1,n2,n3,n4,n5,n6,n7,n8,n9,n10,n11,n12,n13,1+n14)*rat(-n2+n5,1)
+Z(n1,-1+n2,-1+n3,n4,n5,n6,n7,n8,n9,n10,n11,1+n12,n13,1+n14)*rat(n12,1)
+Z(n1,-1+n2,-1+n3,n4,n5,n6,n7,n8,n9,n10,n11,n12,n13,2+n14)*rat(1+n14,1)
+Z(n1,-1+n2,1+n3,n4,n5,n6,n7,n8,-1+n9,n10,n11,n12,n13,1+n14)*rat(n3,1)

```

+Z(n1, -1+n2, n3, -1+n4, n5, n6, n7, n8, n9, n10, 1+n11, n12, n13, 1+n14)\*rat(n11, 1)  
+Z(n1, -1+n2, n3, n4, -1+n5, n6, n7, n8, n9, n10, n11, 1+n12, n13, 1+n14)\*rat(n12, 1)  
+Z(n1, -1+n2, n3, n4, n5, -1+n6, n7, n8, n9, n10, n11, n12, n13, 2+n14)\*rat(-n14-1, 1)  
+Z(n1, -1+n2, n3, n4, n5, n6, -1+n7, 1+n8, n9, n10, n11, n12, n13, 1+n14)\*rat(2\*n8, 1)  
+Z(n1, -1+n2, n3, n4, n5, n6, -1+n7, n8, n9, n10, 1+n11, n12, n13, 1+n14)\*rat(-n11, 1)  
+Z(n1, -1+n2, n3, n4, n5, n6, -1+n7, n8, n9, n10, n11, n12, 1+n13, 1+n14)\*rat(-n13, 1)  
+Z(n1, -1+n2, n3, n4, n5, n6, n7, -1+n8, n9, n10, n11, 1+n12, n13, 1+n14)\*rat(-2\*n12, 1)  
+Z(n1, -1+n2, n3, n4, n5, n6, n7, -1+n8, n9, n10, n11, n12, 1+n13, 1+n14)\*rat(n13, 1)  
+Z(n1, -1+n2, n3, n4, n5, n6, n7, -1+n8, n9, n10, n11, n12, n13, 2+n14)\*rat(-2\*n14-2, 1)  
+Z(n1, -1+n2, n3, n4, n5, n6, n7, 1+n8, -1+n9, n10, n11, n12, n13, 1+n14)\*rat(-n8, 1)  
+Z(n1, -1+n2, n3, n4, n5, n6, n7, n8, -1+n9, n10, n11, 1+n12, n13, 1+n14)\*rat(-2\*n12, 1)  
+Z(n1, -1+n2, n3, n4, n5, n6, n7, n8, n9, 1+n10, -1+n11, n12, n13, 1+n14)\*rat(n10, 1)  
+Z(n1, -1+n2, n3, n4, n5, n6, n7, n8, n9, 1+n10, n11, n12, n13, 1+n14)\*rat(-2\*n10, 1)  
+Z(n1, -1+n2, n3, n4, n5, n6, n7, n8, n9, n10, 1+n11, n12, n13, 1+n14)\*rat(-n11, 1)  
+Z(n1, -1+n2, n3, n4, n5, n6, n7, n8, n9, n10, n11, 1+n12, n13, n14)\*rat(-n12, 1)  
+Z(n1, 1+n2, n3, n4, -1+n5, n6, n7, n8, n9, -1+n10, n11, n12, n13, 1+n14)\*rat(-n2, 1)  
+Z(n1, 1+n2, n3, n4, -1+n5, n6, n7, n8, n9, n10, n11, n12, n13, 1+n14)\*rat(-n2, 1)  
+Z(n1, 1+n2, n3, n4, n5, n6, n7, -1+n8, n9, -1+n10, n11, n12, n13, 1+n14)\*rat(2\*n2, 1)  
+Z(n1, 1+n2, n3, n4, n5, n6, n7, -1+n8, n9, n10, n11, n12, n13, 1+n14)\*rat(n2, 1)  
+Z(n1, n2, -1+n3, n4, n5, n6, n7, -1+n8, n9, n10, n11, 1+n12, n13, 1+n14)\*rat(-n12, 1)  
+Z(n1, n2, -1+n3, n4, n5, n6, n7, -1+n8, n9, n10, n11, n12, n13, 2+n14)\*rat(-n14-1, 1)  
+Z(n1, n2, 1+n3, n4, n5, n6, n7, n8, -1+n9, -1+n10, n11, n12, n13, 1+n14)\*rat(n3, 1)

+Z(n1,n2,n3,n4,-1+n5,n6,-1+n7,n8,n9,n10,n11,n12,1+n13,1+n14)\*rat(n13,1)  
+Z(n1,n2,n3,n4,-1+n5,n6,n7,n8,-1+n9,n10,n11,n12,1+n13,1+n14)\*rat(-n13,1)  
+Z(n1,n2,n3,n4,-1+n5,n6,n7,n8,-1+n9,n10,n11,n12,n13,2+n14)\*rat(1+n14,1)  
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+Z(n1,n2,n3,n4,n5,n6,-1+n7,-1+n8,n9,n10,1+n11,n12,n13,1+n14)\*rat(n11,1)  
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+Z(n1,n2,n3,n4,n5,n6,-1+n7,n8,n9,n10,n11,n12,n13,1+n14)\*rat(-2\*ep-2\*n4-1,1)  
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+Z(n1,n2,n3,n4,n5,n6,n7,-1+n8,-1+n9,n10,n11,n12,n13,2+n14)\*rat(-2\*n14-2,1)  
+Z(n1,n2,n3,n4,n5,n6,n7,-1+n8,n9,-1+n10,1+n11,n12,n13,1+n14)\*rat(-n11,1)  
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+Z(n1,n2,n3,n4,n5,n6,n7,-1+n8,n9,-1+n10,n11,n12,n13,2+n14)\*rat(-2\*n14-2,1)  
+Z(n1,n2,n3,n4,n5,n6,n7,-1+n8,n9,1+n10,n11,n12,n13,1+n14)\*rat(2\*n10,1)

```
+Z(n1,n2,n3,n4,n5,n6,n7,-1+n8,n9,n10,-1+n11,n12,1+n13,1+n14)*rat(-2*n13,1)
+Z(n1,n2,n3,n4,n5,n6,n7,-1+n8,n9,n10,1+n11,n12,n13,1+n14)*rat(n11,1)
+Z(n1,n2,n3,n4,n5,n6,n7,-1+n8,n9,n10,n11,-1+n12,n13,2+n14)*rat(-n14-1,1)
+Z(n1,n2,n3,n4,n5,n6,n7,-1+n8,n9,n10,n11,1+n12,n13,1+n14)*rat(-2*n12,1)
+Z(n1,n2,n3,n4,n5,n6,n7,-1+n8,n9,n10,n11,n12,1+n13,1+n14)*rat(-3*n13,1)
+Z(n1,n2,n3,n4,n5,n6,n7,-1+n8,n9,n10,n11,n12,n13,1+n14)*rat(10*ep+2*n1,1)
+Z(n1,n2,n3,n4,n5,n6,n7,-1+n8,n9,n10,n11,n12,n13,2+n14)*rat(-2*n14-2,1)
+Z(n1,n2,n3,n4,n5,n6,n7,n8,-1+n9,-1+n10,n11,1+n12,n13,1+n14)*rat(-n12,1)
+Z(n1,n2,n3,n4,n5,n6,n7,n8,-1+n9,-1+n10,n11,n12,1+n13,1+n14)*rat(-n13,1)
+Z(n1,n2,n3,n4,n5,n6,n7,n8,-1+n9,n10,-1+n11,n12,n13,2+n14)*rat(-n14-1,1)
+Z(n1,n2,n3,n4,n5,n6,n7,n8,-1+n9,n10,n11,n12,1+n13,1+n14)*rat(-n13,1)
+Z(n1,n2,n3,n4,n5,n6,n7,n8,-1+n9,n10,n11,n12,n13,1+n14)*rat(-4*ep-2*n1-n3,1)
+Z(n1,n2,n3,n4,n5,n6,n7,n8,n9,-1+n10,n11,n12,n13,1+n14)*rat(-n5+1,1)
+Z(n1,n2,n3,n4,n5,n6,n7,n8,n9,1+n10,-1+n11,n12,n13,1+n14)*rat(n10,1)
+Z(n1,n2,n3,n4,n5,n6,n7,n8,n9,n10,-1+n11,n12,n13,1+n14)*rat(2*ep+n5+2*n8
+n9+n10+n11-5,1)
+Z(n1,n2,n3,n4,n5,n6,n7,n8,n9,n10,n11,1+n12,n13,n14)*rat(n12,1)
+Z(n1,n2,n3,n4,n5,n6,n7,n8,n9,n10,n11,n12,-1+n13,1+n14)*rat(-2*ep-n5-2*n8
-n9-n11-n14+3,1)
+Z(n1,n2,n3,n4,n5,n6,n7,n8,n9,n10,n11,n12,n13,1+n14)*rat(2*ep+n2+n7+
2*n8+n9+n11+n14-4,1)
);
```

# First results and benchmarks

- Reproduced 4-loop QCD  $\beta$ -function [Ritbergen, Vermaseren, Larin '97; Czakon '04]
- All  $\xi$  and  $\epsilon$  4-loop solution for QCD propagators and vertices [new!]

$\beta_3$ no gauge	10 minutes
$\beta_3$ 1 gauge	38 minutes
$\beta_3$ all gauge	8.5 hours
no1(2,2,2,2,2,2,2,2,2,2,2,-1,-1,-1)	<b>42 minutes</b>

Table: Benchmark on 24 core machine

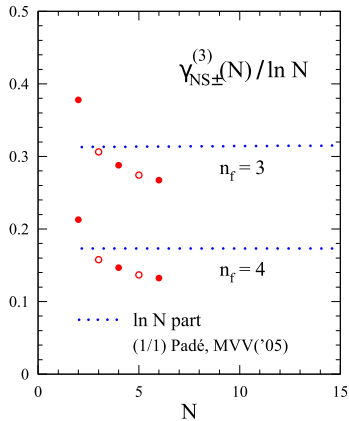
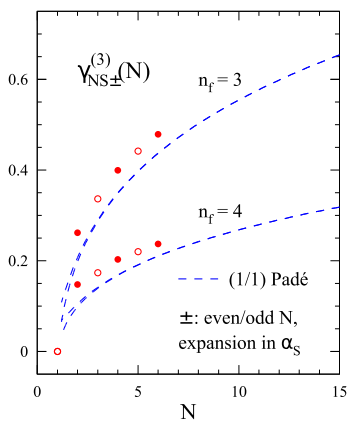
## 4-loop splitting functions

- Can we calculate the 4-loop corrections to the PDF evolution?
- Calculate Mellin moments  $N$  for as many  $N$  as possible
- Each  $N \rightarrow N + 2$  increases weight by 4
- 4-loop time: more than 1000 times 3-loop time

$$\gamma_{ij} = -P_{ij} = \sum_n \left(\frac{\alpha_s}{4\pi}\right)^n P_{ij}^{(n-1)} \quad i, j = q, g$$

Non-singlet (NS): combinations of  $\gamma_{qq}$



N3LO corrections to  $\gamma_{\text{NS}}$ 

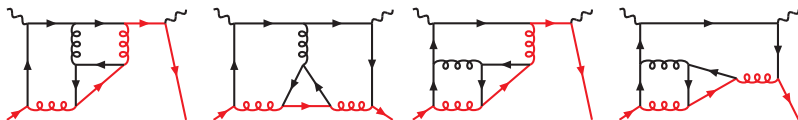
- $N = 2, 3, 4$  in agreement with [Baikov,Chetyrkin '06, Velizhanin '13,'14]
- Not enough moments yet to improve Padé of  $\gamma_{\text{cusp}}^{(3)}$

# Example result

$$\begin{aligned}
 \gamma_{\text{gg}}^{(3)}(N=4) = & C_A^4 \left( \frac{1502628149}{3375000} + \frac{1146397}{11250} \zeta_3 - \frac{504}{5} \zeta_5 \right) + \frac{d_A^{abcd} d_A^{abcd}}{n_a} \left( \frac{21623}{150} \right. \\
 & + \frac{15596}{15} \zeta_3 - \frac{6048}{5} \zeta_5 \left. \right) - n_f C_A^3 \left( \frac{20580892841}{72900000} + \frac{12550223}{22500} \zeta_3 - \frac{8613}{25} \zeta_4 - \frac{4316}{27} \zeta_5 \right) \\
 & + n_f \frac{d_R^{abcd} d_A^{abcd}}{n_a} \left( \frac{160091}{675} + \frac{80072}{225} \zeta_3 - \frac{48016}{45} \zeta_5 \right) - n_f C_A^2 C_F \left( \frac{4212122951}{41006250} \right. \\
 & - \frac{1170784}{5625} \zeta_3 + \frac{418198}{1125} \zeta_4 - \frac{17636}{45} \zeta_5 \left. \right) + n_f C_A C_F^2 \left( \frac{1913110089023}{26244000000} + \frac{39313783}{101250} \zeta_3 \right. \\
 & + \frac{26741}{750} \zeta_4 - \frac{3082}{5} \zeta_5 \left. \right) + n_f C_F^3 \left( \frac{34764568601}{2099520000} - \frac{958343}{40500} \zeta_3 - \frac{18997}{2250} \zeta_4 + \frac{908}{45} \zeta_5 \right) \\
 & - n_f^2 C_A^2 \left( \frac{3250393649}{218700000} - \frac{2969291}{20250} \zeta_3 + \frac{1566}{25} \zeta_4 + \frac{1276}{135} \zeta_5 \right) - n_f^2 C_F^2 \left( \frac{275622924731}{26244000000} \right. \\
 & - \frac{253369}{10125} \zeta_3 + \frac{1078}{225} \zeta_4 \left. \right) + n_f^2 C_A C_F \left( \frac{136020246173}{3280500000} - \frac{1672751}{10125} \zeta_3 + \frac{15172}{225} \zeta_4 \right) \\
 & + n_f^2 \frac{d_R^{abcd} d_R^{abcd}}{n_a} \left( \frac{75788}{675} + \frac{3008}{15} \zeta_3 - \frac{20416}{45} \zeta_5 \right) + n_f^3 C_F \left( \frac{1780699}{24300000} - \frac{484}{675} \zeta_3 \right) \\
 & - n_f^3 C_A \left( \frac{20440457}{21870000} - \frac{1888}{405} \zeta_3 \right)
 \end{aligned}$$

All- $N$  results for  $n_f^3$  and  $n_f^2$ 

All 3-loop graphs (with insertion on gluon line) for  $n_f^2$  in  $\gamma_{qq}$ :



- Simple topologies for forcer  $\rightarrow$  high moments
- Analytic form in  $N$  reconstructed with LLL-method  
[Velizhanin '12; Moch,Vermaseren,Vogt '14]
- Recent success:  $n_f^3$  for  $\gamma_{qg}$  required  $N = 44$
- Singlet splitting function matrix is now known at leading  $n_f$   
[new; Davies,RUVV]

# Quark cusp anomalous dimension

Large  $n_c$  contribution example:

$$\begin{aligned} \gamma_{\text{ns}}^{(3)}(N)|_{C_F n_c n_f^2} = & \frac{127}{18} + \frac{1}{81} \left( \frac{20681}{2} \eta + 2119 S_1 - 2275 \eta^2 - 20460 D_1^2 + 3392 S_1 \eta \right. \\ & - 5036 S_2 \left. \right) + \frac{4}{81} (118 \eta^3 - 886 D_1^3 - 914 S_1 \eta^2 - 848 S_1 D_1^2 - 152 S_{1,2} - 416 S_2 \eta \\ & - 152 S_{2,1} + 1148 S_3) + \frac{8}{27} (-57 D_1^4 + 18 S_1 \eta^3 - 24 S_1 D_1^3 + 2 S_2 \eta^2 + 128 S_2 D_1^2 \\ & - 8 S_3 \eta + 40 S_{1,3} + 80 S_{2,2} + 120 S_{3,1} - 159 S_4) + \frac{8}{9} (-6 \eta^5 - 12 D_1^5 + 10 S_1 \eta^4 \\ & - 24 S_1 D_1^4 + 8 S_2 \eta^3 + 4 S_3 \eta^2 - 8 S_3 D_1^2 + 4 S_{3,1} \eta - 8 S_{1,3,1} + 4 S_{1,4} - 8 S_{2,3} \\ & - 16 S_{3,2} - 2 S_4 \eta - 20 S_{4,1} + 24 S_5) + \text{much simpler } \zeta_3, \zeta_4 \text{ terms} \end{aligned}$$

From  $\ln N$  coefficient in large- $N$  limit:

$$\begin{aligned} \gamma_{\text{cusp}}^{(3)} = & \dots + C_F C_A n_f^2 \left( \frac{923}{81} - \frac{608}{81} \zeta_2 + \frac{2240}{27} \zeta_3 - \frac{112}{3} \zeta_4 \right) \\ & + C_F^2 n_f^2 \left( \frac{2392}{81} - \frac{640}{9} \zeta_3 + 32 \zeta_4 \right) - C_F n_f^3 \left( \frac{32}{81} - \frac{64}{27} \zeta_3 \right) \end{aligned}$$

# Five loops

- Can we go to five loops?
- Recompute 5-loop QCD  $\beta$ -function [Baikov, Chetyrkin, Kühn '16] with general colour group and gauge? [see also: Luthe, Maier, Marquard, Schröder '16]
- 5-loop Forcer is hard:
  - Unknown master integrals
  - More than 200 manual reductions!

# Five loops

- We are interested in anomalous dimensions
- Thus, we only need the UV pole part
- Can we IR rearrange to create a *carpet* diagram?

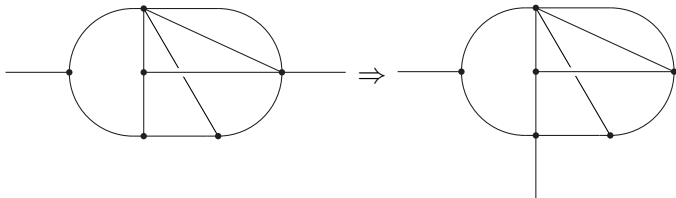
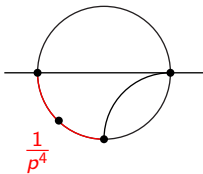


Figure: The graphs have the **same** UV poles

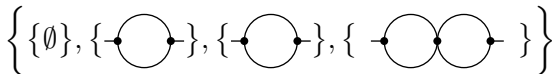
# Five loops

- In general, this creates IR divergences
- Subtract UV and IR all using  $R^*$  [Chetyrkin,Smirnov,Larin,...]
- This involves carefully identifying and subtracting all counter-terms
- $R^*$  has its own problems: commutativity issues, high-rank tensor reductions, ...

# Subdivergences

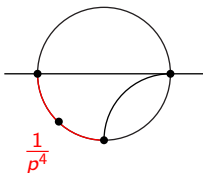


UV subdiagrams:

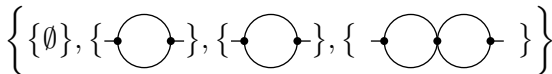




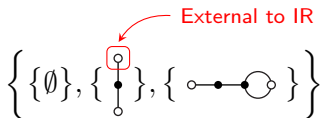
# Subdivergences



UV subdiagrams:



IR subdiagrams:



# Superficial UV divergence

Pole part:  $K(\frac{1}{\epsilon} + 1 + \epsilon) = \frac{1}{\epsilon}$

$$\begin{aligned}
 K\bar{R}^* \left( \text{Diagram 1} \right) &= K \left( \text{Diagram 2} \right) - K \left( \text{Diagram 3} \right) \cdot \text{Diagram 4} \\
 &+ K\underline{R}^* \left( \text{Diagram 5} \right) K \left( \text{Diagram 6} \right) \cdot \text{Diagram 7} \\
 &- K\underline{R}^* \left( \text{Diagram 8} \right) \cdot \text{Diagram 9} \\
 &+ K\underline{R}^* \left( \text{Diagram 10} \right) K \left( \text{Diagram 11} \right) \\
 &+ K\underline{R}^* \left( \text{Diagram 12} \right) K\bar{R} \left( \text{Diagram 13} \right)
 \end{aligned}$$

see *Critical Properties of  $\phi^4$ -Theories* by Kleinert and Schulte for an introduction

# Conclusion

- Very fast four-loop reductions beyond the scope of Laporta methods
- Calculated Mellin moments, including all- $N$  generalizations
- Extensions to five loop is difficult, but possible
- $R^*$  of interest for 4-loop high  $N$  or 5-loop splitting functions

# Acknowledgements

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“HEPGAME”