

Anisotropic dissipative fluid dynamics

– theory and applications in heavy-ion physics

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thanks to: **Etele Molnár, Harri Niemi**

based on: **PRD 93 (2016) 11, 114025;**
arXiv:1606.09019 [nucl-th]

Microscopic foundations of ideal fluid dynamics

Ideal fluid dynamics: fluid is in **local thermodynamical equilibrium**

⇒ **single-particle distribution function:**

$$f_{0k} = [\exp(-\alpha_0 + \beta_0 E_{ku}) + a]^{-1}$$

where: $\beta_0 = 1/T$, T temperature, $\alpha_0 = \beta_0 \mu$, μ chemical potential,

$E_{ku} = k^\mu u_\mu$, with k^μ particle 4-momentum, $u^\mu = \gamma(1, \vec{v})$ fluid 4-velocity,

$a = \pm 1, 0$ for fermions/bosons, Boltzmann particles

Boltzmann equation:

$$k^\mu \partial_\mu f_k = C[f]$$

⇒ **0th and 1st moment of the Boltzmann equation:**

$$\partial_\mu N^\mu = \mathcal{C}$$

$$\partial_\mu T^{\mu\nu} = \mathcal{C}^\nu$$

where: $N^\mu \equiv \int_k k^\mu f_k$ particle no. 4-current,

$T^{\mu\nu} \equiv \int_k k^\mu k^\nu f_k$ energy-momentum tensor,

$\int_k \equiv g \int \frac{d^3k}{(2\pi)^3 k_0}$, g : internal quantum no. degeneracy of momentum state

Note: $\mathcal{C} \equiv \int_k C[f] = 0$ and $\mathcal{C}^\nu \equiv \int_k k^\nu C[f] \equiv 0$ for binary elastic collisions
(particle no. and 4-momenta are microscopic collisional invariants)

⇒ **macroscopic conservation of particle no., energy, and momentum!**

⇒ **set $f_k \equiv f_{0k}$ (Note: f_{0k} is **not** a solution of the Boltzmann equation!)**

⇒ **equations of motion closed – 5 eqs., 5 unknowns: $\alpha_0, \beta_0, u^\mu(3)$**

Microscopic foundations of dissipative fluid dynamics (I)

Consider **small deviations** from local thermodynamical equilibrium:

$$\mathbf{f}_k = \mathbf{f}_{0k} + \delta \mathbf{f}_k \quad |\delta \mathbf{f}_k| \ll 1$$

\Rightarrow **irreducible moments of $\delta \mathbf{f}_k$:**

$$\rho_r^{\mu_1 \dots \mu_\ell} \equiv \int_k E_{ku}^r k^{\langle \mu_1} \dots k^{\mu_\ell \rangle} \delta \mathbf{f}_k$$

where: $A^{\langle \mu_1 \dots \mu_\ell \rangle} \equiv \Delta_{\nu_1 \dots \nu_\ell}^{\mu_1 \dots \mu_\ell} A^{\nu_1 \dots \nu_\ell}$,

$\Delta_{\nu_1 \dots \nu_\ell}^{\mu_1 \dots \mu_\ell}$ projectors onto subspaces orthogonal to u^μ , formed from $\Delta^{\mu\nu}$, symmetric in μ_i, ν_j , traceless,

$\Delta^{\mu\nu} \equiv g^{\mu\nu} - u^\mu u^\nu$ 3-space projector onto direction orthogonal to u^μ

Note: $-\frac{m^2}{3} \rho_0 \equiv \Pi$ bulk viscous pressure, $\rho_0^\mu \equiv n^\mu$ particle diffusion current,

$\rho_0^{\mu\nu} \equiv \pi^{\mu\nu}$ shear-stress tensor, matching conditions in Landau frame: $\rho_1 = \rho_2 = \rho_1^\mu = 0$

\Rightarrow derive equations of motion for **irreducible moments:**

$$\dot{\rho}_r^{\langle \mu_1 \dots \mu_\ell \rangle} \equiv \Delta_{\nu_1 \dots \nu_\ell}^{\mu_1 \dots \mu_\ell} u^\alpha \partial_\alpha \int_k E_{ku}^r k^{\langle \nu_1} \dots k^{\nu_\ell \rangle} \delta \mathbf{f}_k$$

\Rightarrow use **Boltzmann equation:**

$$\delta \dot{\mathbf{f}}_k = -\dot{\mathbf{f}}_{0k} - \frac{1}{E_{ku}} \{ k^\mu \nabla_\mu (f_{0k} + \delta \mathbf{f}_k) - \mathbf{C}[f] \}$$

where: $\dot{A} \equiv u^\mu \partial_\mu A$, $\nabla^\mu \equiv \Delta^{\mu\nu} \partial_\nu$

\Rightarrow system of infinitely many coupled equations for **irreducible moments** $\rho_r^{\mu_1 \dots \mu_\ell}$, completely equivalent to Boltzmann equation \Rightarrow truncation required!

Microscopic foundations of dissipative fluid dynamics (II)

systematic power counting:

$$\text{Kn} \equiv \frac{\ell_{\text{mfp}}}{L_{\text{fluid}}} \sim \ell_{\text{mfp}} \partial_{\mu} \quad \text{Knudsen number}$$

$$\text{Re}^{-1} \equiv \frac{\Pi}{p} \sim \frac{n^{\mu}}{n} \sim \frac{\pi^{\mu\nu}}{p} \quad \text{inverse Reynolds number}$$

with pressure p , particle density n

⇒ for $\ell \geq 3$: $\rho_r^{\mu_1 \dots \mu_\ell} \sim O(\text{Kn}^2, \text{Kn Re}^{-1})$ ⇒ will be neglected

⇒ linearize collision integral: $\int_k E_{ku}^{r-1} k^{\langle \mu_1} \dots k^{\mu_\ell \rangle} C[f] = - \sum_{n=0}^{N_\ell} \mathcal{A}_{rn}^{(\ell)} \rho_n^{\mu_1 \dots \mu_\ell} + O(\delta f_k^2)$

⇒ linearized equations of motion for irreducible moments:

$$\begin{aligned} \dot{\vec{\rho}} + \mathcal{A}^{(0)} \vec{\rho} &= \vec{\alpha}^{(0)} \theta + O(\rho \times \text{Kn}) \\ \dot{\vec{\rho}}^{\langle \mu \rangle} + \mathcal{A}^{(1)} \vec{\rho}^{\mu} &= \vec{\alpha}^{(1)} \nabla^{\mu} \alpha + O(\rho \times \text{Kn}) \\ \dot{\vec{\rho}}^{\langle \mu\nu \rangle} + \mathcal{A}^{(2)} \vec{\rho}^{\mu\nu} &= 2 \vec{\alpha}^{(2)} \sigma^{\mu\nu} + O(\rho \times \text{Kn}) \end{aligned}$$

where $\theta \equiv \partial_{\mu} u^{\mu}$, $\sigma^{\mu\nu} \equiv \nabla^{\langle \mu} u^{\nu \rangle}$

⇒ diagonalize collision matrix: $(\Omega^{-1})^{(\ell)} \mathcal{A}^{(\ell)} \Omega^{(\ell)} = \text{diag}(\chi_0^{(\ell)}, \dots, \chi_i^{(\ell)}, \dots) \equiv \chi^{(\ell)}$

⇒ equations of motion for eigenmodes $\vec{X}^{\mu_1 \dots \mu_\ell} = (\Omega^{-1})^{(\ell)} \vec{\rho}^{\mu_1 \dots \mu_\ell}$ decouple:

$$\begin{aligned} \dot{\vec{X}} + \chi^{(0)} \vec{X} &= \vec{\beta}^{(0)} \theta + O(X \times \text{Kn}) \\ \dot{\vec{X}}^{\langle \mu \rangle} + \chi^{(1)} \vec{X}^{\mu} &= \vec{\beta}^{(1)} \nabla^{\mu} \alpha + O(X \times \text{Kn}) \\ \dot{\vec{X}}^{\langle \mu\nu \rangle} + \chi^{(2)} \vec{X}^{\mu\nu} &= \vec{\beta}^{(2)} \sigma^{\mu\nu} + O(X \times \text{Kn}) \end{aligned}$$

where $\vec{\beta}^{(\ell)} = (\Omega^{-1})^{(\ell)} \vec{\alpha}^{(\ell)}$

Microscopic foundations of dissipative fluid dynamics (III)

\implies slowest eigenmodes (w/o r.o.g. $X_0, X_0^\mu, X_0^{\mu\nu}$) remain dynamical, faster ones ($i \neq 0$) are replaced by their asymptotic values:

$$X_i \simeq \frac{\beta_i^{(0)}}{\chi_i^{(0)}} \theta, \quad X_i^\mu \simeq \frac{\beta_i^{(1)}}{\chi_i^{(1)}} \nabla^\mu \alpha, \quad X_i^{\mu\nu} \simeq \frac{\beta_i^{(2)}}{\chi_i^{(2)}} \sigma^{\mu\nu}$$

Note: systematic improvement possible by making faster eigenmodes dynamical

G.S. Denicol, H. Niemi, I. Bouras, E. Molnar, Z. Xu, DHR, C. Greiner, PRD 89 (2014) 7, 074005

\implies since $\vec{\rho}^{\mu_1 \dots \mu_\ell} = \Omega^{(\ell)} \vec{X}^{\mu_1 \dots \mu_\ell}$:

$$\rho_i \simeq \Omega_{i0}^{(0)} X_0 + \sum_{j=3}^{N_0} \Omega_{ij}^{(0)} \frac{\beta_j^{(0)}}{\chi_j^{(0)}} \theta$$

$$\rho_i^\mu \simeq \Omega_{i0}^{(1)} X_0^\mu + \sum_{j=2}^{N_1} \Omega_{ij}^{(1)} \frac{\beta_j^{(1)}}{\chi_j^{(1)}} \nabla^\mu \alpha$$

$$\rho_i^{\mu\nu} \simeq \Omega_{i0}^{(2)} X_0^{\mu\nu} + \sum_{j=1}^{N_2} \Omega_{ij}^{(2)} \frac{\beta_j^{(2)}}{\chi_j^{(2)}} \sigma^{\mu\nu}$$

\implies for $i = 0$: express $X_0, X_0^\mu, X_0^{\mu\nu}$ in terms of $\Pi, n^\mu, \pi^{\mu\nu}$ as well as $\theta, \nabla^\mu \alpha, \sigma^{\mu\nu}$

\implies reinsert back, express $\rho_i, \rho_i^\mu, \rho_i^{\mu\nu}$ in terms of $\Pi, n^\mu, \pi^{\mu\nu}$ as well as $\theta, \nabla^\mu \alpha, \sigma^{\mu\nu}$:

$$\frac{m^2}{3} \rho_i \simeq -\Omega_{i0}^{(0)} \Pi + \left(\zeta_i - \Omega_{i0}^{(0)} \zeta_0 \right) \theta$$

$$\rho_i^\mu \simeq \Omega_{i0}^{(1)} n^\mu + \left(\kappa_i - \Omega_{i0}^{(1)} \kappa_0 \right) \nabla^\mu \alpha$$

$$\rho_i^{\mu\nu} \simeq \Omega_{i0}^{(2)} \pi^{\mu\nu} + 2 \left(\eta_i - \Omega_{i0}^{(2)} \eta_0 \right) \sigma^{\mu\nu}$$

where $\zeta_i = \frac{m^2}{3} \sum_{r=0, \neq 1, 2}^{N_0} \tau_{ir}^{(0)} \alpha_r^{(0)}$, $\kappa_i = \sum_{r=0, \neq 1}^{N_1} \tau_{ir}^{(1)} \alpha_r^{(1)}$, $\eta_i = \sum_{r=0}^{N_2} \tau_{ir}^{(2)} \alpha_r^{(2)}$, $\tau^{(\ell)} = \Omega^{(\ell)} (\chi^{-1})^{(\ell)} (\Omega^{-1})^{(\ell)}$

Microscopic foundations of dissipative fluid dynamics (IV)

⇒ equations of motion for Π , n^μ , $\pi^{\mu\nu}$:

$$\begin{aligned} \tau_\Pi \dot{\Pi} + \Pi &= -\zeta_0 \theta + \mathcal{K} + \mathcal{J} + \mathcal{R} \\ \tau_n \dot{n}^{\langle\mu} + n^\mu &= \kappa_0 \nabla^\mu \alpha + \mathcal{K}^\mu + \mathcal{J}^\mu + \mathcal{R}^\mu \\ \tau_\pi \dot{\pi}^{\langle\mu\nu} + \pi^{\mu\nu} &= 2\eta_0 \sigma^{\mu\nu} + \mathcal{K}^{\mu\nu} + \mathcal{J}^{\mu\nu} + \mathcal{R}^{\mu\nu} \end{aligned}$$

Kn²: $\mathcal{K} = \bar{\zeta}_1 \omega_{\mu\nu} \omega^{\mu\nu} + \bar{\zeta}_2 \sigma^{\mu\nu} \sigma_{\mu\nu} + \bar{\zeta}_3 \theta^2 + \bar{\zeta}_4 (\nabla\alpha)^2 + \bar{\zeta}_5 (\nabla p)^2 + \bar{\zeta}_6 \nabla_\mu \alpha \nabla^\mu p + \bar{\zeta}_7 \nabla^2 \alpha + \bar{\zeta}_8 \nabla^2 p$,

$\mathcal{K}^\mu = \bar{\kappa}_1 \sigma^{\mu\nu} \nabla_\nu \alpha + \bar{\kappa}_2 \sigma^{\mu\nu} \nabla_\nu p + \bar{\kappa}_3 \theta \nabla^\mu \alpha + \bar{\kappa}_4 \theta \nabla^\mu p + \bar{\kappa}_5 \omega^{\mu\nu} \nabla_\nu \alpha + \bar{\kappa}_6 \Delta^{\mu\lambda} \partial^\nu \sigma_{\lambda\nu} + \bar{\kappa}_7 \nabla^\mu \theta$,

$\mathcal{K}^{\mu\nu} = \bar{\eta}_1 \omega_\lambda^{\langle\mu} \omega^{\nu\rangle\lambda} + \bar{\eta}_2 \theta \sigma^{\mu\nu} + \bar{\eta}_3 \sigma_\lambda^{\langle\mu} \sigma^{\nu\rangle\lambda} + \bar{\eta}_4 \sigma_\lambda^{\langle\mu} \omega^{\nu\rangle\lambda} + \bar{\eta}_5 \nabla^{\langle\mu} \alpha \nabla^{\nu\rangle} \alpha$
 $+ \bar{\eta}_6 \nabla^{\langle\mu} p \nabla^{\nu\rangle} p + \bar{\eta}_7 \nabla^{\langle\mu} \alpha \nabla^{\nu\rangle} p + \bar{\eta}_8 \nabla^{\langle\mu} \nabla^{\nu\rangle} \alpha + \bar{\eta}_9 \nabla^{\langle\mu} \nabla^{\nu\rangle} p$

Re⁻¹Kn: $\mathcal{J} = -\ell_{\Pi n} \nabla_\mu n^\mu - \tau_{\Pi n} n^\mu \nabla_\mu p - \delta_{\Pi\Pi} \theta \Pi - \lambda_{\Pi n} n^\mu \nabla_\mu \alpha + \lambda_{\Pi\pi} \pi^{\mu\nu} \sigma_{\mu\nu}$
 $\mathcal{J}^\mu = \tau_n \omega^{\mu\nu} n_\nu - \delta_{nn} \theta n^\mu - \ell_{n\Pi} \nabla^\mu \Pi + \ell_{n\pi} \Delta^{\mu\nu} \nabla^\lambda \pi_{\nu\lambda} + \tau_{n\Pi} \Pi \nabla^\mu p - \tau_{n\pi} \pi^{\mu\nu} \nabla_\nu p - \lambda_{nn} \sigma^{\mu\nu} n_\nu$
 $+ \lambda_{n\Pi} \Pi \nabla^\mu \alpha - \lambda_{n\pi} \pi^{\mu\nu} \nabla_\nu \alpha$
 $\mathcal{J}^{\mu\nu} = 2\tau_\pi \pi_\lambda^{\langle\mu} \omega^{\nu\rangle\lambda} - \delta_{\pi\pi} \theta \pi^{\mu\nu} - \tau_{\pi\pi} \pi_\lambda^{\langle\mu} \sigma^{\nu\rangle\lambda} + \lambda_{\pi\Pi} \Pi \sigma^{\mu\nu} - \tau_{\pi n} n^{\langle\mu} \nabla^{\nu\rangle} p + \ell_{\pi n} \nabla^{\langle\mu} n^{\nu\rangle}$
 $+ \lambda_{\pi n} n^{\langle\mu} \nabla^{\nu\rangle} \alpha$ where $\omega^{\mu\nu} \equiv (\nabla^\mu u^\nu - \nabla^\nu u^\mu) / 2$

Re⁻²: $\mathcal{R} = \varphi_1 \Pi^2 + \varphi_2 n_\mu n^\mu + \varphi_3 \pi^{\mu\nu} \pi_{\mu\nu}$ G.S. Denicol, H. Niemi, E. Molnar, DHR,
 $\mathcal{R}^\mu = \varphi_4 \pi^{\mu\nu} n_\nu + \varphi_5 \Pi n^\mu$ PRD 85 (2012) 114047,
 $\mathcal{R}^{\mu\nu} = \varphi_6 \Pi \pi^{\mu\nu} + \varphi_7 \pi_\lambda^{\langle\mu} \pi^{\nu\rangle\lambda} + \varphi_8 n^{\langle\mu} n^{\nu\rangle}$ Erratum PRD 91 (2015) 3, 039902

Microscopic foundations of dissipative fluid dynamics (V)

Single-particle distribution function:

$$f_{\mathbf{k}} = f_{0\mathbf{k}} \left[1 + (1 - a f_{0\mathbf{k}}) \sum_{\ell=0}^{\infty} \sum_{n=0}^{N_{\ell}} \mathcal{H}_{\mathbf{k}n}^{(\ell)} \rho_n^{\mu_1 \dots \mu_{\ell}} k_{\langle \mu_1} \dots k_{\mu_{\ell} \rangle} \right]$$

where $\mathcal{H}_{\mathbf{k}n}^{(\ell)} = \frac{W^{(\ell)}}{\ell!} \sum_{m=n}^{N_{\ell}} a_{mn}^{(\ell)} P_{\mathbf{k}m}^{(\ell)}$, with $P_{\mathbf{k}n}^{(\ell)} = \sum_{r=0}^n a_{nr}^{(\ell)} E_{\mathbf{k}u}^r$ polynomials of order n in $E_{\mathbf{k}u}$,

with coefficients $a_{nr}^{(\ell)}$ determined such that $\frac{W^{(\ell)}}{(2\ell+1)!!} \int_{\mathbf{k}} (\Delta^{\alpha\beta} k_{\alpha} k_{\beta})^{\ell} P_{\mathbf{k}n}^{(\ell)} P_{\mathbf{k}m}^{(\ell)} f_{0\mathbf{k}} (1 - a f_{0\mathbf{k}}) = \delta_{mn}$

\Rightarrow explicitly for $\ell \leq 2$:

$$\begin{aligned} \delta f_{\mathbf{k}} = f_{0\mathbf{k}} (1 - a f_{0\mathbf{k}}) & \left(-\frac{3}{m^2} \left\{ \mathcal{H}_{\mathbf{k}0}^{(0)} \Pi + \sum_{n=3}^{N_0} \mathcal{H}_{\mathbf{k}n}^{(0)} \left[-\Omega_{n0}^{(0)} \Pi + (\zeta_n - \Omega_{n0}^{(0)} \zeta_0) \theta \right] \right\} \right. \\ & + \mathcal{H}_{\mathbf{k}0}^{(1)} n^{\mu} k_{\mu} + \sum_{n=2}^{N_1} \mathcal{H}_{\mathbf{k}n}^{(1)} \left[\Omega_{n0}^{(1)} n^{\mu} + (\kappa_n - \Omega_{n0}^{(1)} \kappa_0) \nabla^{\mu} \alpha \right] k_{\mu} \\ & \left. + \mathcal{H}_{\mathbf{k}0}^{(2)} \pi^{\mu\nu} k_{\mu} k_{\nu} + \sum_{n=1}^{N_2} \mathcal{H}_{\mathbf{k}n}^{(2)} \left[\Omega_{n0}^{(2)} \pi^{\mu\nu} + 2 (\eta_n - \Omega_{n0}^{(2)} \eta_0) \sigma^{\mu\nu} \right] k_{\mu} k_{\nu} \right) \\ \mathcal{H}_{\mathbf{k}0}^{(2)} & = \frac{1}{2 J_{42}} \left(1 + \sum_{m=1}^{N_2} \sum_{r=0}^m a_{m0}^{(2)} a_{mr}^{(2)} E_{\mathbf{k}u}^r \right) \end{aligned}$$

usually: $\delta f_{\mathbf{k}} = f_{0\mathbf{k}} (1 - a f_{0\mathbf{k}}) \frac{1}{2T^2(\epsilon+p)} \pi^{\mu\nu} k_{\mu} k_{\nu}$ with energy density ϵ

Anisotropic fluid dynamics

Initial gradients in heavy-ion collisions are large

⇒ deviations from local thermodynamical equilibrium are large!

⇒ may invalidate dissipative fluid dynamics

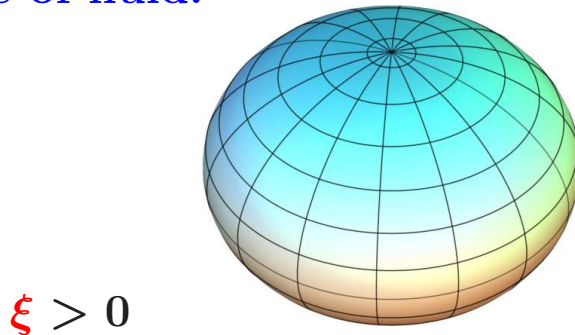
Idea: “resum” dissipative corrections into single-particle distribution function,
 e.g.: W. Florkowski, PLB 668 (2008) 32; M. Martinez, M. Strickland, PRC 81 (2010) 024906

$$\hat{f}_{0k} = \left[\exp \left(-\hat{\alpha} + \hat{\beta}_u \sqrt{E_{ku}^2 + \xi E_{kl}^2} \right) + a \right]^{-1}$$

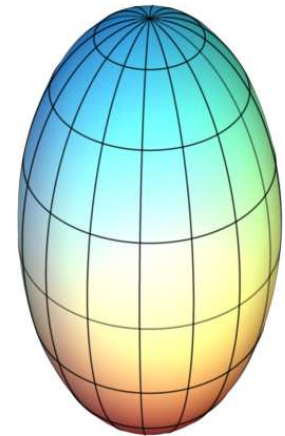
where $E_{kl} \equiv -l^\mu k_\mu$, with l^μ direction of anisotropy, $l^\mu l_\mu = -1$, $l^\mu u_\mu = 0$,
 usually: $l^\mu = \gamma_z(v_z, 0, 0, 1)$, $\gamma_z = (1 - v_z^2)^{-1/2}$,

ξ anisotropy parameter

⇒ in LR frame of fluid:



$\xi > 0$



$\xi < 0$

⇒ 5 conservation equations determine $\hat{\alpha}$, $\hat{\beta}_u$, u^μ (3)

⇒ need additional equation to determine ξ !

Microscopic foundations of anisotropic dissipative fluid dynamics (I)

$$f_{\mathbf{k}} = f_{0\mathbf{k}} + \delta f_{\mathbf{k}} \equiv \hat{f}_{0\mathbf{k}} + \delta \hat{f}_{\mathbf{k}}$$

If $|\delta f_{\mathbf{k}}| \sim 1$, choose $\hat{f}_{0\mathbf{k}}$ such that $|\delta \hat{f}_{\mathbf{k}}| \ll 1$

⇒ improved convergence properties of expansion around $\hat{f}_{0\mathbf{k}}$!

D. Bazow, U.W. Heinz, M. Strickland, PRC 90 (2014) 5, 054910

E. Molnár, H. Niemi, DHR, PRD 93 (2016) 11, 114025

⇒ irreducible moments of $\delta \hat{f}_{\mathbf{k}}$:

$$\hat{\rho}_{rs}^{\mu_1 \dots \mu_\ell} \equiv \int_{\mathbf{k}} E_{\mathbf{k}u}^r E_{\mathbf{k}l}^s k^{\{\mu_1 \dots \mu_\ell\}} \delta \hat{f}_{\mathbf{k}}$$

where: $A^{\{\mu_1 \dots \mu_\ell\}} \equiv \Xi_{\nu_1 \dots \nu_\ell}^{\mu_1 \dots \mu_\ell} A^{\nu_1 \dots \nu_\ell}$,

$\Xi_{\nu_1 \dots \nu_\ell}^{\mu_1 \dots \mu_\ell}$ projectors onto subspaces orthogonal to both u^μ and l^μ , formed from $\Xi^{\mu\nu}$, symmetric in μ_i, ν_j , traceless,

$\Xi^{\mu\nu} \equiv g^{\mu\nu} - u^\mu u^\nu + l^\mu l^\nu$ 2-space projector onto direction orthogonal to both u^μ and l^μ

⇒ derive equations of motion for irreducible moments:

$$\dot{\hat{\rho}}_{rs}^{\{\mu_1 \dots \mu_\ell\}} \equiv \Xi_{\nu_1 \dots \nu_\ell}^{\mu_1 \dots \mu_\ell} u^\alpha \partial_\alpha \int_{\mathbf{k}} E_{\mathbf{k}u}^r E_{\mathbf{k}l}^s k^{\{\nu_1 \dots \nu_\ell\}} \delta \hat{f}_{\mathbf{k}}$$

⇒ use Boltzmann equation:

$$\delta \dot{\hat{f}}_{\mathbf{k}} = -\dot{\hat{f}}_{0\mathbf{k}} - \frac{1}{E_{\mathbf{k}u}} \left\{ -E_{\mathbf{k}l} D_l (\hat{f}_{0\mathbf{k}} + \delta \hat{f}_{\mathbf{k}}) + k^\mu \tilde{\nabla}_\mu (\hat{f}_{0\mathbf{k}} + \delta \hat{f}_{\mathbf{k}}) - C[f] \right\}$$

where: $D_l \equiv -l^\mu \partial_\mu$, $\tilde{\nabla}^\mu \equiv \Xi^{\mu\nu} \partial_\nu$

Microscopic foundations of anisotropic dissipative fluid dynamics (II)

Truncation: so far, no eigenmode analysis, only 14-moment approximation

Define
$$\hat{I}_{nrq}(\hat{\alpha}, \hat{\beta}_u, \xi) \equiv \frac{1}{(2q)!!} \int_k E_{ku}^n E_{kl}^r (-\Xi^{\alpha\beta} k_\alpha k_\beta)^q \hat{f}_{0k}$$

⇒ the 14 moments are:

particle density

$$n \equiv \hat{n} = \hat{I}_{100} \iff \hat{\rho}_{10} = 0 \quad (1^{\text{st}} \text{ Landau matching cond.})$$

particle diffusion in l^μ -direction

$$n_l \equiv \hat{n}_l + \hat{\rho}_{01} = \hat{I}_{110} + \hat{\rho}_{01}$$

energy density

$$e \equiv \hat{e} = \hat{I}_{200} \iff \hat{\rho}_{20} = 0 \quad (2^{\text{nd}} \text{ Landau matching cond.})$$

heat flow in l^μ -direction

$$M \equiv \hat{M} + \hat{\rho}_{11} = \hat{I}_{210} + \hat{\rho}_{11}$$

pressure in l^μ -direction

$$P_l \equiv \hat{P}_l = \hat{I}_{220} \iff \hat{\rho}_{02} = 0 \quad (3^{\text{rd}} \text{ Landau matching cond.})$$

transverse pressure

$$P_\perp \equiv \hat{P}_\perp + \frac{3}{2}\Pi = \hat{I}_{201} - \frac{m_0^2}{2}\hat{\rho}_{00}$$

particle diffusion in transverse direction

$$V_\perp^\mu \equiv \hat{\rho}_{00}^\mu$$

heat flow in transverse direction

$$W_{\perp u}^\mu \equiv \hat{\rho}_{10}^\mu$$

shear-stress current in l^μ -direction

$$W_{\perp l}^\mu \equiv \hat{\rho}_{01}^\mu$$

shear-stress tensor in transverse direction $\pi_\perp^{\mu\nu} \equiv \hat{\rho}_{00}^{\mu\nu}$

⇒ Landau frame: $M = W_{\perp u}^\mu = 0 \iff \hat{\rho}_{11} = -\hat{M}, \hat{\rho}_{10}^\mu = 0$

⇒ eliminate all other moments by linear relation:

$$\hat{\rho}_{ij}^{\mu_1 \dots \mu_\ell} = (-1)^\ell \ell! \sum_{n=0}^{N_\ell} \sum_{m=0}^{N_\ell - n} \hat{\rho}_{nm}^{\mu_1 \dots \mu_\ell} \gamma_{injm}^{(\ell)} \quad \text{where } \gamma_{injm}^{(\ell)} \text{ function of } \hat{\alpha}, \hat{\beta}_u, \xi$$

Note: for $\hat{f}_{0k}(\xi) : \hat{n}_l = \hat{M} \equiv 0!$

Microscopic foundations of anisotropic dissipative fluid dynamics (III)

⇒ 5 conservation equations:

$$\begin{aligned}
0 &= \dot{\hat{n}} + \hat{n} (l_\mu D_l u^\mu + \tilde{\theta}) - D_l n_l + n_l (\tilde{\theta}_l - l_\mu \dot{u}^\mu) - V_\perp^\mu (\dot{u}_\mu + D_l l_\mu) + \tilde{\nabla}_\mu V_\perp^\mu \\
0 &= \dot{\hat{e}} + (\hat{e} + \hat{P}_l) l_\mu D_l u^\mu + \left(\hat{e} + \hat{P}_\perp + \frac{3}{2} \Pi \right) \tilde{\theta} + W_{\perp l}^\mu (D_l u_\mu - l_\nu \tilde{\nabla}_\mu u^\nu) - \pi_\perp^{\mu\nu} \tilde{\sigma}_{\mu\nu} \\
0 &= (\hat{e} + \hat{P}_l) l_\mu \dot{u}^\mu + D_l \hat{P}_l + \left(\hat{P}_\perp - \hat{P}_l + \frac{3}{2} \Pi \right) \tilde{\theta}_l + W_{\perp l}^\mu (\dot{u}_\mu + 2 D_l l_\mu + l_\nu \tilde{\nabla}_\mu u^\nu) - \tilde{\nabla}_\mu W_{\perp l}^\mu - \pi_\perp^{\mu\nu} \tilde{\sigma}_{l,\mu\nu} \\
0 &= \left(\hat{e} + \hat{P}_\perp + \frac{3}{2} \Pi \right) \Xi_\nu^\alpha \dot{u}^\nu - \tilde{\nabla}^\alpha \left(\hat{P}_\perp + \frac{3}{2} \Pi \right) + \left(\hat{P}_\perp - \hat{P}_l + \frac{3}{2} \Pi \right) \Xi_\nu^\alpha D_l l^\nu - \Xi_\nu^\alpha D_l W_{\perp l}^\nu + W_{\perp l}^\alpha \left(\frac{3}{2} \tilde{\theta}_l - l_\mu \dot{u}^\mu \right) \\
&+ W_{\perp l,\nu} (\tilde{\sigma}_l^{\alpha\nu} - \tilde{\omega}_l^{\alpha\nu}) - \pi_\perp^{\mu\alpha} (\dot{u}_\mu + D_l l_\mu) + \Xi_\nu^\alpha \tilde{\nabla}_\mu \pi_\perp^{\mu\nu}
\end{aligned}$$

where $\tilde{\theta} \equiv \tilde{\nabla}_\mu u^\mu$, $\tilde{\theta}_l \equiv \tilde{\nabla}_\mu l^\mu$, $\tilde{\sigma}^{\mu\nu} \equiv \partial^{\{\mu} u^{\nu\}}$, $\tilde{\sigma}_l^{\mu\nu} \equiv \partial^{\{\mu} l^{\nu\}}$, $\tilde{\omega}_l^{\mu\nu} \equiv \frac{1}{2} \Xi^{\mu\alpha} \Xi^{\nu\beta} (\partial_\alpha l_\beta - \partial_\beta l_\alpha)$

+ 9 relaxation equations for Π , n_l , \hat{P}_l , V_\perp^μ , $W_{\perp l}^\mu$, $\tilde{\pi}^{\mu\nu}$

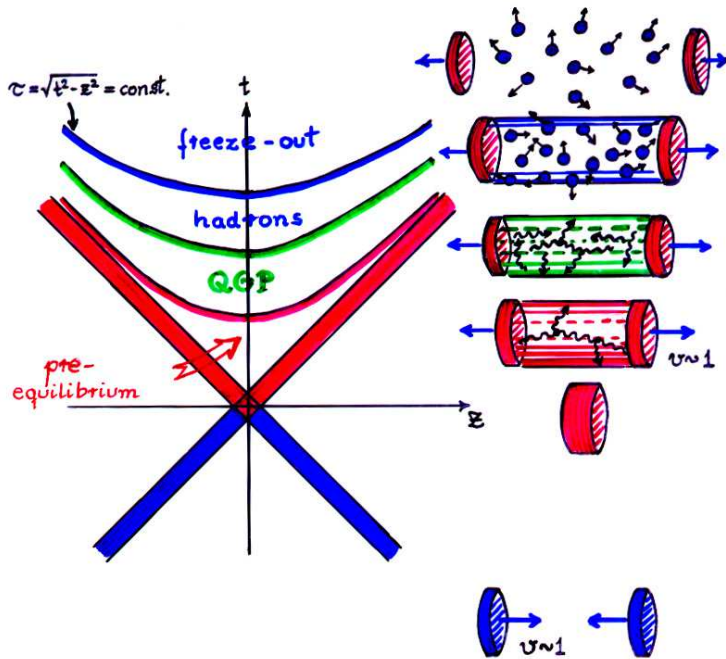
for details, see E. Molnár, H. Niemi, DHR, PRD 93 (2016) 11, 114025

Application to heavy-ion collisions (I)

Bjorken flow:

J.D. Bjorken, PRD 27 (1983) 140

The space-time picture:



“Pure” anisotropic fluid dynamics

$$(\delta \hat{f}_k \equiv 0 \iff \text{all } \rho_{rs}^{\mu_1 \dots \mu_\ell} \equiv 0)$$

\implies eqs. of motion for irreducible moments become eqs. of motion for moments \hat{I}_{nrq} :

$$\partial_\tau \hat{I}_{i+j,j,0} + \frac{(j+1)\hat{I}_{i+j,j,0} + (i-1)\hat{I}_{i+j,j+2,0}}{\tau} = \hat{C}_{i-1,j}$$

\implies conservation equations:

$$i = 1, j = 0 : \partial_\tau \hat{n} + \frac{\hat{n}}{\tau} = 0$$

$$i = 2, j = 0 : \partial_\tau \hat{\epsilon} + \frac{\hat{\epsilon} + \hat{P}_l}{\tau} = 0$$

\implies 2 eqs., 3 unknowns: $\hat{\alpha}, \hat{\beta}_u, \xi$

\implies need add. eq. to close eqs. of motion!

\implies in principle, eq. of motion for **any** moment $\hat{I}_{i+j,j,0}$ suffices

\implies but which one is the **best choice?**

E. Molnár, H. Niemi, DHR, arXiv:1606.09019 [nucl-th]

Application to heavy-ion collisions (II)

assume **relaxation-time approximation** for collision term: $\hat{\mathcal{C}}_{i-1,j} \equiv -\frac{\hat{I}_{i+j,j,0} - I_{i+j,j,0}}{\tau_{\text{eq}}}$
 where $I_{i+j,j,0} = \lim_{\xi \rightarrow 0} \hat{I}_{i+j,j,0}$

⇒ study the following choices:

$$(1) \quad i = 0, j = 2: \quad \partial_\tau \hat{P}_l + \frac{3\hat{P}_l - \hat{I}_{240}}{\tau} = -\frac{\hat{P}_l - I_{220}}{\tau_{\text{eq}}}$$

$$(2) \quad i = 3, j = 0: \quad \partial_\tau \hat{I}_{300} + \frac{\hat{I}_{300} - 2\hat{I}_{320}}{\tau} = -\frac{\hat{I}_{300} - I_{300}}{\tau_{\text{eq}}}$$

$$(3) \quad i = 1, j = 2: \quad \partial_\tau \hat{I}_{320} + \frac{3\hat{I}_{320}}{\tau} = -\frac{\hat{I}_{320} - I_{320}}{\tau_{\text{eq}}}$$

$$(4) \quad i = 0, j = 0: \quad \partial_\tau \hat{I}_{000} + \frac{\hat{I}_{000} - \hat{I}_{020}}{\tau} = -\frac{\hat{I}_{000} - I_{000}}{\tau_{\text{eq}}}$$

$$(5) \quad i = 0, j = 4: \quad \partial_\tau \hat{I}_{440} + \frac{5\hat{I}_{440} - \hat{I}_{460}}{\tau} = -\frac{\hat{I}_{440} - I_{440}}{\tau_{\text{eq}}}$$

$$(6) \quad i = 1, j = 4: \quad \partial_\tau \hat{I}_{540} + \frac{5\hat{I}_{540}}{\tau} = -\frac{\hat{I}_{540} - I_{540}}{\tau_{\text{eq}}}$$

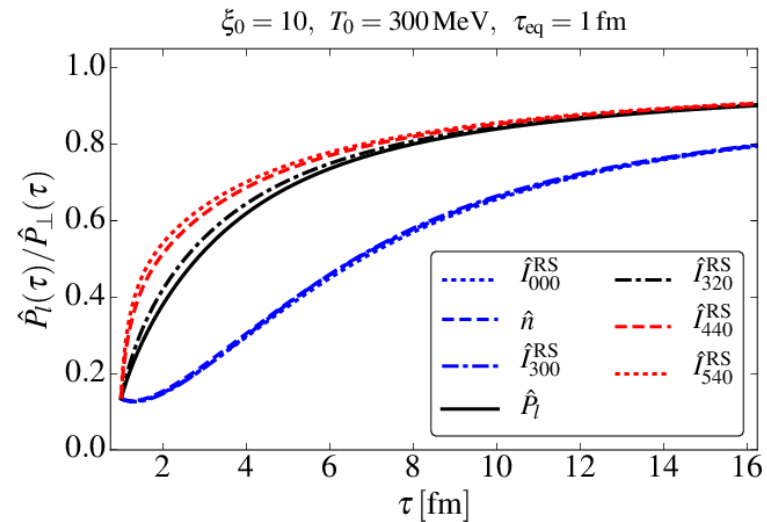
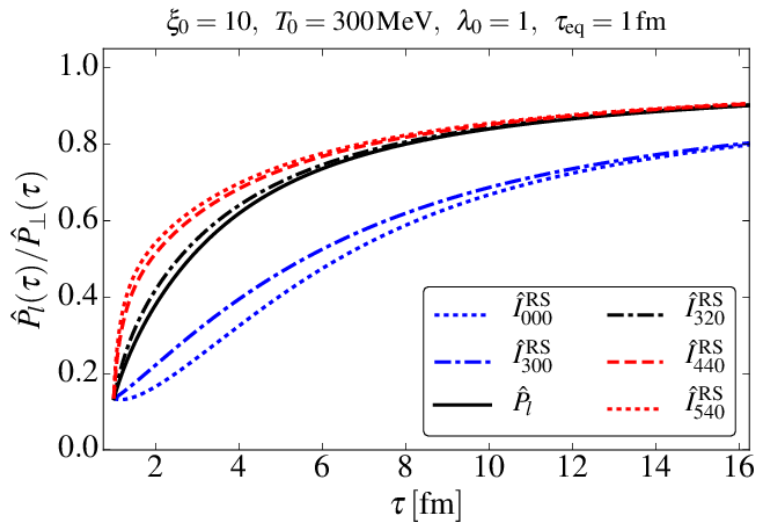
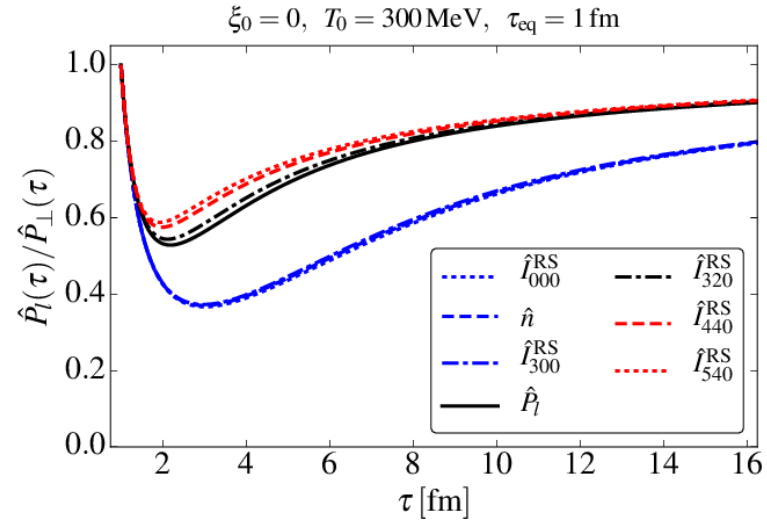
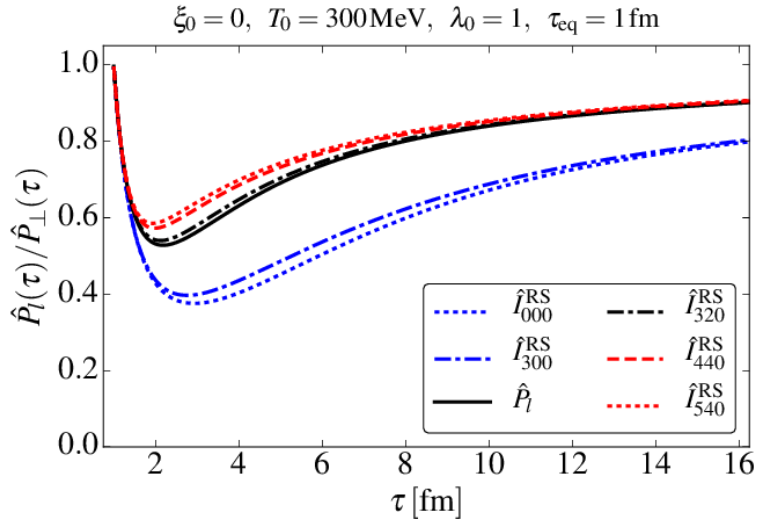
$$(7) \quad \text{in case particle no. is not conserved: } i = 1, j = 0: \quad \partial_\tau \hat{n} + \frac{\hat{n}}{\tau} = -\frac{\hat{n} - I_{100}}{\tau_{\text{eq}}}$$

Note: different moments probe \hat{f}_{0k} in different regions of momentum space!

Application to heavy-ion collisions (III)

particle no. conservation:

no particle no. conservation:

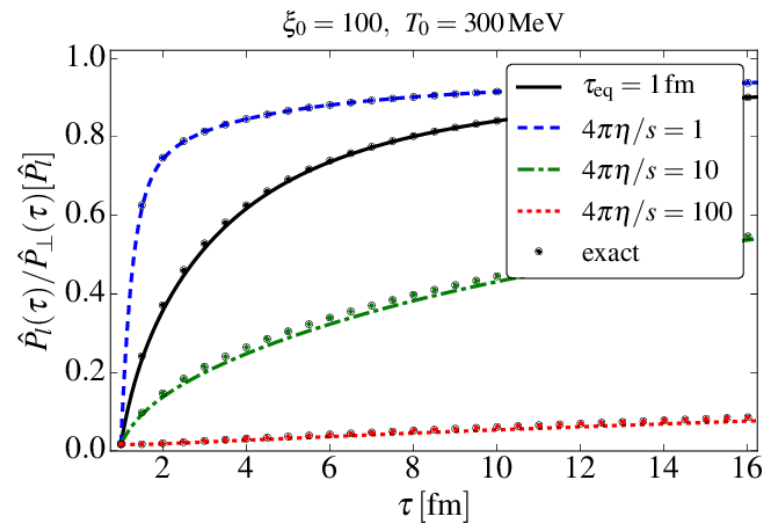
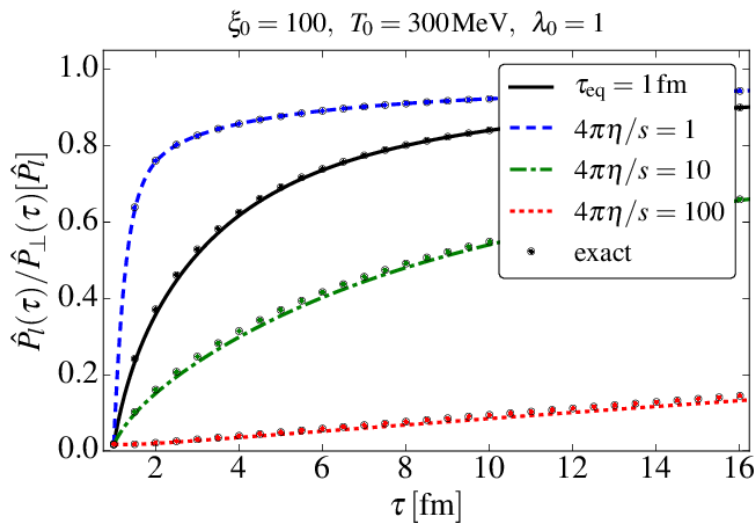
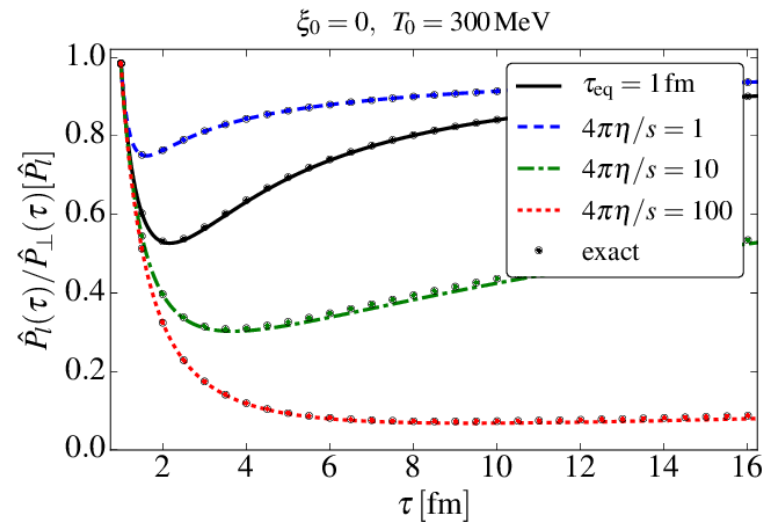
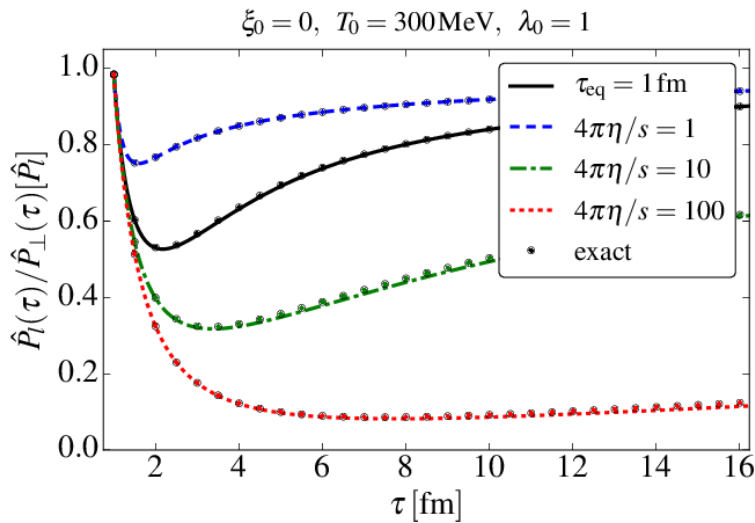


⇒ all cases (1) – (7) give different results! ⇒ which one is the best?

Application to heavy-ion collisions (IV)

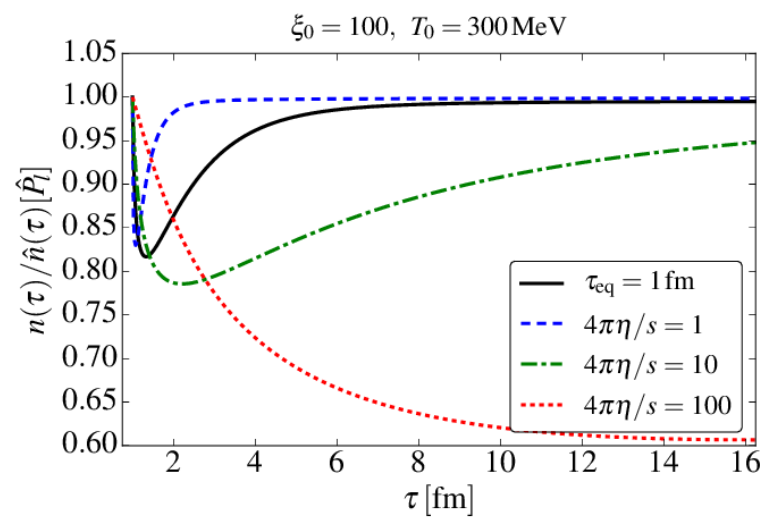
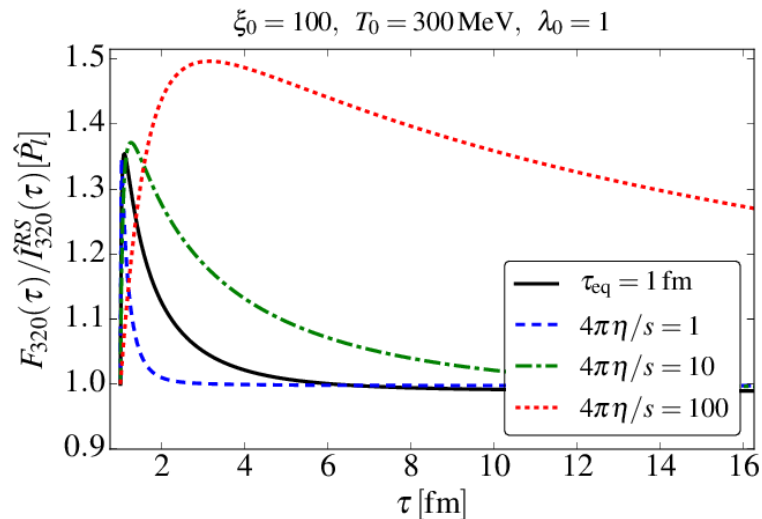
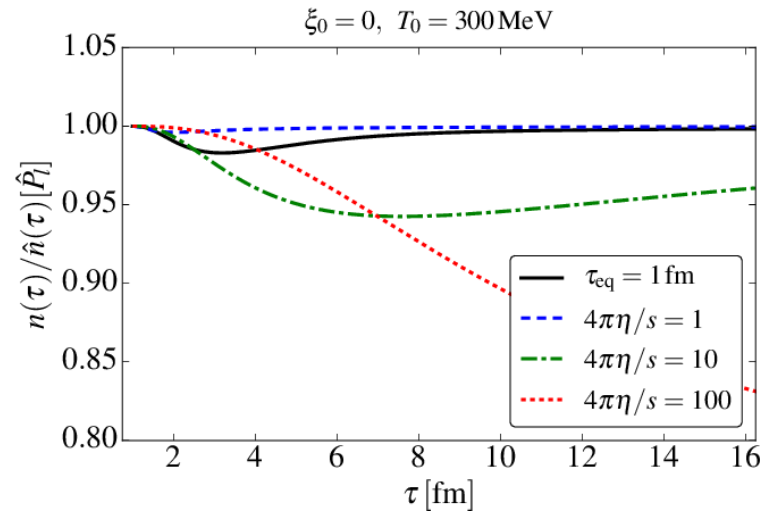
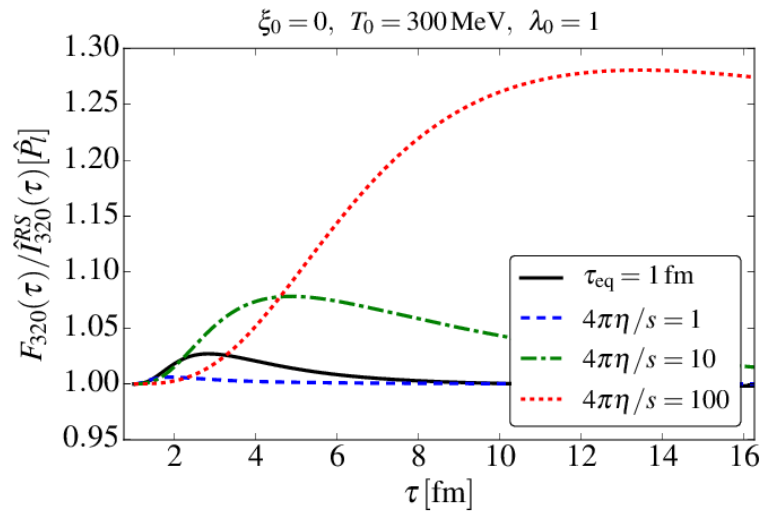
⇒ comparison of case (1) to solution of Boltzmann equation

W. Florkowski, R. Ryblewski, M. Strickland, PRC 88 (2013) 024903



Application to heavy-ion collisions (V)

⇒ relaxation eq. for \hat{P}_l gives **best match** to solution of Boltzmann equation!
However: other moments not necessarily also agree well with Boltzmann eq.



Conclusions and Outlook

1. Derivation of equations of motion of anisotropic dissipative fluid dynamics from Boltzmann equation

E. Molnár, H. Niemi, DHR, PRD 93 (2016) 11, 114025

⇒ still need to do eigenmode analysis!

2. Closure of equations of motion of “pure” anisotropic fluid dynamics

⇒ best agreement to solution of Boltzmann equation provided by \hat{P}_l

but: not all moments agree with solution of Boltzmann equation

E. Molnár, H. Niemi, DHR, arXiv:1606.09019 [nucl-th]

⇒ need to improve $\hat{f}_{0k}?!$

Conclusions and Outlook



Happy Birthday, Ulrich!

**And many healthy, productive,
and successful years!**

**...there's still a lot of fun to have
with interesting, unsolved problems!**