Hard Hydro Loops (HHLs) and long-time tails from hydro-kinetics, for a Bjorken expansion

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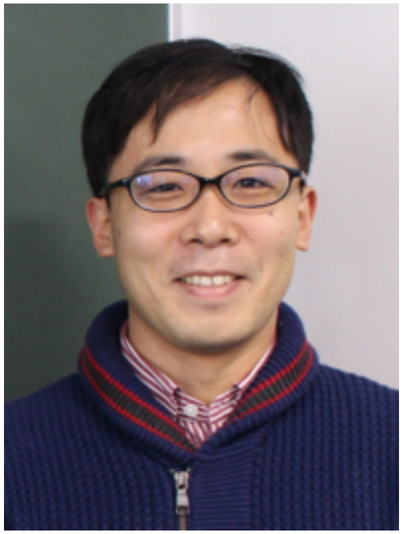
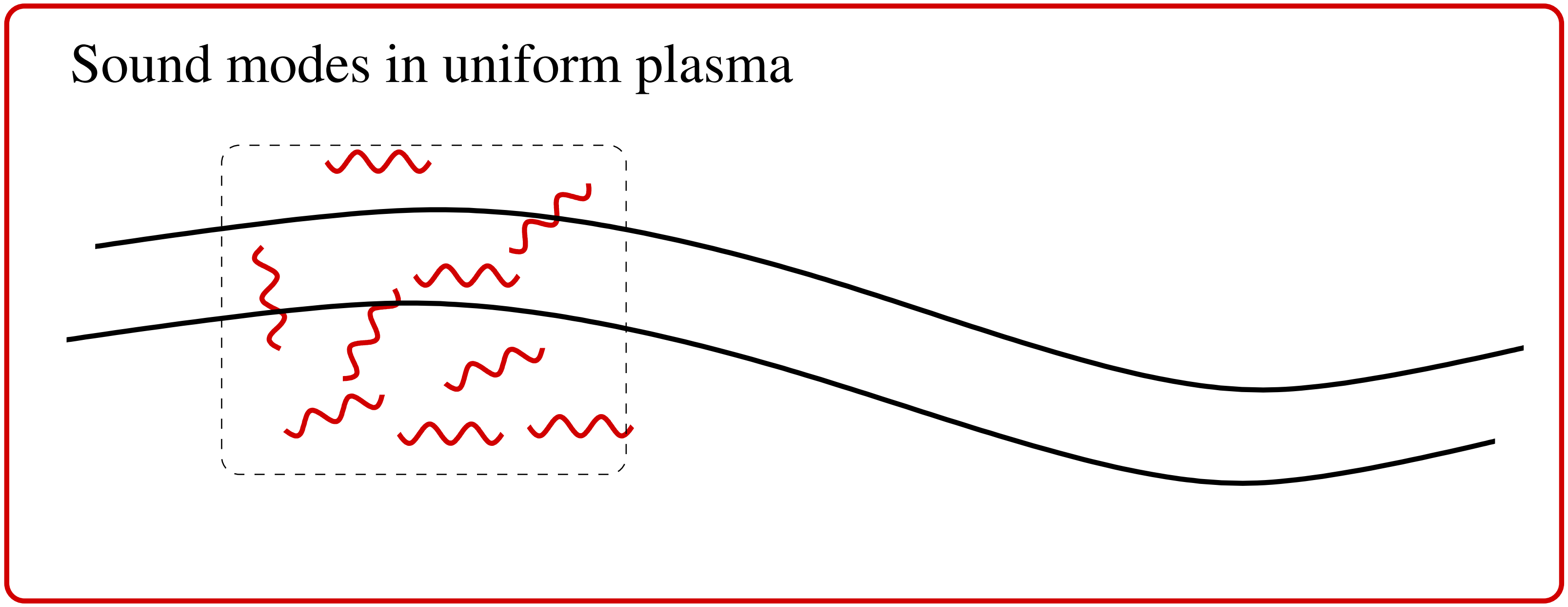
- Yukinao Akamatsu, Aleksas Mazeliauskas, DT, arXiv:1606.07742



Thermal fluctuations:



These hard sound modes are part of the bath, giving to the pressure and shear viscosity

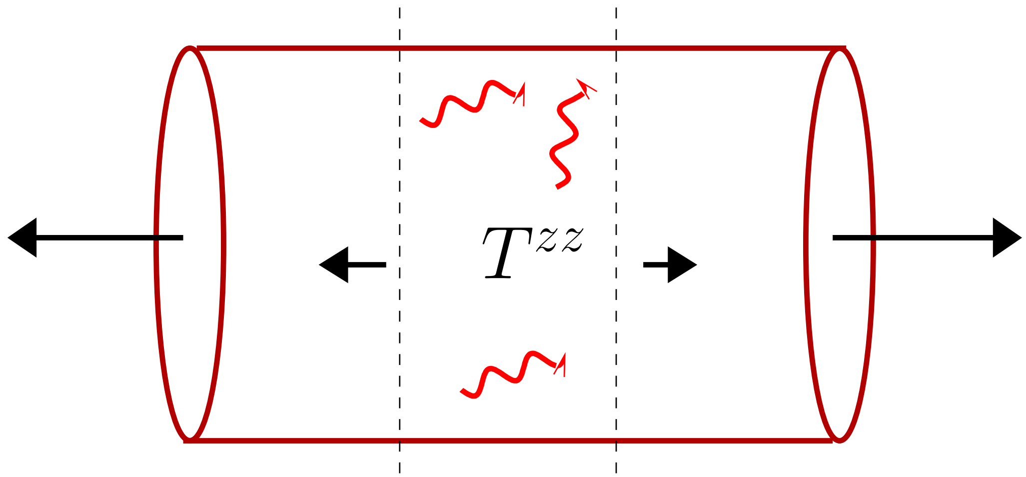
$$N_{ee}(\boldsymbol{k},t) \equiv \underbrace{\langle e^*(\boldsymbol{k},t) e(\boldsymbol{k},t)\rangle}_{\text{energy-density flucts}} = (e+p)T/c_s^2$$

$$N_{gg}(\boldsymbol{k},t) \equiv \underbrace{\left\langle g^{*i}(\boldsymbol{k},t) g^{j}(\boldsymbol{k},t)\right\rangle}_{\text{momentum, } g^i \equiv T^{0i}} = (e+p)T\delta^{ij}$$

In an expanding system these correlators will be driven out of equilibrium.

This changes the evolution of the slow modes.

A Bjorken expansion



1. The system has an expansion rate of $\partial_\mu u^\mu = 1/\tau$

2. The hydrodynamic expansion parameter is

$$\epsilon \equiv \frac{\gamma_\eta}{\tau} \ll 1 \qquad \gamma_\eta \equiv \frac{\eta}{e+p}$$

and corrections to hydrodynamics are organized in powers of ϵ

$$T^{zz} = p\Big[1 + \underbrace{\mathcal{O}(\epsilon)}_{\text{1st order}} + \underbrace{\mathcal{O}(\epsilon^2)}_{\text{2nd order}} + \ldots\Big]$$

High k modes are brought to equilibrium by the dissipation and noise

The transition regime:

- There is a wave number where the damping rate competes with the expansion

$$\underbrace{\gamma_\eta k^2}_{\text{damping rate}} \sim \underbrace{\frac{1}{\tau}}_{\text{expansion rate}}$$

and thus the transition happens for: $\gamma_\eta \equiv \eta/(e+p)$

$$k \sim k_* \equiv \frac{1}{\sqrt{\gamma_\eta \tau}}$$

need $k \gg k_*$ to reach equilibrium!

- This is an intermediate scale $k_* \equiv 1/(\tau\sqrt{\epsilon})$, $\epsilon \equiv \eta/(e+p)\tau$

These inequalities are the same and hold whenever hydro applies

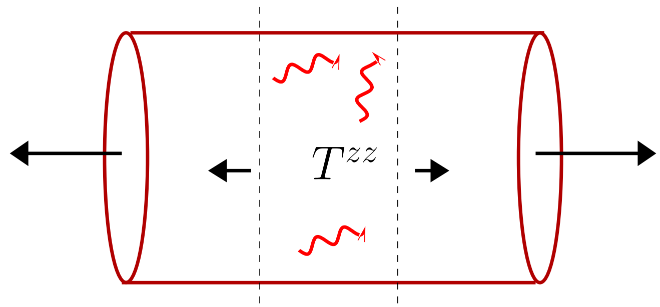
$$\frac{1}{\tau} \ll k_* \ll \frac{1}{\ell_{\rm mfp}}$$

$$\frac{1}{\tau} \ll \frac{1}{\tau}\frac{1}{\sqrt{\epsilon}} \ll \frac{1}{\tau}\frac{1}{\epsilon}$$

Want to develop a set of hydro-kinetic equations for $k \sim k_*$

using the scale separation $\epsilon \ll \sqrt{\epsilon} \ll 1$

Estimate of longitudinal pressure from non-equilibrium modes



$$\Delta T^{zz} \sim \underbrace{\tfrac{1}{2}T}_{\text{energy / mode}} \times \underbrace{k_*^3}_{\text{number of non-eq modes}}$$

- Using $e + p = sT$ we estimate

$$\frac{\Delta T^{zz}}{e+p} \sim \frac{1}{s}\,\frac{1}{(\gamma_\eta \tau)^{3/2}}$$

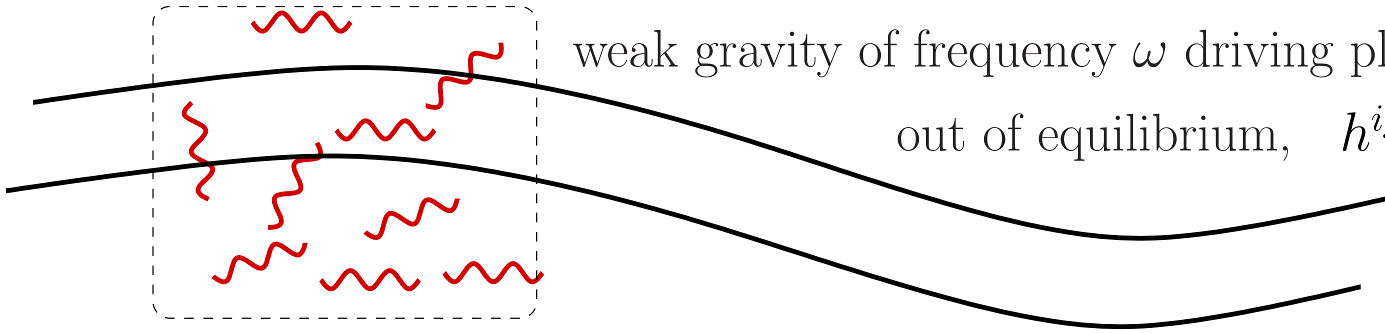
- The full result will be:

$$\frac{\langle T^{zz}\rangle}{e+p} = \Big[\underbrace{\frac{p}{e+p}}_{\sim 1} - \underbrace{\frac{4}{3}\frac{\gamma_\eta}{\tau}}_{\text{1st order}} + \underbrace{\frac{1.08318}{s\,(4\pi\gamma_\eta\tau)^{3/2}}}_{\text{3/2 order}} + \underbrace{\frac{(\lambda_1 - \eta\tau_\pi)}{e+p}\frac{8}{9\tau^2}}_{\text{2nd order}}\Big]$$

The correction is suppressed by $s =$ the number of degrees of freedom

Equilibrium Linear Response

Kovtun, Yaffe; Kovtun, Moore, Romatschke



weak gravity of frequency ω driving plasma out of equilibrium, $h^{ij}(\omega)$

Hydro prediction

$$\langle T^{ij} \rangle = p h^{ij} - \eta \big(\overbrace{\nabla^i u^j}^{\text{cov-deriv}} + \nabla^j u^i - \frac{2}{3} \nabla \cdot u \big) + \text{2nd order}$$

So

$$\langle T^{xy}(\omega) \rangle = \Big[p - \overbrace{i\omega\eta}^{\text{1st order}} + \overbrace{(\eta\tau_\pi - \tfrac{1}{2}\kappa)\omega^2}^{\text{2nd order}} \Big] h^{xy}(\omega)$$

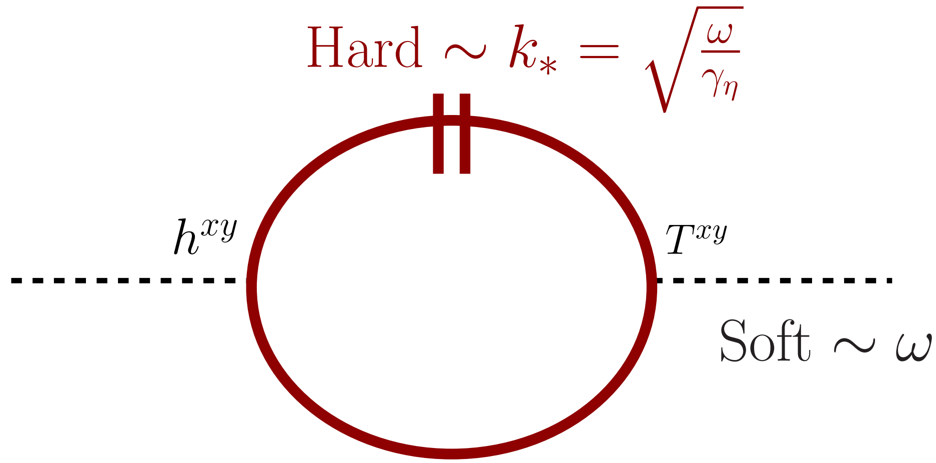
Thermal flucts. are not included, and are driven slightly out of equilibrium for $k \sim k_*$

$$\gamma_\eta k_*^2 \sim \omega \qquad \text{and they are hard} \qquad \omega \ll k_* \sim \sqrt{\frac{\omega}{\gamma_\eta}} \ll \frac{1}{\ell_{\rm mfp}}$$

Include hard thermal fluctuations with $k_* \sim \sqrt{\omega/\gamma_\eta}$ as loops

Hydro Hard Thermal Loops (HHTLs)

Kovtun, Yaffe; Kovtun, Moore, Romatschke



Evaluate the "Hard Hydro Thermal Loop"

$$\langle T^{xy}(\omega)\rangle = \Big[p - \underbrace{i\omega\eta}_{\text{1st order}} + \underbrace{\frac{(7+(3/2)^{3/2})}{240\pi} T \left(\frac{\omega}{\gamma_\eta}\right)^{3/2}}_{\text{3/2 order}} + \mathcal{O}(\omega^2)\Big] h^{xy}(\omega)$$

The correction is of order

$$\Delta T^{xy} \sim \tfrac{1}{2} T\, k_*^3\, h^{xy}$$

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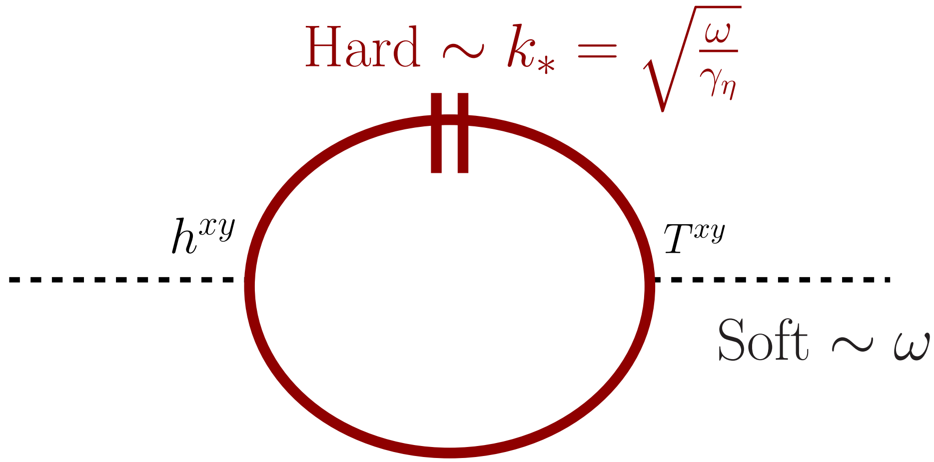

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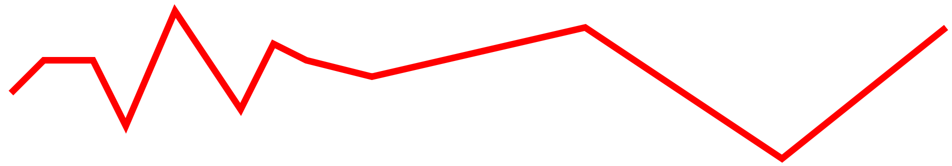
I will follow Uli with hydrodynamics!

(by deriving HHTL loops from hydro Kinetic Theory)

Developing hydro-kinetics – Brownian motion

Random Walk



$$\frac{dp}{dt} = -\eta p + \xi \qquad \left\langle \xi(t)\xi(t')\right\rangle = 2TM\eta\,\delta(t-t')$$

1. Then we want to calculate

$$N(t) = \left\langle p^2(t)\right\rangle$$

2. Integrate the equation for short times

$$p(t+\Delta t) = -\eta\, p(t)\Delta t + \int_t^{t+\Delta t} \xi(t')dt'$$

3. Compute $\langle p(t+\Delta t)\, p(t+\Delta t)\rangle$ and find an equation

$$\frac{\Delta N}{\Delta t} = -2\eta\Big[N - \underbrace{TM}_{\text{equilibrium}}\Big]$$

Developing hydro-kinetics – linearized hydro in a uniform system

1. Evolve fields of linearized hydro with bare parameters $p_0(\Lambda)$, $\eta_0(\Lambda)$, $s_0(\Lambda)$ etc

$$\phi_a(\boldsymbol{k}) \equiv \Big(e(\boldsymbol{k}), g^x(\boldsymbol{k}), g^y(\boldsymbol{k}), g^z(\boldsymbol{k})\Big)$$

2. Then the equations are schematically exactly the same

$$\frac{d\phi_a(\boldsymbol{k})}{dt} = \underbrace{\mathcal{L}_{ab}(\boldsymbol{k})}_{\text{ideal}}\,\phi_b(\boldsymbol{k}) + \underbrace{D_{ab}\phi_b}_{\text{1st visc}} + \xi_a \qquad \langle \xi_a \xi_b \rangle = 2T\mathcal{D}_{ab}(\boldsymbol{k})\delta(t-t')$$

3. Break up the equations into eigen modes of $\mathcal{L}_{ab}$, and analyze exactly same way:

$$\overbrace{\lambda_+ = +ic_s k}^{\text{right moving sound}} \qquad \overbrace{\lambda_- = -ic_s k-}^{\text{left moving sound}} \qquad \overbrace{\lambda_T = 0}^{\text{two diffusion modes}}$$

So for k in the z direction, work with the following linear combos (eigenvects)

$$\phi_A \equiv \Big[\underbrace{c_s e(\boldsymbol{k}) \pm g^z(\boldsymbol{k})}_{\phi_+ \text{ and } \phi_-} \;,\; \underbrace{g^x(\boldsymbol{k})}_{\equiv \phi_{T_1}} \;,\; \underbrace{g^y(\boldsymbol{k})}_{\equiv \phi_{T_2}} \Big]$$

The kinetic equations in flat space

1. The relevant correlators are e.g.

$$N_{++}(\boldsymbol{k},t) = \left\langle \phi^*_{+}(\boldsymbol{k})\phi_{+}(\boldsymbol{k}) \right\rangle \qquad N_{T_1T_1} = \left\langle \phi^*_{T_1}(\boldsymbol{k})\phi_{T_1}(\boldsymbol{k}) \right\rangle$$

2. Thus

$$\frac{dN_{++}}{dt} = -\tfrac{4}{3}\gamma_\eta k^2 \left[N_{++} - N^{\rm eq}_{++}\right]$$
$$\frac{dN_{T_1T_1}}{dt} = -2\gamma_\eta k^2 \left[N_{T_1T_1} - N^{\rm eq}_{T_1T_1}\right]$$

and similar equations for N_{--} and $N_{T_2T_2}$. Here

$$N^{\rm eq}_{T_1T_1} \equiv (e+p)T \qquad \text{and} \qquad N^{\rm eq}_{++} \equiv (e+p)T$$

3. Neglect off diagonal components of density matrix in eigen-basis

$$\frac{dN_{+T_1}}{dt} = \underbrace{-ic_s k}_{\lambda_A - \lambda_B} \; N_{+T_1} \qquad \Leftarrow \text{Rapidly rotating.}$$

Now we will do the same for a perturbed and expanding system

Case 1: Kinetic equations for perturbed system (HHTLs)

1. Turn on a weak gravitational perturbations, $h_{ij} = h(t)\,\mathrm{diag}(1,1,-2)$

$$\partial_t N_{++}(k) = \underbrace{-\tfrac{4}{3}\gamma_\eta k^2 \left[N_{++} - N_{++}^{\rm eq}\right]}_{\text{damping}} + \underbrace{\partial_t h \left(\sin^2\theta_k - 2\cos^2\theta_k\right)}_{\text{perturbation} \quad h_{ij}\hat{k}^i\hat{k}^j} N_{++}$$

2. Solve the equations to first order in the gravitational, e.g.

$$\delta N_{++} = \frac{i\omega h \left(\sin^2\theta_k - 2\cos^2\theta_k\right)}{-i\omega + \frac{4}{3}\gamma_\eta K^2}$$

3. Calculate the stress tensor

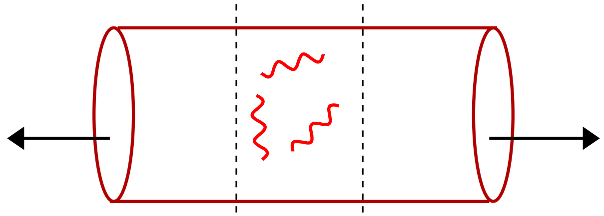
$$\delta T^{ij} = (e+p)\left\langle v^i v^j \right\rangle = \int \frac{d^3K}{(2\pi)^3} \frac{\left\langle g^i(\boldsymbol{k}) g^j(-\boldsymbol{k})\right\rangle}{e+p}$$

4. Find an HTL like expression

$$\langle \delta T^{xx} + \delta T^{yy} - 2\delta T^{zz}\rangle \supset h \int \frac{d^3K}{(2\pi)^3}\, \delta N_{++} \underbrace{\left(\sin^2\theta - 2\cos^2\theta\right)}_{\hat{k}^x\hat{k}^x + \hat{k}^y\hat{k}^y - 2\hat{k}^z\hat{k}^z}$$

Precisely reproduces hard hydro loop calculation!

Case 2: Kinetic equations for a Bjorken expansion – Hard Hydro Loops (HHLs)



- The hydrodynamic field fields $\phi_a = (c_s e, g^x, g^y, \tau g^\eta)$ are:

$$\phi_a(\tau, \boldsymbol{k}_\perp, \kappa) = \int d^2\boldsymbol{x} \int d\eta \, e^{i\boldsymbol{k}_\perp \cdot \boldsymbol{x}_\perp + i\kappa\,\eta} \, \phi_a(\tau, \boldsymbol{x}_\perp, \eta)$$

- The equations take the form:

$$\frac{d}{d\tau}\phi_a(\boldsymbol{k}_\perp, \kappa) = \underbrace{\mathcal{L}_{ab}}_{\text{ideal}} \, \phi_b + \underbrace{D_{ab}\phi_b}_{\text{viscous}} + \underbrace{\mathcal{P}_{ab}}_{\text{perturb}} + \underbrace{\xi_a}_{\text{noise}}$$

The previous analysis goes through with a no complications, $\lambda = \pm i c_s k, 0$

$$\mathcal{P}_{ab} = \frac{1}{\tau}\begin{pmatrix} 1 + c_s^2 & & & \\ & 1 & & \\ & & 1 & \\ & & & 2 \end{pmatrix}$$

The kinetic equations and approach to equilibrium:

- The kinetic equations and approach to equilibrium

$$\frac{\partial}{\partial\tau}N_{++} = -\frac{1}{\tau}\Big[\underbrace{2 + c_{s0}^2 + \frac{\kappa^2/\tau^2}{k_\perp^2+\kappa^2/\tau^2}}_{\text{perturbation } = 2\mathcal{P}_{++}}\Big]N_{++} \underbrace{- \frac{\frac{4}{3}\eta_0}{s_0T_0}\left(k_\perp^2+\frac{\kappa^2}{\tau^2}\right)\left[N_{++}-\frac{s_0T_0^2}{2c_{s0}^2\tau}\right]}_{\text{damping to equilibrium}},$$

$$\frac{\partial}{\partial\tau}N_{T_2T_2} = -\frac{2}{\tau}\Big[\underbrace{1 + \frac{k_\perp^2}{k_\perp^2+\kappa^2/\tau^2}}_{\text{perturbation } = 2\mathcal{P}_{T_2T_2}}\Big]N_{T_2T_2} \underbrace{- \frac{2\eta_0}{s_0T_0}\left(k_\perp^2+\frac{\kappa^2}{\tau^2}\right)\left[N_{T_2T_2}-\frac{s_0T_0^2}{\tau}\right]}_{\text{damping to equilibrium}}.$$

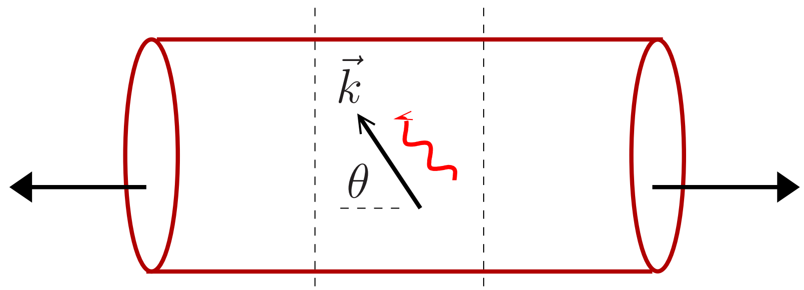
  and similar equations for the other modes

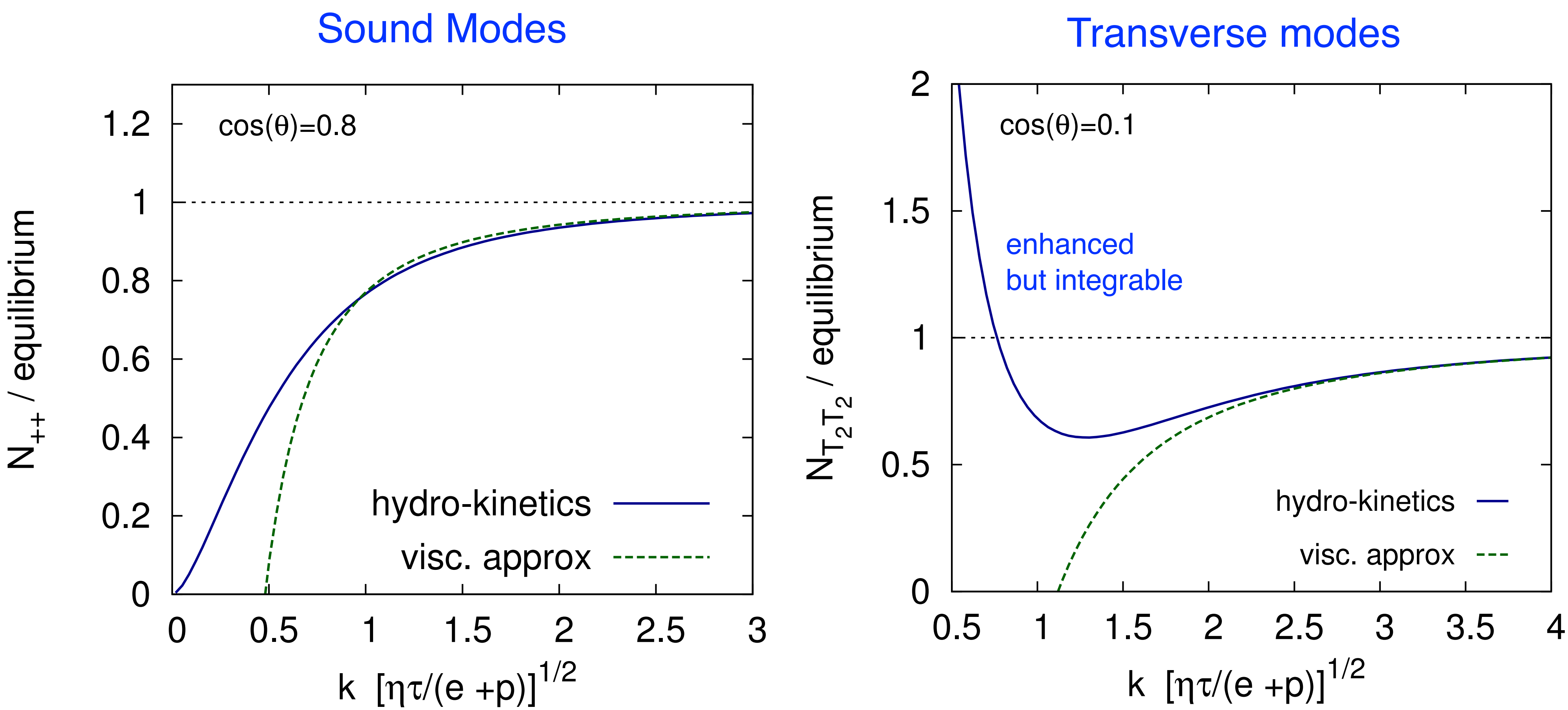
- For large k, we solve, and the modes approximately equilibrate:

$$N_{++} \simeq \frac{s_0T_0^2}{2c_{s0}^2\tau}\Big[\underbrace{1}_{\text{equilibrium}} + \underbrace{\frac{s_0T_0}{\frac{4}{3}\eta_0(k_\perp^2+\kappa^2/\tau^2)}\left(c_{s0}^2 - \frac{\kappa^2/\tau^2}{k_\perp^2+\kappa^2/\tau^2}\right)}_{\text{first viscous correction analogous to } \delta f} + \ldots\Big]$$

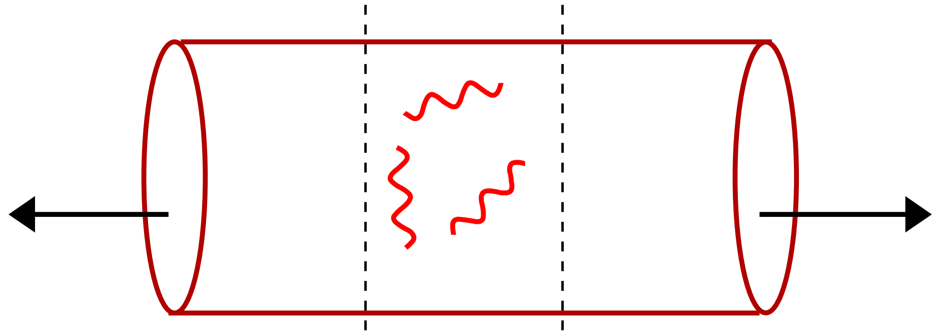
Now we solved these kinetic equations numerically

The non-equilibrium steady state at late times:





The evolution of the background



$$\frac{dT^{\tau\tau}}{d\tau} = -\frac{T^{\tau\tau} + T^{zz}}{\tau}$$

where

$$T^{zz}_{\rm hydro} = \underbrace{p_0(\Lambda)}_{\text{Ideal}} - \underbrace{\frac{\frac{4}{3}\eta_0(\Lambda)}{\tau}}_{\text{first order}} + \underbrace{(\lambda_1 - \eta\tau_\pi)\frac{8}{9\tau^2}}_{\text{second order}} + \ldots$$

In addition the fluctuations give another contribution:

$$T^{zz}_{\rm flucts} = (e+p)\,\langle v^z v^z \rangle$$

Evaluating the fluctuation contribution:

$$\frac{T^{zz}_{\rm flucts}}{e+p} = \langle v^z v^z \rangle$$
$$= \int \frac{d^2k_\perp d\kappa}{(2\pi)^3} \frac{1}{(e_o+p_o)^2} \left[N_{++}\cos^2\theta + N_{T_2T_2}\sin^2\theta \right]$$
$$= \frac{T_o\Lambda^3}{6\pi^2} - \left(\frac{17\Lambda}{120\pi^2} \frac{s_o T_o^2}{\eta_o(\Lambda)} \right) \frac{\frac{4}{3}}{\tau} + \text{finite}$$

Thus the full stress is then:

compare Kovtun, Moore, Romatschke

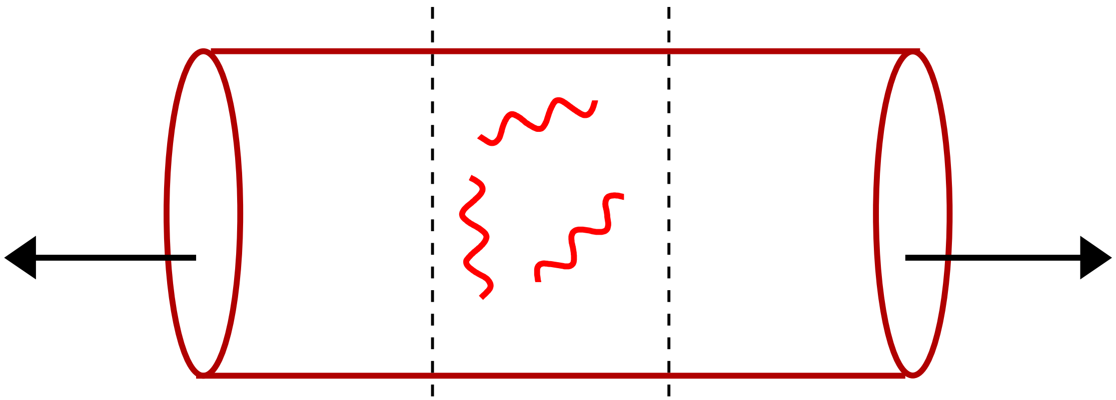
$$T^{zz} = T^{zz}_{\rm hydro} + T^{zz}_{\rm flucts}$$
$$= p_{\rm phys} - \frac{\frac{4}{3}\eta_{\rm phys}}{\tau} + \text{finite}$$

$$p_{\rm phys} \equiv p_0(\Lambda) + \frac{T_o\Lambda^3}{6\pi^2}$$
$$\eta_{\rm phys} \equiv \eta_0(\Lambda) + \left(\frac{17\Lambda}{120\pi^2} \frac{s_o T_o^2}{\eta_o(\Lambda)} \right)$$

where the physical quantities, $p_{\rm phys}$ and $\eta_{\rm phys}$, are independent of Λ

What's the finite correction?

Final result for a Bjorken expansion:



$$\frac{dT^{\tau\tau}}{d\tau} = -\frac{T^{\tau\tau} + T^{zz}}{\tau}$$

$$\frac{\langle T^{zz} \rangle}{e+p} = \Big[\underbrace{\frac{p}{e+p}}_{\sim 1} - \underbrace{\frac{4}{3}\frac{\gamma_\eta}{\tau}}_{\text{1st order}} + \underbrace{\frac{1.08318}{s\,(4\pi\gamma_\eta\tau)^{3/2}}}_{\text{3/2 order}} + \underbrace{\frac{(\lambda_1 - \eta\tau_\pi)}{e+p}\frac{8}{9\tau^2}}_{\text{2nd order}} \Big]$$

From which much can be wrought or wrung . . .

Numerical results:

Take representative numbers

$$\frac{(\lambda_1 - \eta\tau\pi)}{e+p} \simeq -0.8 \left(\frac{\eta}{e+p}\right)^2 \qquad \frac{T^3}{s} \simeq \frac{1}{13.5}$$

For $\eta/s = 1/4\pi$ find:

$$\frac{T^{zz}}{e+p} = \frac{1}{4}\left[1. - \underbrace{0.092}_{\text{first}}\left(\frac{4.5}{\tau T}\right) + \underbrace{0.034}_{\text{3/2 order} \simeq 30\%}\left(\frac{4.5}{\tau T}\right)^{3/2} - \underbrace{0.0009}_{\text{second}}\left(\frac{4.5}{\tau T}\right)^2\right]$$

while for $\eta/s = 2/4\pi$ we have:

$$\frac{T^{zz}}{e+p} = \frac{1}{4}\left[1. - \underbrace{0.185}_{\text{first}}\left(\frac{4.5}{\tau T}\right) + \underbrace{0.013}_{\text{3/2 order} \simeq 10\%}\left(\frac{4.5}{\tau T}\right)^{3/2} - \underbrace{0.0034}_{\text{second}}\left(\frac{4.5}{\tau T}\right)^2\right]$$

Fluctuation contribution is a correction to first order hydro

but larger than second order in practice

Summary

1. For wavenumbers of order

$$k \sim \sqrt{\frac{e+p}{\eta\tau}}$$

   the system transitions to equilibrium

2. Worked out an alternate description of hydro with noise:

   - Hydro + hydro-kinetics

$$\begin{aligned} \partial_\mu (T^{\mu\nu}_{\rm hydro} + T^{\mu\nu}_{\rm flucts}) =& 0 \\ \partial_\tau N_{\rm flucts}(\boldsymbol{k},\tau) =& \ldots \end{aligned}$$

   This should be generalized to a general flows.

3. How is the non-linear $T^{\mu\nu}_{\rm flucts}$ imprinted on the particles?

$$\delta f_{\rm flucts} = ????$$

4. Fluctuating hydro is much more important than second order hydro in practice!