Electroweak radiative corrections for the LHC

Lecture I

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Overview

Lecture I

Relevance and calculational techniques

- Relevance of electroweak (EW) corrections at LHC
- Calculational techniques for virtual corrections
- Calculational techniques for real corrections

Lecture II

Unstable particles and results for specific LHC processes

- Unstable particles
- Higgs production in vector-boson fusion
- Single gauge-boson production
- Gauge-boson pair production
- Higgs production in gluon fusion

Introductory remarks

Disclaimer

- selection of topics subjective
- references incomplete
- apologies for any omissions

Thanks

- to collaborators, in particular
  A. Bredenstein, M. Ciccolini, S. Dittmaier, S. Pozzorini
- for discussions
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- special thanks to S. Dittmaier and S. Pozzorini for part of slides

Relevance of EWRC

Situation at LEP

events

1.5×10^7 $e^+e^- \to Z \to$ hadrons, 1.7×10^6 $e^+e^- \to Z \to$ leptons

4.8×10^4 $e^+e^- \to W^+W^-$

typical accuracies

$e^+e^- \to Z \to$ hadrons: $\Delta\sigma/\sigma \sim 0.09\%$

$\Delta \sin^2 \theta_{\rm eff}^{\rm lept} \sim 0.03\%$, $\Delta M_{\rm W}/M_{\rm W} \sim 0.05\%$

$e^+e^- \to W^+W^-$: $\Delta\sigma/\sigma \sim 1\%$

predictions are calculated within perturbation theory

expansion parameter: $\alpha/\pi \sim 0.23\%$, $\alpha/(\pi \sin^2 \theta_{\rm w}) \sim 1\%$

generic one-loop corrections: $\sim 2\%$

generic two-loop corrections: $\sim 0.05\%$

implications

complete one-loop corrections required

for $e^+e^- \to Z$ at least leading two-loop corrections needed

Relevance of electroweak radiative corrections

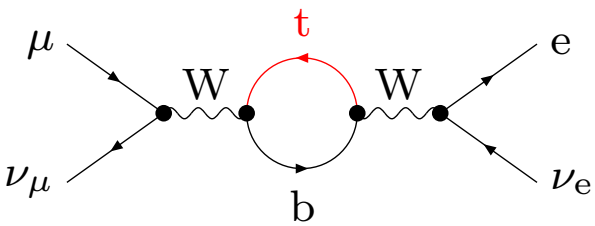
order of magnitude

$$\sim \left[\frac{\alpha}{\pi} \ldots \frac{\alpha}{\pi} \ln \frac{E^2}{m_{\rm e}^2}\right] \times \mathcal{O}(1) \sim [0.2\% \ldots 6\%] \times \mathcal{O}(1) \text{ for } E = 100\,\text{GeV}$$

$\Rightarrow$ mandatory for precision tests of the electroweak (EW) Standard Model (SM)

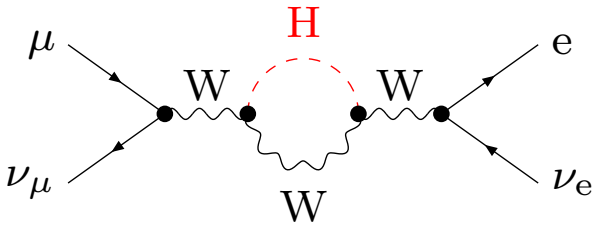
depend on all details of the theory

- top quark



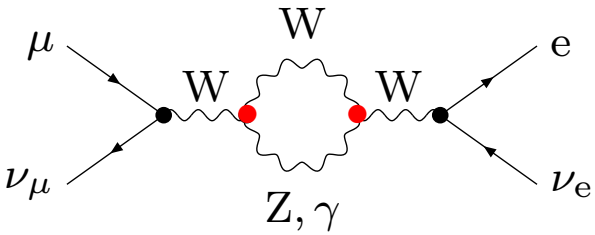
$$\sim \frac{\alpha}{\pi} \frac{m_{\rm t}^2}{s_{\rm w}^2 M_{\rm W}^2} \approx 0.05 \; (m_{\rm t} = 175\,\text{GeV})$$

- Higgs boson

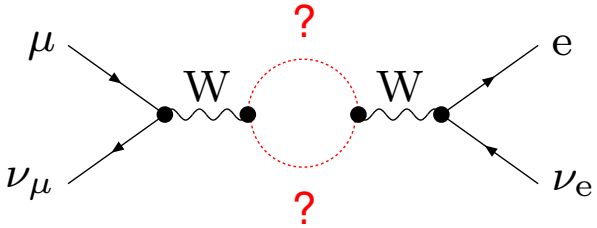


$$\sim \frac{\alpha}{\pi} \ln \frac{M_{\rm H}}{M_{\rm W}}$$

- gauge-boson self couplings



- New physics (supersymmetry) ?



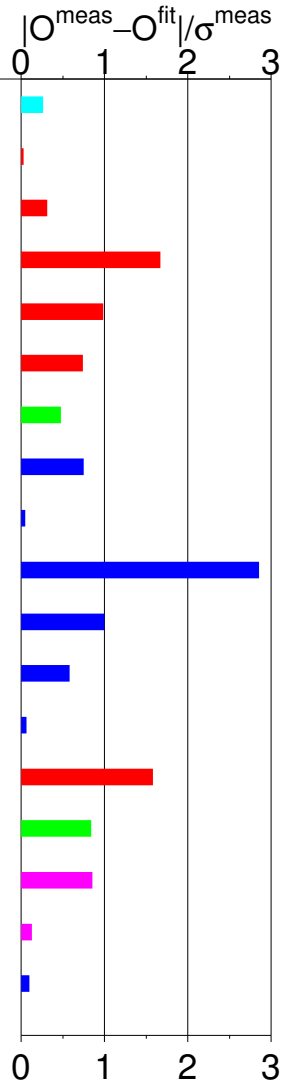
$\Rightarrow$ allow for indirect experimental tests of not directly accessible quantities

test the SM at its quantum level ($\propto \hbar$)

Global fit of LEP data

LEPEWWG '09

	Measurement	Fit
$\Delta\alpha^{(5)}_{had}(m_Z)$	0.02758 ± 0.00035	0.02767
m_Z [GeV]	91.1875 ± 0.0021	91.1874
Γ_Z [GeV]	2.4952 ± 0.0023	2.4959
σ^0_{had} [nb]	41.540 ± 0.037	41.478
R_l	20.767 ± 0.025	20.742
$A^{0,l}_{fb}$	0.01714 ± 0.00095	0.01643
$A_l(P_\tau)$	0.1465 ± 0.0032	0.1480
R_b	0.21629 ± 0.00066	0.21579
R_c	0.1721 ± 0.0030	0.1723
$A^{0,b}_{fb}$	0.0992 ± 0.0016	0.1038
$A^{0,c}_{fb}$	0.0707 ± 0.0035	0.0742
A_b	0.923 ± 0.020	0.935
A_c	0.670 ± 0.027	0.668
A_l(SLD)	0.1513 ± 0.0021	0.1480
$\sin^2\theta^{lept}_{eff}(Q_{fb})$	0.2324 ± 0.0012	0.2314
m_W [GeV]	80.399 ± 0.025	80.378
Γ_W [GeV]	2.098 ± 0.048	2.092
m_t [GeV]	173.1 ± 1.3	173.2



March 2009

good agreement

$\chi^2/\mathrm{d.o.f.} = 17.3/13 \quad (18\%)$

best fit for Higgs boson mass:

$M_H = 84\,^{+34}_{-26}\,\mathrm{GeV}$,

$M_H < 154\,\mathrm{GeV}$ @ 95% CL

direct search at LEP2

($e^+e^- \not\to ZH$)

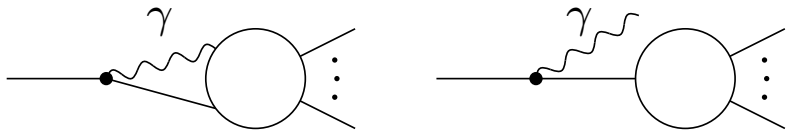
$M_H > 114.4\,\mathrm{GeV}$

Leading EW corrections at LEP energies

leading electromagnetic logarithms

from ISR and FSR: (10%) QED

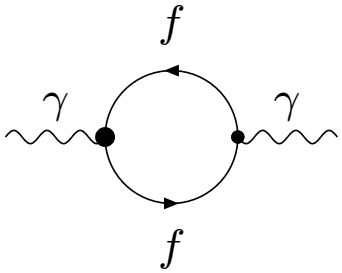
real and virtual corrections have to be added



$\sim \frac{\alpha}{\pi} \ln \frac{E^2}{m_\mathrm{e}^2} \sim 6\%$ at $E \sim M_\mathrm{Z}$

running of α: $\sim 6\%$ "QED"

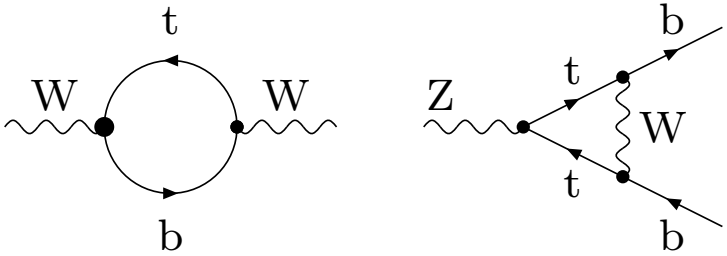
renormalization of electromagnetic coupling



$\sim \frac{\alpha}{3\pi} \sum_f Q_f^2 \ln \frac{E^2}{m_f^2}, \quad E \sim M_\mathrm{Z}$

corrections $\propto m_\mathrm{t}^2/M_\mathrm{W}^2$: $\sim 3\%$

renormalization of weak mixing angle, ...



$\sim \frac{\alpha}{\pi} \frac{m_\mathrm{t}^2}{s_\mathrm{w}^2 M_\mathrm{W}^2} \sim 0.05, \quad m_\mathrm{t} = 175\,\mathrm{GeV}$

Relevance of higher orders at LHC

QCD corrections: substantial part of predictions

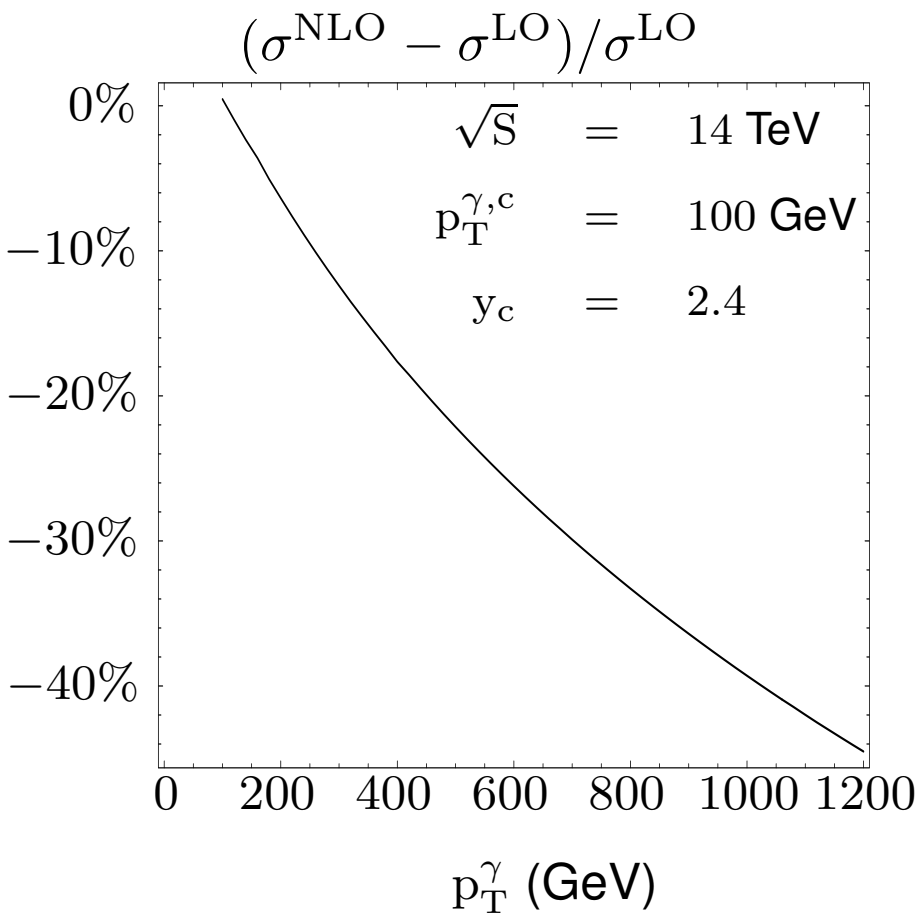
- LO predictions suffer from large scale uncertainty
  $\Rightarrow$ often no quantitative prediction possible
- NLO predictions: reduced scale uncertainty
  $\Rightarrow$ basically needed for all scattering processes at the LHC
  $\mathcal{O}(\alpha_s) \times \log(\ldots) \sim 10 - 100\%$
- NNLO predictions: reduced scale uncertainty
  $\Rightarrow$ needed for some light-candle processes like single W/Z production
  $\mathcal{O}(\alpha_s^2) \times \log^2(\ldots) \sim \text{few} - 20\%$
- uncertainties from parton distribution functions $\sim 3 - 10\%$

NLO electroweak corrections

- generically: $\mathcal{O}(\alpha) \sim \mathcal{O}(\alpha_s^2) \sim \text{few}\%$
- EW corrections can be enhanced by large energy or kinematic effects
- leading NNLO EW corrections might be relevant for some processes
  (Drell-Yan, Z+jet, W+jet)

Example: electroweak corrections to $pp \to Z\gamma$

Hollik, Meier '04



small p_T

- corrections of $\mathcal{O}(\alpha) \sim 1\%$

$p_T > 100$ GeV

- large negative corrections $\gg 1\%$
- increase with p_T
- -40% at $p_T \sim 1$ TeV !

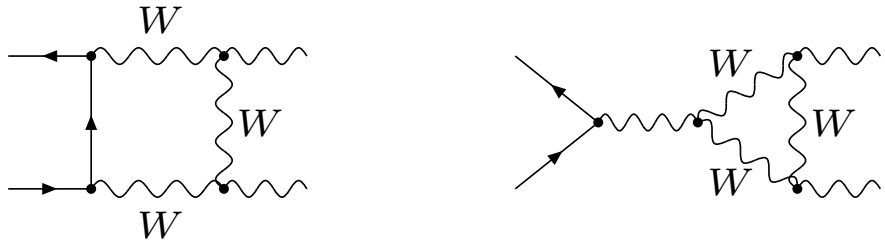
Origin of large electroweak corrections

Scattering energy $\gg$ characteristic scale of EW corrections: $\sqrt{s} \gg M_{\mathrm{W}}$

$\Rightarrow$ large double logarithms

$$\ln^2\left(\frac{s}{M_{\mathrm{W}}^2}\right) \sim 25 \qquad \text{at} \quad \sqrt{s} \sim 1\,\mathrm{TeV}$$

from vertex and box diagrams involving virtual weak bosons



Kuroda, Moultaka, Schildknecht '91; Degrassi, Sirlin '92; Beenakker et al. '93; Denner, Dittmaier, Schuster '95; Denner, Dittmaier, Hahn '97, Beccaria et al. '98; Ciafaloni, Comelli '99

general feature of hard scattering processes!

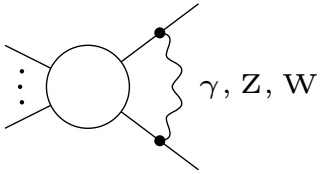
key to understand electroweak logarithms

$$M_{\mathrm{W}}^2/s \ll 1 \quad \Rightarrow \quad \ln\left(\frac{s}{M_{\mathrm{W}}^2}\right) \qquad \text{mass singularities!}$$

Universal origin of leading EW logarithms

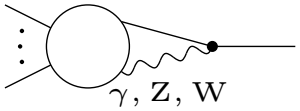
Large EW logarithms are of universal origin

- infrared logarithms $\Leftrightarrow$ external particles of the process
  mass singularities in virtual corrections related to external lines
  - soft and collinear virtual gauge bosons

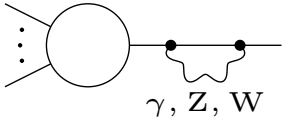


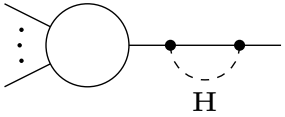
double logarithms $\alpha \ln^2 \frac{s}{M_{\mathrm{W}}^2}$

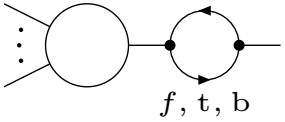
  - collinear or soft virtual gauge bosons, wave-function renormalization



single logarithms $\alpha \ln \frac{s}{M_{\mathrm{W}}^2}$







- ultraviolet logarithms $\Leftrightarrow$ parameter renormalization at scale $M_{\mathrm{W}}^2 \ll s$
  $\Rightarrow$ running of electroweak couplings from M_{W} to $\sqrt{s}$
  single logarithms $\alpha \ln \frac{s}{M_{\mathrm{W}}^2}$

Typical size at $\sqrt{s} \simeq 1\,\mathrm{TeV}$

Generic estimate for $2 \to 2$ processes

$$\left(\frac{\delta\sigma_1}{\sigma_0}\right)_{\mathrm{LL}} \simeq -\frac{\alpha}{\pi s_{\mathrm{w}}^2}\ln^2\frac{s}{M_{\mathrm{W}}^2} \simeq -26\%$$

$$\left(\frac{\delta\sigma_1}{\sigma_0}\right)_{\mathrm{LL}} \simeq +\frac{3\alpha}{\pi s_{\mathrm{w}}^2}\ln\frac{s}{M_{\mathrm{W}}^2} \simeq +16\%$$

one-loop electroweak corrections

- impact on LHC and ILC observables $\geq \mathcal{O}(10\%)$
- typically cancellations between leading and subleading logarithms
- actual size depends on the process ...

per-cent-level precision requires also two-loop EW effects

Results on electroweak logarithms

- General recipe for arbitrary processes at one loop Denner, Pozzorini '01
- explicit two-loop calculations of electroweak logarithms
  - $\ln^4(s/M_\mathrm{W}^2)$ for arbitrary processes: Melles; Hori, Kawamura, Kodaira '00
  - $\ln^3(s/M_\mathrm{W}^2)\ln(t/s)$ for arbitrary processes: Denner, Melles, Pozzorini '03
  - $\ln^3(s/M_\mathrm{W}^2)$ for fermion processes: Pozzorini '04; Denner, Jantzen, Pozzorini '06,'08
- resummation based on infrared evolution equations
  - $\ln^4(s/M_\mathrm{W}^2)$ for arbitrary processes: Fadin et al. '00
  - $\ln^3(s/M_\mathrm{W}^2)$ for arbitrary processes: Melles '01–'04
  - $\ln^2(s/M_\mathrm{W}^2)$ for massless fermionic processes: Kühn et al. '00–'03
  - $\ln(s/M_\mathrm{W}^2)$ for massless fermionic processes: Jantzen et al. '04,'05
- resummation within soft-collinear effective field theory
  - $\ln^3(s/M_\mathrm{W}^2)$ for fermionic processes: Chiu et al. '08

Summary on electroweak logarithms

Electroweak (Sudakov) logarithms

- provide estimate for one-loop corrections at level of 5–10%
  $\hookrightarrow$ good for estimates
- useful to estimate electroweak two-loop corrections
- real corrections should be included in addition
  - real photon radiation
  - real massive vector-boson radiation Baur '06

  $\Rightarrow$ partial cancellation of enhanced corrections
- not reliable for processes with other sources of large contributions
  e.g. large logarithms $\log(t/s)$ not included (important for processes with large contributions in forward/backward directions)
- at LHC many cross sections get sizeable contributions from energies below $1\,\mathrm{TeV}$

$\Rightarrow$ exact calculations of NLO EWRC preferable if possible

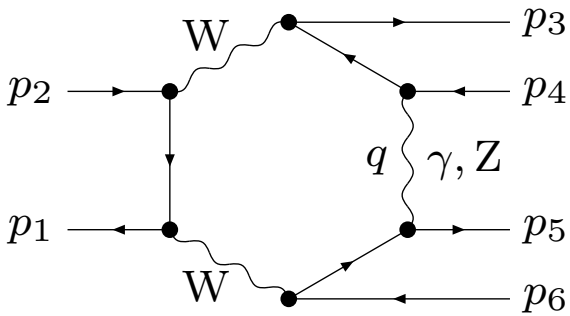
Calculational techniques I

virtual corrections

A typical 6-point diagram

for $\mathrm{e^+e^-} \to 4f$ or $\mathrm{pp(q\bar{q})} \to \mathrm{t\bar{t}b\bar{b}}$

$$f_2(p_2,\rho) + \bar{f}_1(p_1,-\rho) \;\to\; f_3(p_3,-\sigma) + \bar{f}_4(p_4,\sigma) + f_5(p_5,\tau) + \bar{f}_6(p_6,-\tau)$$



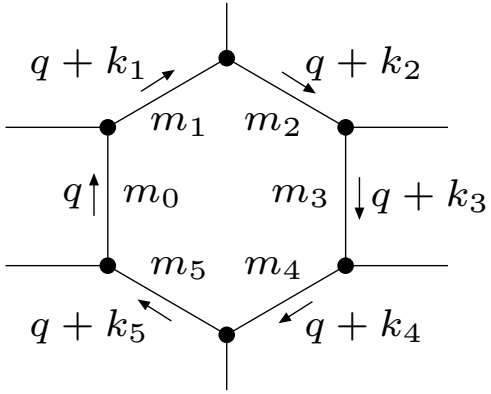
ρ, σ, τ chiralities

analytic expression for massless fermions ($V = \gamma, \mathrm{Z}$)

$$\mathcal{M}^{\rho\sigma\tau} = g_{\mathrm{W}}^4 g_V^\sigma g_V^\tau \int \mathrm{d}^D q \frac{\bar{v}(p_1)\gamma^\nu(\not{q} + \not{p}_5 + \not{p}_6 - \not{p}_1)\gamma^\lambda \omega_\rho u(p_2)}{(q^2 - M_V^2)(q+p_5)^2[(q+p_5+p_6)^2 - M_{\mathrm{W}}^2]}$$
$$\times \frac{\bar{u}(p_3)\gamma^\lambda(\not{q} - \not{p}_4)\gamma^\mu \omega_\sigma v(p_4) \times \bar{u}(p_5)\gamma^\mu(\not{q} + \not{p}_3)\gamma^\nu \omega_\tau v(p_6)}{[(q+p_5+p_6-p_1)^2][(q-p_3-p_4)^2 - M_{\mathrm{W}}^2](q-p_4)^2}$$

$\omega_\pm = \frac{1\pm\gamma_5}{2}$: projector on chiralities = helicities for massless fermions

6-point tensor integrals



denominator factors:

$$N_i = (q+k_i)^2 - m_i^2 + \mathrm{i}\epsilon, \quad k_0 = 0$$

$$i = 0, \ldots, 5$$

integral definition

$$F^{\mu_1 \cdots \mu_P}(k_1, \ldots, k_5, m_0, \ldots, m_5) = \frac{(2\pi\mu)^{4-D}}{\mathrm{i}\pi^2} \int \mathrm{d}^D q \, \frac{q^{\mu_1} \cdots q^{\mu_P}}{N_0 N_1 \ldots N_5}$$

decomposition of tensor integral into covariants: (scalar integral $F = F_0$)

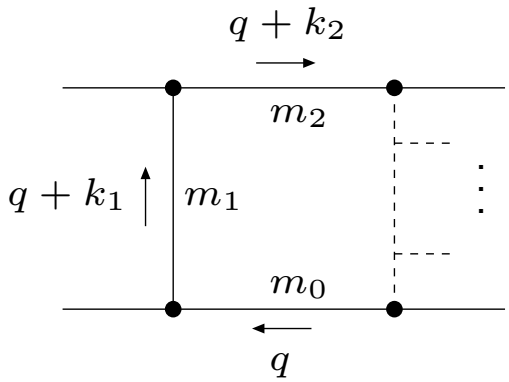
$$F^\mu = \sum_{i=1}^{5} k_i^\mu F_i, \qquad F^{\mu\nu} = \sum_{i,j=1}^{5} k_i^\mu k_j^\nu F_{ij} + g^{\mu\nu} F_{00}$$

$$F^{\mu\nu\rho} = \sum_{i,j,k=1}^{5} k_i^\mu k_j^\nu k_k^\rho F_{ijk} + \sum_{i=1}^{5} (g^{\mu\nu} k_i^\rho + g^{\nu\rho} k_i^\mu + g^{\rho\mu} k_i^\nu) F_{00i}$$

F_i, F_{ij}, F_{ijk}, F_{00i} tensor coefficients

General one-loop tensor integrals

General N-point one-loop tensor integrals of rank P



N denominator factors $1/N_k$:

$$N_k = (q+k_k)^2 - m_k^2 + \mathrm{i}\epsilon, \quad k_0 = 0$$
$$k = 0, \ldots, N-1$$

integral definition

$$T^{N,\mu_1\ldots\mu_P} = \frac{(2\pi\mu)^{4-D}}{\mathrm{i}\pi^2} \int \mathrm{d}^D q \, \frac{q^{\mu_1}\cdots q^{\mu_P}}{N_0 N_1 \ldots N_{N-1}}$$

($A \equiv T^1$, $B \equiv T^2$, $C \equiv T^3$, $D \equiv T^4$, $E \equiv T^5$, $F \equiv T^6$, etc.)

decomposition of tensor integral into covariants:

$$T^{N,\mu_1\ldots\mu_P} = \sum_{i_1,\ldots,i_P=1}^{N-1} p_{i_1}^{\mu_1}\ldots p_{i_P}^{\mu_P} T^N_{i_1\ldots i_P} + \sum_{i_3,\ldots,i_P=1}^{N-1} \{g^{\mu_1\mu_2} p_{i_3}^{\mu_3}\ldots p_{i_P}^{\mu_P} + \ldots\} T^N_{00 i_3\ldots i_P}$$
$$+ \sum_{i_5,\ldots,i_P=1}^{N-1} \{ggp\ldots p\}^{\mu_1\ldots\mu_P}_{i_5\ldots i_P} T^N_{0000 i_5\ldots i_P} + \ldots$$

Step 1: insertion of tensor integrals

$$
\begin{aligned}
\mathcal{M}^{\rho\sigma\tau} = {} & \mathrm{i}\pi^2(2\pi\mu)^{D-4} g_{\mathrm{W}}^4 g_V^\sigma g_V^\tau \\
& \times \bar{v}(p_1)\gamma^\nu\gamma^\alpha\gamma^\lambda\omega_\rho u(p_2) \times \bar{u}(p_3)\gamma^\lambda\gamma^\beta\gamma^\mu\omega_\sigma v(p_4) \times \bar{u}(p_5)\gamma^\mu\gamma^\gamma\gamma^\nu\omega_\tau v(p_6) \\
& \times \Big[-(p_5+p_6-p_1)_\alpha p_{4,\beta} p_{3,\gamma} F_0 \\
& \quad - p_{4,\beta} p_{3,\gamma} F_\alpha + (p_5+p_6-p_1)_\alpha p_{3,\gamma} F_\beta - (p_5+p_6-p_1)_\alpha p_{4,\beta} F_\gamma \\
& \quad + p_{3,\gamma} F_{\alpha\beta} + (p_5+p_6-p_1)_\alpha F_{\beta\gamma} - p_{4,\beta} F_{\alpha\gamma} \\
& \quad + F_{\alpha\beta\gamma}\Big] \\
= {} & \mathrm{i}\pi^2(2\pi\mu)^{D-4} g_{\mathrm{W}}^4 g_V^\sigma g_V^\tau \\
& \times \bar{v}(p_1)\gamma^\nu\gamma^\alpha\gamma^\lambda\omega_\rho u(p_2) \times \bar{u}(p_3)\gamma^\lambda\gamma^\beta\gamma^\mu\omega_\sigma v(p_4) \times \bar{u}(p_5)\gamma^\mu\gamma^\gamma\gamma^\nu\omega_\tau v(p_6) \\
& \times \Big[-(p_5+p_6-p_1)_\alpha p_{4,\beta} p_{3,\gamma} F_0 \\
& + [-p_{5,\alpha} p_{4,\beta} p_{3,\gamma} + (p_5+p_6-p_1)_\alpha p_{5,\beta} p_{3,\gamma} - (p_5+p_6-p_1)_\alpha p_{4,\beta} p_{5,\gamma}] F_1 \\
& + \ldots \\
& + p_{5,\alpha} p_{5,\beta} p_{5,\gamma} F_{111} + \ldots + [p_{5,\alpha} g_{\beta\gamma} + g_{\alpha\gamma} p_{5,\beta} + g_{\beta\gamma} p_{5,\alpha}] F_{001} + \ldots\Big]
\end{aligned}
$$

Step 1: insertion of tensor integrals

$\Rightarrow \mathcal{M}^{\rho\sigma\tau} = \sum_n F_n^{\rho\sigma\tau}(\{s_{ij}\})\hat{\mathcal{M}}_n^{\rho\sigma\tau}(p_1,\ldots,p_6)$

standard matrix elements (SME) $\hat{\mathcal{M}}_n^{\rho\sigma\tau}$:
involve all tensorial and spinorial objects
depend on helicities of external particles, purely kinematical objects

examples:

$$\begin{aligned}
\hat{\mathcal{M}}_1^{\rho\sigma\tau} &= \bar{v}(p_1)\gamma^\nu \not{p}_3 \gamma^\lambda \omega_\rho u(p_2) \times \bar{u}(p_3)\gamma^\lambda \not{p}_1 \gamma^\mu \omega_\sigma v(p_4) \times \bar{u}(p_5)\gamma^\mu \not{p}_1 \gamma^\nu \omega_\tau v(p_6) \\
\hat{\mathcal{M}}_2^{\rho\sigma\tau} &= \bar{v}(p_1)\gamma^\nu \not{p}_6 \gamma^\lambda \omega_\rho u(p_2) \times \bar{u}(p_3)\gamma^\lambda \not{p}_1 \gamma^\mu \omega_\sigma v(p_4) \times \bar{u}(p_5)\gamma^\mu \not{p}_3 \gamma^\nu \omega_\tau v(p_6) \\
\hat{\mathcal{M}}_3^{\rho\sigma\tau} &= \bar{v}(p_1)\gamma^\mu \omega_\rho u(p_2) \times \bar{u}(p_3)\gamma_\mu \omega_\sigma v(p_4) \times \bar{u}(p_5)\not{p}_2 \omega_\tau v(p_6)
\end{aligned}$$

scalar invariant functions $F_n^{\rho\sigma\tau}(\{s_{ij}\})$:
linear combinations of F_0, F_i, F_{ij}, and F_{ijk}
coefficients depend on invariants, masses, couplings

examples:

$$\begin{aligned}
F_1^{\rho\sigma\tau} &= g_{\mathrm{W}}^4 g_V^\sigma g_V^\tau F_{334} \\
F_2^{\rho\sigma\tau} &= -g_{\mathrm{W}}^4 g_V^\sigma g_V^\tau (F_3 + F_{34} + F_{23} + F_{234})
\end{aligned}$$

Step 2: reduction of standard matrix elements

Processes with 6 external fermions: $\mathcal{O}(10^3)$ SME

goal: reduce SME to small (minimal) set

- express SME as linear combinations of other SME
  minimal set: one SME per helicity amplitude
  in practice reasonably small set
- beware of spurious singularities (denominators)
- use discrete symmetries

$\Rightarrow$

- concise results
- fast and numerically stable code

Step 2: reduction of standard matrix elements (cont.)

Reduction in D dimensions using

- Dirac algebra $\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}$
- Dirac equation $(\not{p} - m)u(p) = 0$, (transversality of polarization vectors)
- momentum conservation $p_1 + p_2 = p_3 + p_4 + p_5 + p_6$

after renormalization and subtraction of IR singularities
reduction in 4 dimensions using four-dimensionality of space time

- Chisholm identity
- Shouten identity
- products of ϵ tensors
- decomposition of metric tensor
- identities for products of spinor chains

$\Rightarrow \mathcal{O}(10)$ SME

4-dimensional identities

- Chisholm identity

$$\gamma^\alpha\gamma^\beta\gamma^\gamma = g^{\alpha\beta}\gamma^\gamma - g^{\alpha\gamma}\gamma^\beta + g^{\beta\gamma}\gamma^\alpha + \mathrm{i}\epsilon^{\alpha\beta\gamma\delta}\gamma_\delta\gamma_5$$

$\Rightarrow$ eliminate $\gamma^\alpha\gamma^\beta\gamma^\gamma$ or $\epsilon^{\alpha\beta\gamma\delta}\gamma_\delta\gamma_5$ terms

consequences

$$\begin{aligned}
\gamma^\mu\gamma^\alpha\gamma^\beta\omega_\sigma \otimes \gamma_\mu\omega_\sigma &= \gamma_\mu\omega_\sigma \otimes \gamma_\mu\gamma^\beta\gamma^\alpha\omega_\sigma \\
\gamma^\mu\gamma^\alpha\gamma^\beta\omega_\sigma \otimes \gamma_\mu\omega_{-\sigma} &= \gamma_\mu\omega_\sigma \otimes \gamma^\alpha\gamma^\beta\gamma_\mu\omega_{-\sigma}
\end{aligned}$$

$$\begin{aligned}
\gamma^\mu\gamma^\alpha\gamma^\nu\omega_\sigma \otimes \gamma_\mu\gamma^\beta\gamma_\nu\omega_\sigma &= 4g^{\alpha\beta}\gamma^\mu\omega_\sigma \otimes \gamma_\mu\omega_\sigma \\
\gamma^\mu\gamma^\alpha\gamma^\nu\omega_\sigma \otimes \gamma_\mu\gamma^\beta\gamma_\nu\omega_{-\sigma} &= 4\gamma^\beta\omega_\sigma \otimes \gamma^\alpha\omega_{-\sigma}
\end{aligned}$$

$\Rightarrow$ simplify contracted Dirac chains

- Schouten identity

$$\epsilon_{[\alpha\beta\gamma\delta}\, g_{\mu]\nu} = \epsilon_{\alpha\beta\gamma\delta}\, g_{\mu\nu} + \epsilon_{\beta\gamma\delta\mu}\, g_{\alpha\nu} + \epsilon_{\gamma\delta\mu\alpha}\, g_{\beta\nu} + \epsilon_{\delta\mu\alpha\beta}\, g_{\gamma\nu} + \epsilon_{\mu\alpha\beta\gamma}\, g_{\delta\nu} = 0$$

- products of ϵ tensors are eliminated via

$$\epsilon^{\alpha\beta\gamma\delta}\epsilon^{\mu\nu\rho\sigma} = -\begin{vmatrix} g^{\alpha\mu} & g^{\alpha\nu} & g^{\alpha\rho} & g^{\alpha\sigma} \\ g^{\beta\mu} & g^{\beta\nu} & g^{\beta\rho} & g^{\beta\sigma} \\ g^{\gamma\mu} & g^{\gamma\nu} & g^{\gamma\rho} & g^{\gamma\sigma} \\ g^{\delta\mu} & g^{\delta\nu} & g^{\delta\rho} & g^{\delta\sigma} \end{vmatrix}$$

4-dimensional identities (cont.)

- decomposition of metric tensor

$$g^{\mu\nu} = \sum_{i,j=0}^{3} g^{ij}\, n_i^\mu\, n_j^\nu$$

in terms of four independent orthonormal four-vectors n_j^μ
with $n_i^\mu\, n_{j,\mu} = g_{ij}, \quad g_{ij} = g^{ij} = \mathrm{diag}(1,-1,-1,-1)$
n_i^μ can be constructed from three linearly-independent massless momenta p_i, p_j, p_k

$$n_0^\mu(p_i,p_j,p_k) = \frac{1}{\sqrt{2\,(p_ip_j)}}\left(p_i^\mu + p_j^\mu\right), \qquad n_1^\mu(p_i,p_j,p_k) = \frac{1}{\sqrt{2\,(p_ip_j)}}\left(p_i^\mu - p_j^\mu\right)$$

$$n_2^\mu(p_i,p_j,p_k) = -\frac{1}{\sqrt{2\,(p_ip_j)\,(p_ip_k)\,(p_jp_k)}}\left[(p_jp_k)\,p_i^\mu + (p_ip_k)\,p_j^\mu - (p_ip_j)\,p_k^\mu\right]$$

$$n_3^\mu(p_i,p_j,p_k) = -\frac{1}{\sqrt{2\,(p_ip_j)\,(p_ip_k)\,(p_jp_k)}}\,\epsilon^{\mu\alpha\beta\gamma}\,p_{i,\alpha}\,p_{j,\beta}\,p_{k,\gamma}$$

no complicated denominators

$\Rightarrow$ decoupling of spinor chains or other objects $\quad \bar{v}_1\gamma^\mu u_2\;\bar{v}_3\gamma_\mu u_4 = \sum_{i,j=0}^{3}\bar{v}_1\not{n}_i u_2\;\bar{v}_3\not{n}_j u_4$

Step 3: reduction of tensor integrals

According to Denner, Dittmaier NPB734 (2006) 62 [hep-ph/0509141]

- 6-point integrals $\longrightarrow$ six 5-point integrals (see also Binoth et al. '05)
  simultaneous reduction of rank by one, no inverse Gram determinants
- 5-point integrals $\longrightarrow$ five 4-point integrals Melrose '65; Denner, Dittmaier '02, '05
  simultaneous reduction of rank by one, no inverse Gram determinants
- 3-point and 4-point integrals: default: Passarino–Veltman reduction
  to scalar 3-point and 4-point integrals and 1- and 2-point integrals
  $\hookrightarrow$ serious numerical instabilities for small Gram determinants $\det G \to 0$

  exceptional phase space points treated by

  (i) expansions in small Gram and other kinematical determinants
  (see also R.K.Ellis et al. '05) or
  (ii) semi-numerical method (numerical calculation of logarithmic Feynman-parameter integral and algebraic reduction to this basis integral)
  (see also Binoth et al. '05; Ferroglia et al. '02) $\oplus$ analytical special cases
- 1- and 2-point integrals: numerically stable direct calculation

Gram matrix

Gram matrix

$$Z^{(N)}(p_1,\ldots,p_N) = \begin{pmatrix} 2p_1p_1 & \ldots & 2p_1p_N \\ \vdots & \ddots & \vdots \\ 2p_Np_1 & \ldots & 2p_Np_N \end{pmatrix}$$

Gram determinant: $\Delta^{(N)} = \det(Z^{(N)})$
vanishes for linear dependent momenta
in Minkowski space also for other cases

adjoint

$$\tilde{Z}^{(N)}_{ij} = (-1)^{i+j} \left| \begin{array}{ccccc} 2p_1p_1 & \ldots & 2p_1p_j & \ldots & 2p_1p_N \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ \cancel{2p_ip_1} & \cancel{\ldots} & \cancel{2p_ip_j} & \cancel{\ldots} & \cancel{2p_ip_N} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ 2p_Np_1 & \ldots & 2p_Np_j & \ldots & 2p_Np_N \end{array} \right|$$

$$\sum_k Z^{(N)}_{ik} \tilde{Z}^{(N)}_{kj} = \delta_{ij} \Delta^{(N)}$$

$\det(Z^{(N)}) = 0$: $\Rightarrow$ spurious singularities in tensor integral reduction
in general no real singularities

Modified Cayley matrix

Modified Cayley matrix: $\left(Y_{ij}\right) = \left(m_i^2 + m_j^2 - (p_i - p_j)^2\right), \qquad i = 0, \ldots, N$

matrix

$$X^{(N)} = \begin{pmatrix} 2m_0^2 & f_1 & \ldots & f_N \\ f_1 & 2p_1p_1 & \ldots & 2p_1p_N \\ \vdots & \vdots & \ddots & \vdots \\ f_N & 2p_Np_1 & \ldots & 2p_Np_N \end{pmatrix} = \left(\begin{array}{c|ccc} 2m_0^2 & \underline{f_1} & \underline{f_2} & \underline{f_3} \\ f_1 & & & \\ f_2 & & Z & \\ f_3 & & & \end{array}\right)$$

with $f_k = p_k^2 - m_k^2 + m_0^2$

Modified Cayley determinant

$$\det(X^{(N)}) = 2m_0^2\Delta^{(N)} - \sum_{n,m=1}^{N} f_n f_m \tilde{Z}_{nm}^{(N)} = \begin{vmatrix} Y_{00} & Y_{01} & \ldots & Y_{0N} \\ Y_{10} & Y_{11} & \ldots & Y_{1N} \\ \vdots & \vdots & \ddots & \vdots \\ Y_{N0} & Y_{N1} & \ldots & Y_{NN} \end{vmatrix} = \det(Y),$$

$\det(X^{(N)}) = 0$: necessary condition for Landau (real) singularity

Reduction of 6-point functions

General idea: reduce a determinant that is zero in 4 space–time dimensions

$\hookrightarrow$ relation between 6-point and 5-point integrals

starting point

$$\int \mathcal{F} = \frac{(2\pi\mu)^{4-D}}{\mathrm{i}\pi^2} \int \mathrm{d}^D q \, \frac{q^{\mu_1} \cdots q^{\mu_P}}{N_0 N_1 \ldots N_5} \begin{vmatrix} q^\mu & 2qp_1 & \ldots & 2qp_5 \\ p_1^\mu & 2p_1p_1 & \ldots & 2p_1p_5 \\ \vdots & \vdots & \ddots & \vdots \\ p_{k-1}^\mu & 2p_{k-1}p_1 & \ldots & 2p_{k-1}p_5 \\ 0 & f_1 & \ldots & f_5 \\ p_{k+1}^\mu & 2p_{k+1}p_1 & \ldots & 2p_{k+1}p_5 \\ \vdots & \vdots & \ddots & \vdots \\ p_5^\mu & 2p_5p_1 & \ldots & 2p_5p_5 \end{vmatrix}$$

$$= -\tilde{X}_{k0} F^{\alpha\mu_1\ldots\mu_P} \Big(\underbrace{g_\alpha^\mu}_{D\text{-dim}} - \underbrace{g_{(4)}{}_\alpha^\mu}_{4\text{-dim}} \Big) = \begin{cases} \mathcal{O}(D-4) & \text{for } P \leq 6 \\ \text{finite, but simply calculable} & \text{for } P > 6 \end{cases}$$

4-dimensional metric tensor build of momenta

$$g_{(4)}^{\mu\nu} = \sum_{i,j=1}^{4} 2p_i^\mu p_j^\nu (Z^{(4)})_{ji}^{-1} = \sum_{i,j=1}^{4} 2p_i^\mu k_j^\nu (2p_i k_j)^{-1}$$

Reduction of 6-point functions (cont.)

Reduction of determinant in momentum space

use $2qp_i = N_i - N_0 - f_i$, cancel N_i, and expand determinants

$$\Rightarrow \int \mathcal{F} = -\tilde{X}_{k0} F^{\mu\mu_1\cdots\mu_P} - \sum_{n,m=1}^{5} \tilde{\tilde{X}}_{(km)(0n)} p_m^\mu \left[E^{\mu_1\cdots\mu_P}(n) - E^{\mu_1\cdots\mu_P}(0) \right]$$

(index k can be chosen to optimise stability)

$E^{\mu_1\cdots\mu_P}(n)$: tensor integral with n-th denominator cancelled

$\hookrightarrow$ tensor coefficients $F_{\ldots}$ read off upon comparing coefficients of covariants

comments:

- reduction of rank: $F^{\mu\mu_1\cdots\mu_P} \rightarrow$ rank $(P+1)$
  $E^{\mu_1\cdots\mu_P}(n) \rightarrow$ rank-P
- no inverse Gram determinant, but $E^{\mu\mu_1\cdots\mu_P} = [\ldots]/\tilde{X}_{k0}$
  $\tilde{X}_{k0} = 0$ for all $k \Rightarrow \det(X) = 0$
- reduction works for massive/massless case in any IR regularization

Explicit results for 6-point tensor coefficients

$$
\begin{aligned}
F_{i_1} &= \sum_{n=1}^{5} c_{i_1 n}\Big[E_0(n) - E_0(0)\Big] \\
F_{00} &= 0, \\
F_{i_1 i_2} &= \frac{1}{2}\sum_{n=1}^{5}\Big\{c_{i_1 n}\Big[E_{(i_2)_n}(n)\bar{\delta}_{i_2 n} - E_{i_2}(0)\Big] + (i_1 \leftrightarrow i_2)\Big\} \\
F_{00 i_1} &= \frac{1}{3}\sum_{n=1}^{5} c_{i_1 n}\Big[E_{00}(n) - E_{00}(0)\Big] \\
F_{i_1 i_2 i_3} &= \frac{1}{3}\sum_{n=1}^{5}\Big\{c_{i_1 n}\Big[E_{(i_2)_n (i_3)_n}(n)\bar{\delta}_{i_2 n}\bar{\delta}_{i_3 n} - E_{i_2 i_3}(0)\Big] + (i_1 \leftrightarrow i_2) + (i_1 \leftrightarrow i_3)\Big\}
\end{aligned}
$$

etc., $c_{i_1 n} =$ related to inverse matrix X^{-1}

scalar integral via method of Melrose '65 (Denner '93):

$$\det(X) F_0 = -\sum_{n=0}^{5} \det(Y_n) E_0(n),$$

$Y_n = Y$ with entries in ith row replaced by 1

Reduction of 5-point integrals

General idea: reduce a determinant that is zero in 4 space–time dimensions

$\hookrightarrow$ relation between 5-point and 4-point integrals

starting point:

$$\int \mathcal{E} = \frac{(2\pi\mu)^{4-D}}{\mathrm{i}\pi^2} \int \mathrm{d}^D q \, \frac{q^{\mu_1} \cdots q^{\mu_P}}{N_0 N_1 \ldots N_4} \begin{vmatrix} q^\mu & -2q^2 & 2qp_1 & \ldots & 2qp_4 \\ 0 & 2m_0^2 & f_1 & \ldots & f_4 \\ p_1^\mu & -2p_1 q & 2p_1 p_1 & \ldots & 2p_1 p_4 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ p_4^\mu & -2p_4 q & 2p_4 p_1 & \ldots & 2p_4 p_4 \end{vmatrix}$$

$$= 2m_0^2 \det(Z) (g_\alpha^\mu - g_{(4)}{}_\alpha^\mu) E^{\alpha\mu_1\ldots\mu_P}$$

$$+ 2 \sum_{n=1}^{4} \tilde{X}_{n0} \Big[p_n^\mu (\underbrace{g_{\alpha\beta}}_{D\text{-dim}} - \underbrace{g_{(4)}{}_{\alpha\beta}}_{\text{4-dim (built of momenta)}}) - p_{n,\beta} (g_\alpha^\mu - g_{(4)}{}_\alpha^\mu) \Big] E^{\alpha\beta\mu_1\ldots\mu_P}$$

$$= \begin{cases} \mathcal{O}(D-4) & \text{for } P \leq 3 \\ \text{finite, but simply calculable} & \text{for } P > 3 \end{cases}$$

Reduction of 5-point integrals (cont.)

Reduction of determinant in momentum space (simple manipulations)

$\hookrightarrow$ propagator cancellations lead to 4-point integrals

$$\int \mathcal{E} = \det(X) E^{\mu\mu_1\cdots\mu_P} - \sum_{n,m=1}^{4} \tilde{X}_{mn} p_m^\mu \Big[D^{\mu_1\cdots\mu_P}(n) - D^{\mu_1\cdots\mu_P}(0) \Big]$$
$$- \sum_{n=1}^{4} \tilde{X}_{n0} \Big[-p_n^\mu D^{\mu_1\cdots\mu_P}(0) + \mathcal{D}^{\mu\mu_1\cdots\mu_P}(n) \Big]$$
$$+ \sum_{n=1}^{4} \mathcal{D}^{\alpha\mu_1\cdots\mu_P}(n) \sum_{m,l=1}^{4} 2 p_{m,\alpha} p_l^\mu \tilde{\tilde{X}}_{(ln)(0m)}$$

$\hookrightarrow$ tensor coefficients $E_{\ldots}$ read off upon comparing coefficients of covariants

comments:

- reduction of rank: $E^{\mu\mu_1\cdots\mu_P} \longrightarrow$ rank $(P+1)$
  D's / $\mathcal{D}$'s $\longrightarrow$ directly obtained from rank-P integrals
- no inverse Gram determinant, but $E^{\mu\mu_1\cdots\mu_P} = [\ldots]/\det(X)$
- reduction works for massive/massless case in any IR regularization

Explicit results for 5-point tensor coefficients

$$\det(X)E_{i_1} = \sum_{n=1}^{4} \tilde{X}_{i_1 n}\Big[D_0(n) - D_0(0)\Big] - \tilde{X}_{i_1 0} D_0(0)$$

$$\det(X)E_{00} = \sum_{n=1}^{4} \tilde{X}_{n0}\Big[D_{00}(n) - D_{00}(0)\Big]$$

$$2\det(X)E_{i_1 i_2} = \Bigg\{\sum_{n=1}^{4} \tilde{X}_{i_1 n}\Big[D_{(i_2)_n}(n)\bar{\delta}_{i_2 n} - D_{i_2}(0)\Big] - \tilde{X}_{i_1 0} D_{i_2}(0) - 2\sum_{n=1}^{4} \tilde{\tilde{X}}_{(i_1 n)(0 i_2)}\Big[D_{00}(n) - D_{00}(0)\Big]\Bigg\} + (i_1 \leftrightarrow i_2)$$

etc.

scalar integral via method of Melrose '65 (Denner '93; Denner, Dittmaier '02):

$$\det(X)E_0 = -\sum_{n=0}^{4} \det(Y_n) D_0(n),$$

$Y_n = Y$ with entries in ith row replaced by 1

Reduction of 4-point and 3-point integrals

Passarino–Veltman reduction

recursive solution for tensor coefficients:

$$\underbrace{T^N_{00i_3\ldots i_P}}_{\text{rank } P} \propto 2m_0^2 \underbrace{T^N_{i_3\ldots i_P}}_{\text{rank } P-2} + \sum_{n=1}^{N-1} f_n \underbrace{T^N_{ni_3\ldots i_P}}_{\text{rank } P-1} + (T^{N-1} \text{ terms}),$$

$$\Delta^{(N)} \underbrace{T^N_{i_1\ldots i_P}}_{\text{rank } P} = \sum_{\substack{n=1 \\ i_1 \neq 0}}^{N-1} \tilde{Z}_{i_1 n} \left[-f_n \underbrace{T^N_{i_2\ldots i_P}}_{\text{rank } P-1} - 2 \sum_{r=2}^{P} \delta_{ni_r} \underbrace{T^N_{00i_2\ldots\hat{i}_r\ldots i_P}}_{\text{rank } P} + (T^{N-1} \text{ terms}) \right]$$

$\hookrightarrow$ recursive calculation of $T^N_{i_1\ldots i_P}$ from scalar integral T^N_0 and $T^{N-1}_{i_2\ldots i_P}$:

$$T^N_0 = \text{basis integral} \quad \rightarrow \quad T^N_{i_1} \quad \rightarrow \quad T^N_{i_1 i_2} \quad \rightarrow \quad T^N_{i_1 i_2 i_3} \quad \rightarrow \quad \ldots$$

but: $T^N_{i_1\ldots i_P} = [\ldots]/\Delta^{(N)}$

relations involve inverse of Gram matrix

$\hookrightarrow$ potential instabilities for $\det(Z) \to 0$

Example: 4-point tensor integrals

scalar integral $= \quad D_0$

$$D^\mu = \sum_{i=1}^{3} p_i^\mu D_i$$

$$D^{\mu\nu} = \sum_{i,j=1}^{3} p_i^\mu p_j^\nu D_{ij} \qquad + \; g^{\mu\nu} D_{00}$$

$$D^{\mu\nu\rho} = \sum_{i,j,k=1}^{3} p_i^\mu p_j^\nu p_k^\rho D_{ijk} \qquad + \sum_{i=1}^{3} \{g^{\mu\nu} p_i^\rho + \cdots\} D_{00i}$$

$$D^{\mu\nu\rho\sigma} = \sum_{i,j,k,l=1}^{3} p_i^\mu p_j^\nu p_k^\rho p_l^\sigma D_{ijkl} + \sum_{i,j=1}^{3} \{g^{\mu\nu} p_i^\rho p_j^\sigma + \cdots\} D_{00ij} + \{g^{\mu\nu} g^{\rho\sigma} + \cdots\} D_{0000}$$

$$\vdots \qquad \vdots \qquad \vdots \qquad \vdots$$

Example: 4-point tensor integrals

PV reduction: **step 0**

$\boxed{D_0}$ basis integral

D_i

D_{ij} D_{00}

D_{ijk} D_{00i}

D_{ijkl} D_{00ij} D_{0000}

$\vdots$ $\vdots$ $\vdots$

Example: 4-point tensor integrals

PV reduction: **step 1**

D_0

$\downarrow$

$\boxed{D_i}$

D_{ij} $\qquad$ D_{00}

D_{ijk} $\qquad$ D_{00i}

D_{ijkl} $\qquad$ D_{00ij} $\qquad$ D_{0000}

$\vdots$ $\qquad$ $\vdots$ $\qquad$ $\vdots$

Example: 4-point tensor integrals

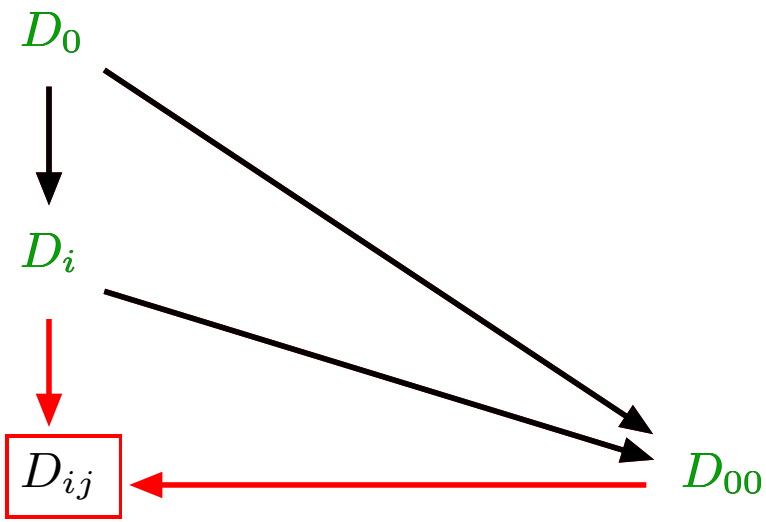
PV reduction:

step 2a

$$
\begin{tikzcd}
D_0 \arrow[d] \arrow[ddr] & & \\
D_i \arrow[dr] & & \\
D_{ij} & \boxed{D_{00}} & \\
D_{ijk} & D_{00i} & \\
D_{ijkl} & D_{00ij} & D_{0000} \\
\vdots & \vdots & \vdots
\end{tikzcd}
$$

Example: 4-point tensor integrals

PV reduction: **step 2b**



D_{ijk} D_{00i}

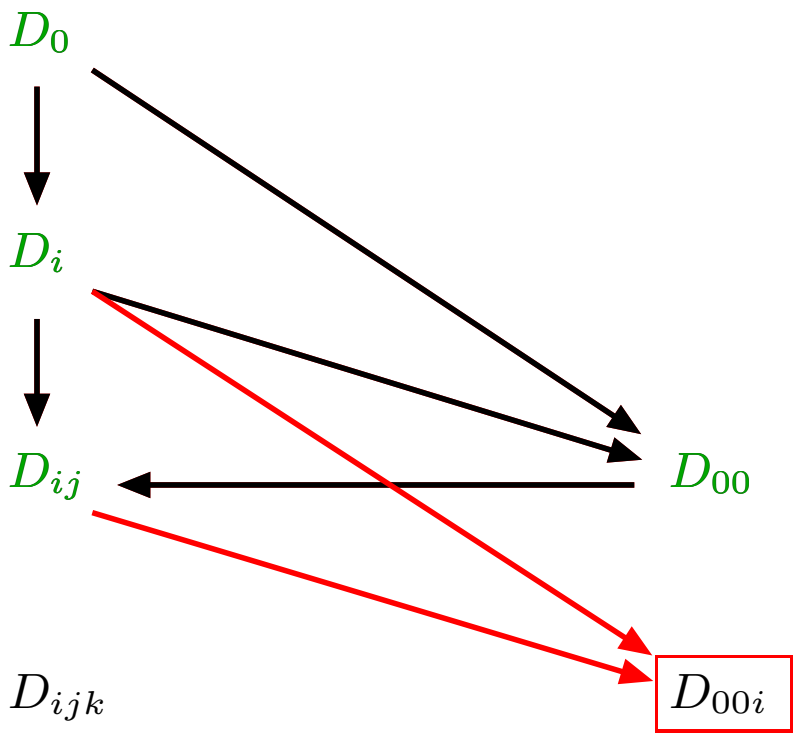
D_{ijkl} D_{00ij} D_{0000}

$\vdots$ $\vdots$ $\vdots$

Example: 4-point tensor integrals

PV reduction:

step 3a

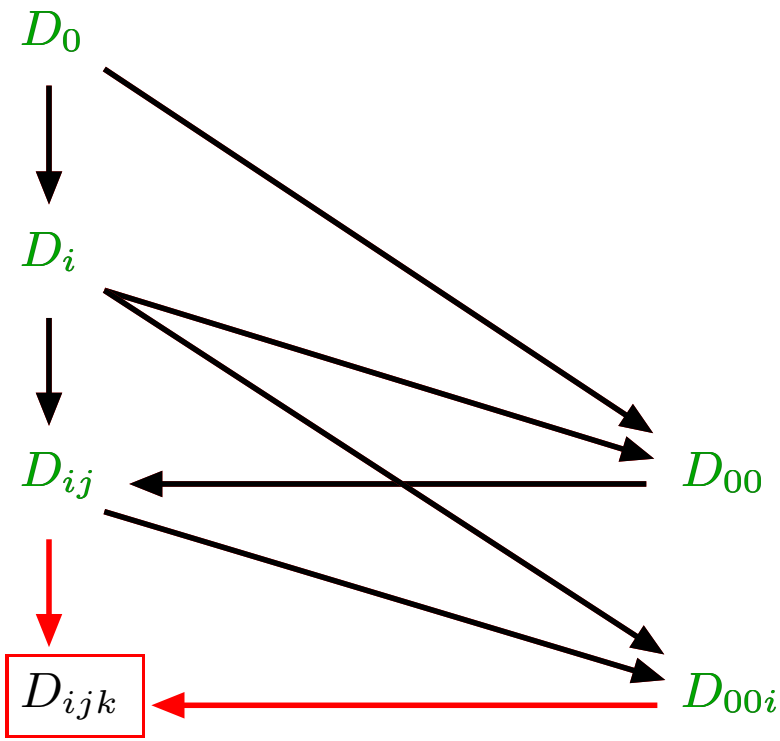


D_{ijkl} $\quad$ D_{00ij} $\quad$ D_{0000}

$\vdots$ $\quad$ $\vdots$ $\quad$ $\vdots$

Example: 4-point tensor integrals

PV reduction: **step 3b**



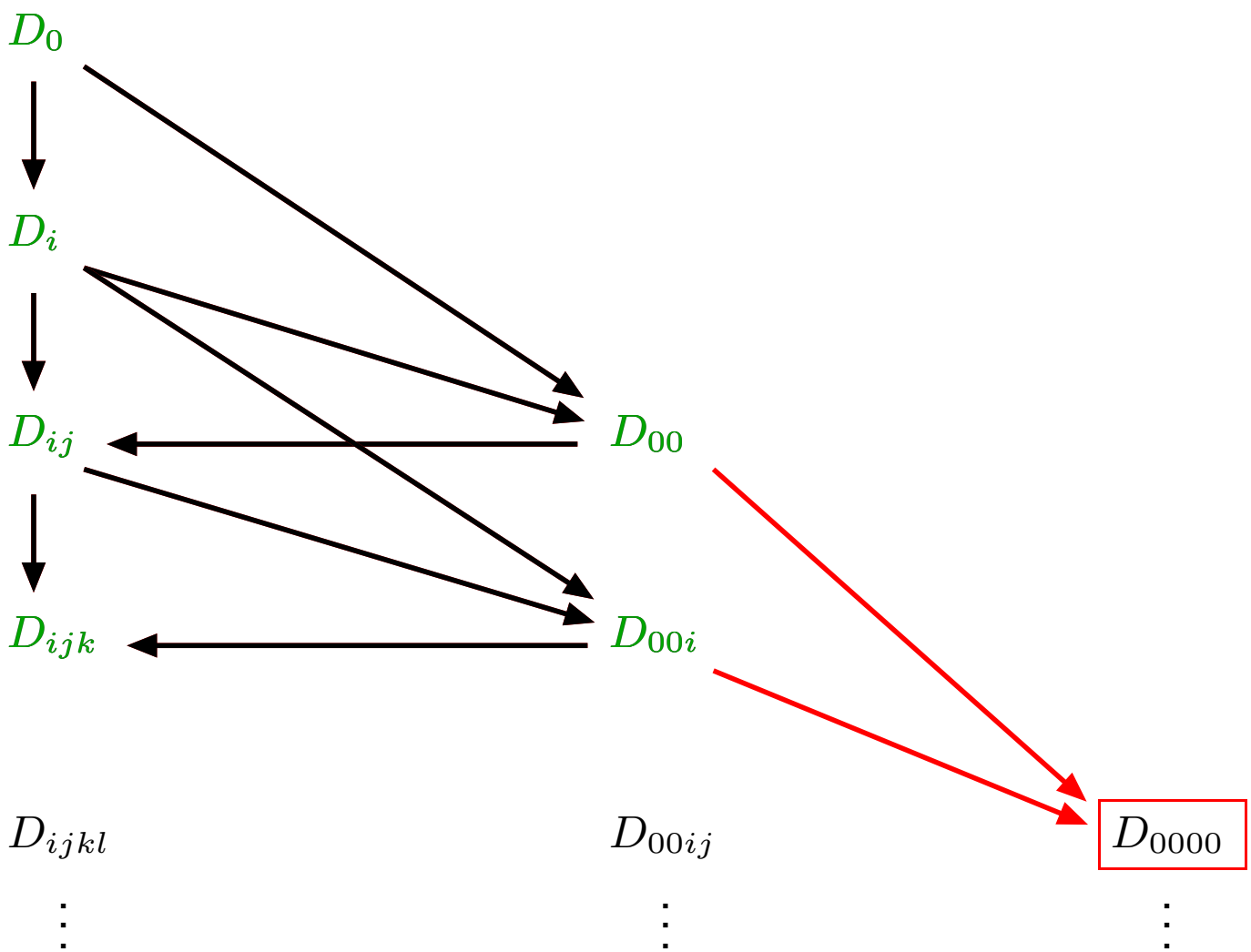
D_{ijkl} $\quad$ D_{00ij} $\quad$ D_{0000}

$\vdots$ $\quad$ $\vdots$ $\quad$ $\vdots$

Example: 4-point tensor integrals

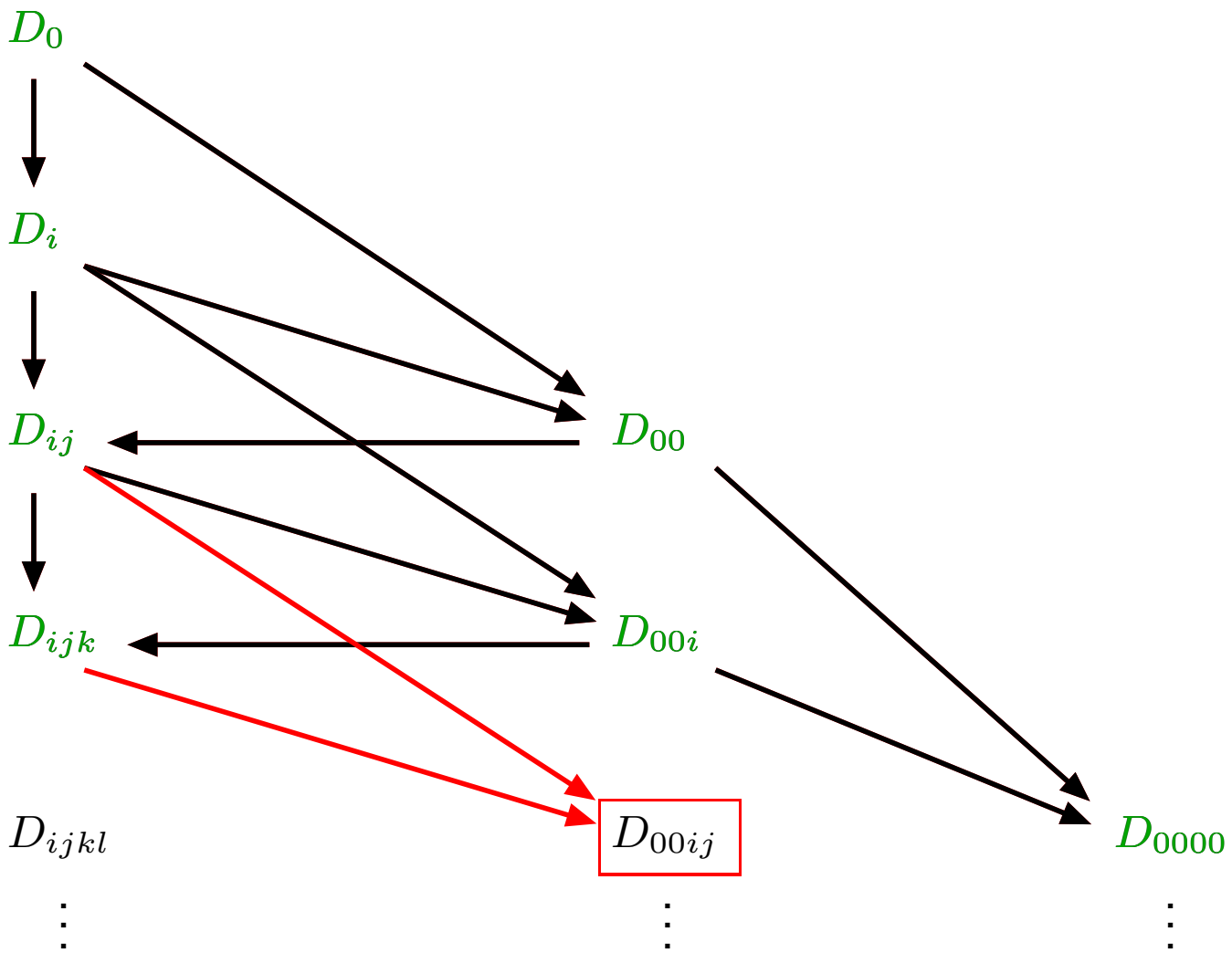
PV reduction:

step 4a



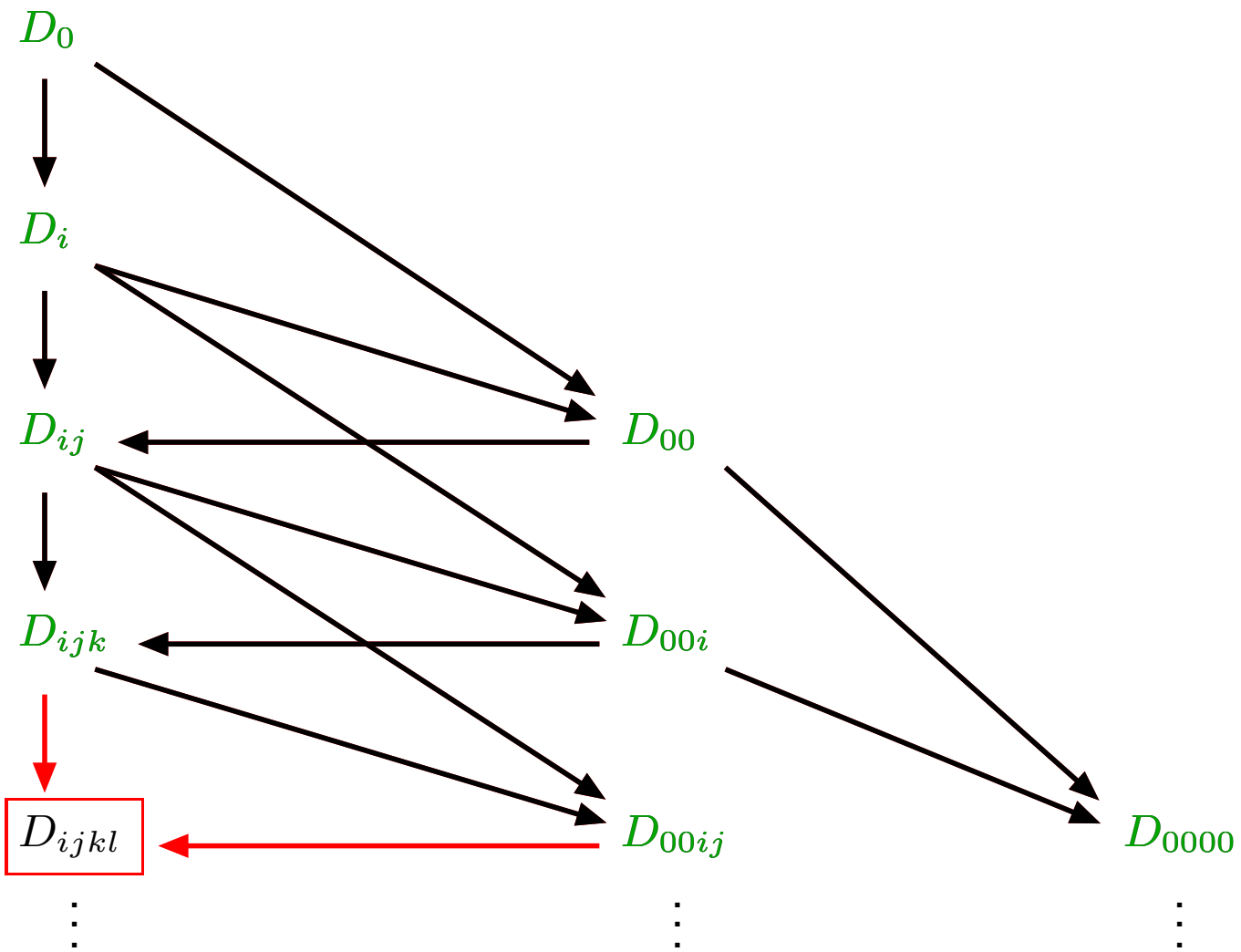
Example: 4-point tensor integrals

PV reduction: **step 4b**



Example: 4-point tensor integrals

PV reduction: **step 4c**



Reduction for small Gram determinant

PV relations rewritten

$$\tilde{X}_{0j} D_{i_1\ldots i_P} = 2\sum_{n=1}^{N-1} \tilde{Z}_{jn} \sum_{r=1}^{P} \delta_{ni_r} D_{00i_1\ldots\hat{i}_r\ldots i_P} + \det(Z) D_{ji_1\ldots i_P} + C\text{'s}$$

$$\tilde{Z}_{kl} D_{00i_1\ldots i_P} \propto \sum_{n,m=1}^{N-1} \tilde{\tilde{Z}}_{(kn)(lm)} \Bigg[f_n f_m D_{i_1\ldots i_P} + 2\sum_{r=1}^{P} (f_n \delta_{mi_r} + f_m \delta_{ni_r}) D_{00i_1\ldots\hat{i}_r\ldots i_P}$$
$$+ 4 \sum_{\substack{r,s=1\\ r\neq s}}^{P} \delta_{ni_r}\delta_{mi_s} D_{0000i_1\ldots\hat{i}_r\ldots\hat{i}_s\ldots i_P} \Bigg] + 2m_0^2 \tilde{Z}_{kl} D_{i_1\ldots i_P} - \det(Z) D_{kli_1\ldots i_P} + C\text{'s}$$

$\tilde{X}$, $\tilde{Z}$, $\tilde{\tilde{Z}}$ = minors (subdeterminants) of X and Z (j, k, l chosen appropriately)

$\hookrightarrow$ coefficients $\underbrace{D_{ij\ldots}}_{\text{rank } P}$ and $\underbrace{D_{00i\ldots}}_{\text{rank } P+1}$ from lower-rank terms $\underbrace{D_{ij\ldots}}_{\text{rank } P-1}$ and $\underbrace{D_{00i\ldots}}_{\text{rank } P}$

up to suppressed higher-rank terms $\det(Z)\ \underbrace{D_{ij\ldots}}_{\text{rank } P+1}$

$\hookrightarrow$ equations suited for iteration for small $\det(Z)$:

D's directly from C's up to terms suppressed by $\det(Z)$

Example: 4-point tensor integrals

Expansion for small Gram determinant: **step 0**

$\boxed{D_0^{(0)}}$

starting point: all D tensors = 0

$\uparrow$

D_i

D_{ij} D_{00}

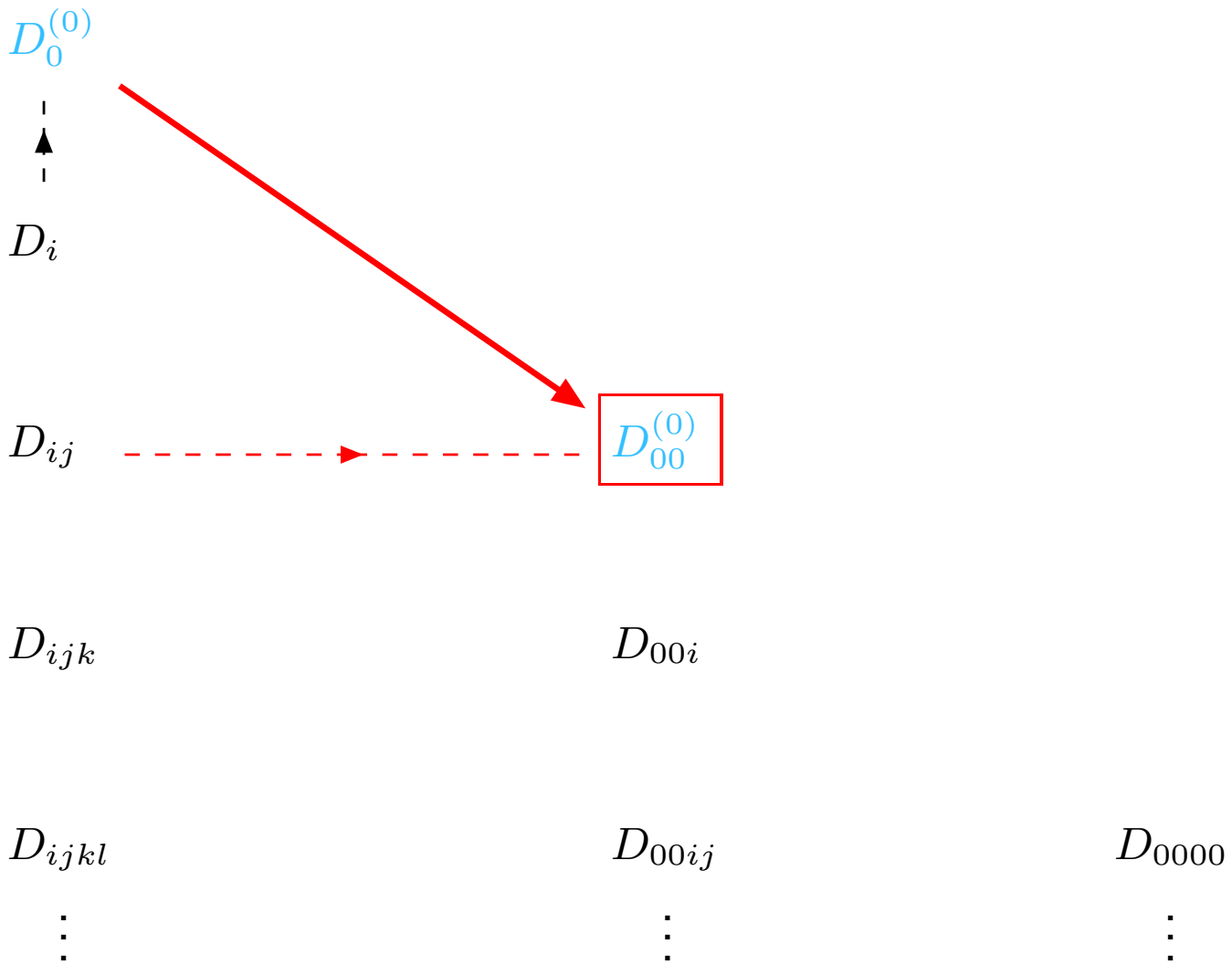
D_{ijk} D_{00i}

D_{ijkl} D_{00ij} D_{0000}

$\vdots$ $\vdots$ $\vdots$

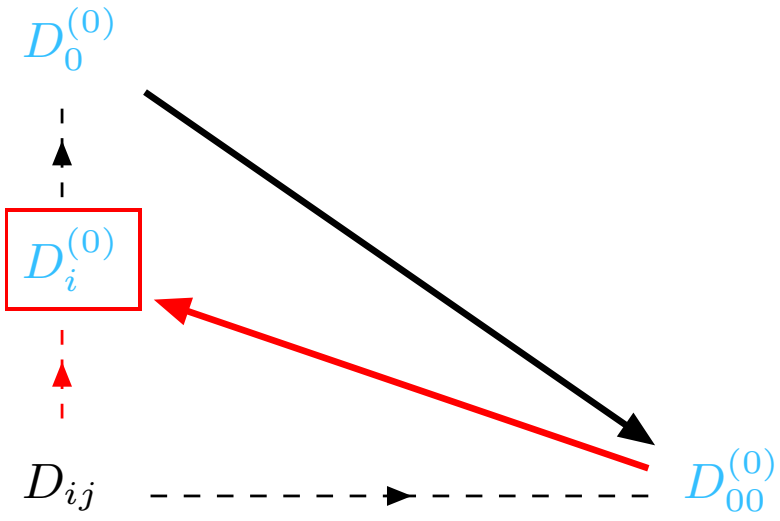
Example: 4-point tensor integrals

Expansion for small Gram determinant: **step 1a**



Example: 4-point tensor integrals

Expansion for small Gram determinant: **step 1b**



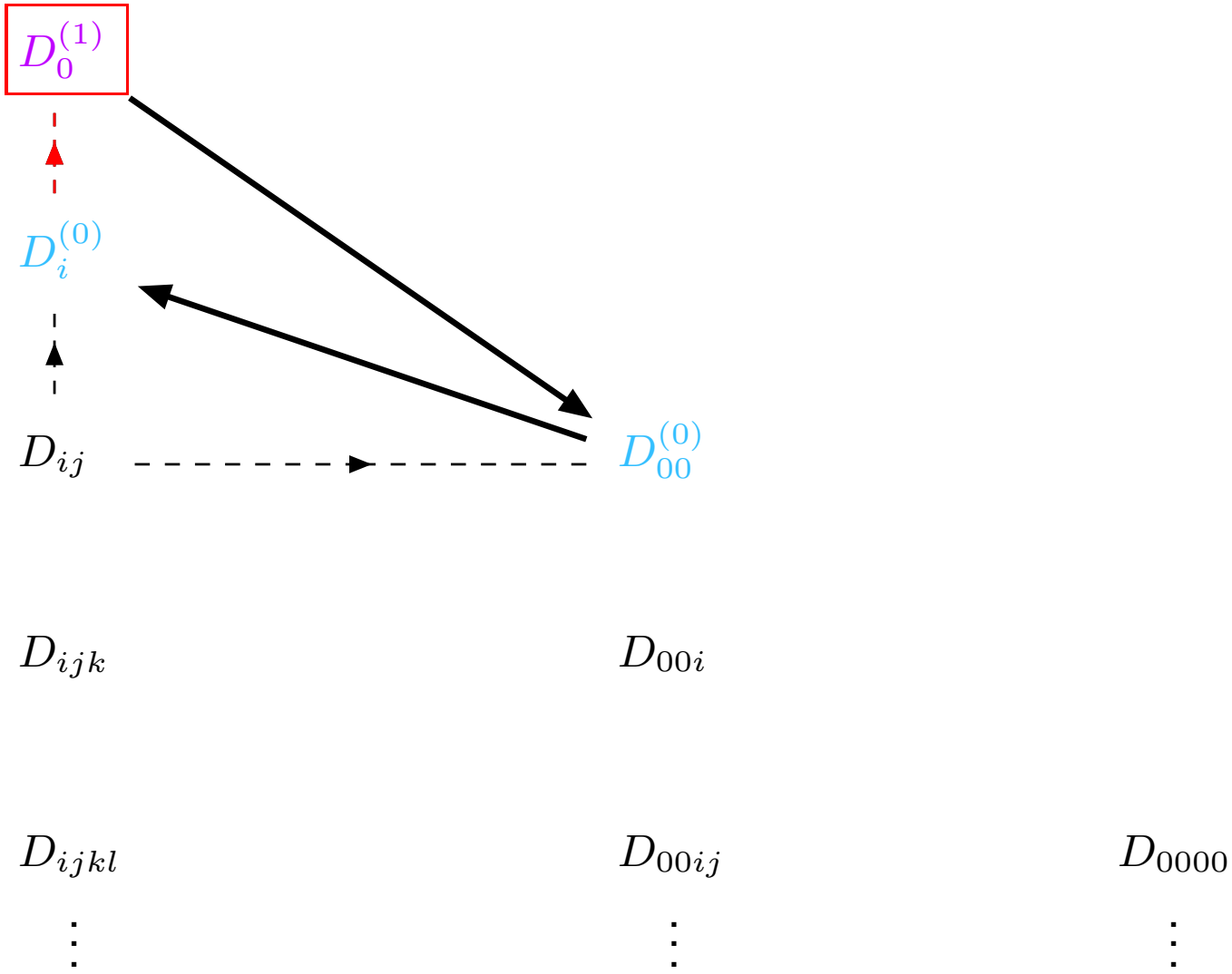
D_{ijk} $\qquad$ D_{00i}

D_{ijkl} $\qquad$ D_{00ij} $\qquad$ D_{0000}

$\vdots$ $\qquad$ $\vdots$ $\qquad$ $\vdots$

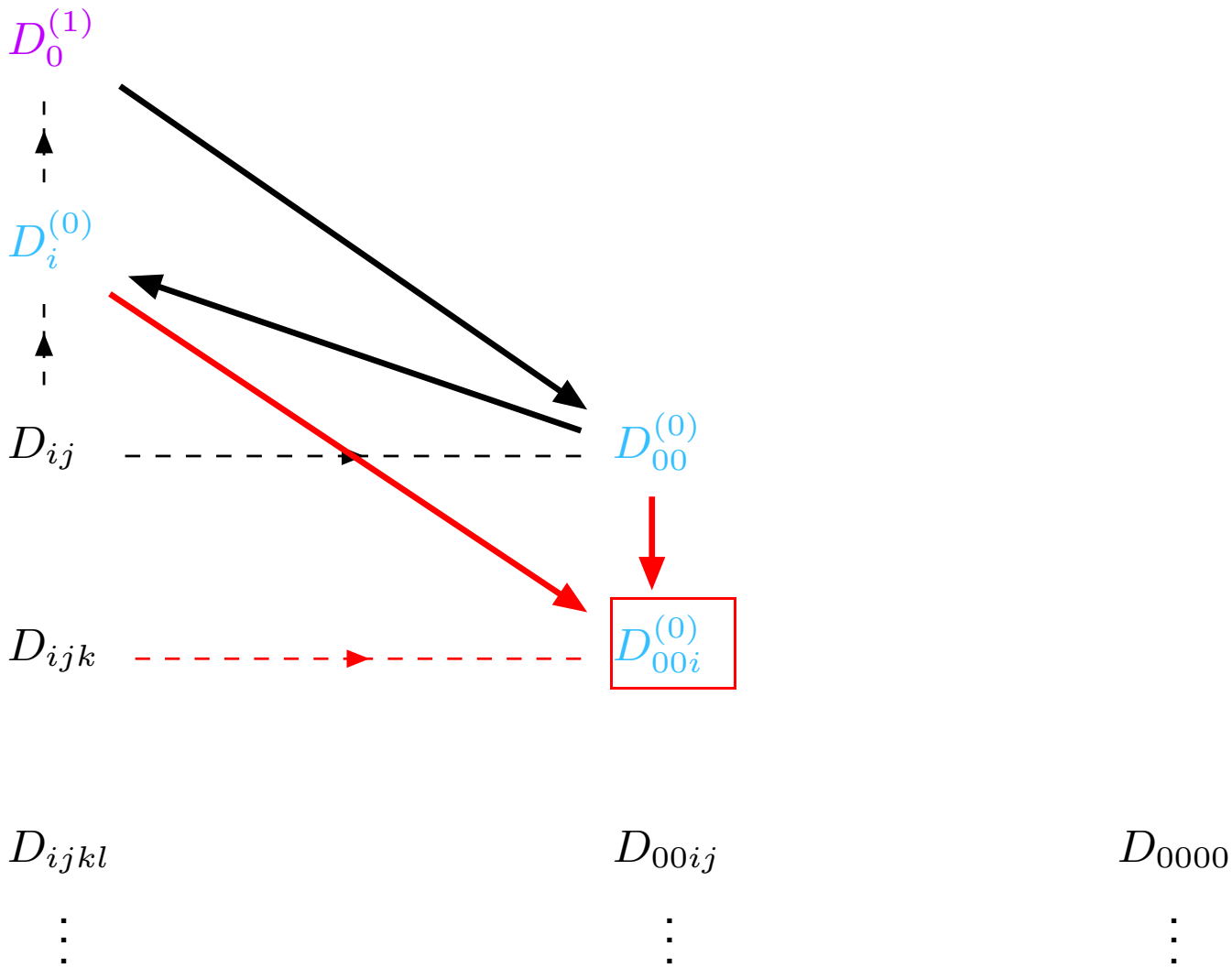
Example: 4-point tensor integrals

Expansion for small Gram determinant: **step 1.0**



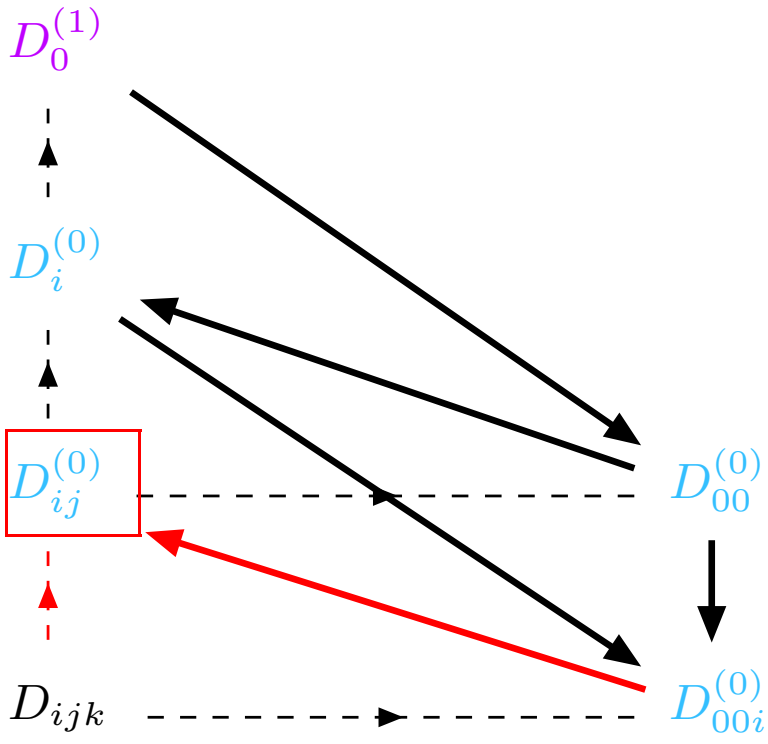
Example: 4-point tensor integrals

Expansion for small Gram determinant: **step 2a**



Example: 4-point tensor integrals

Expansion for small Gram determinant: **step 2b**

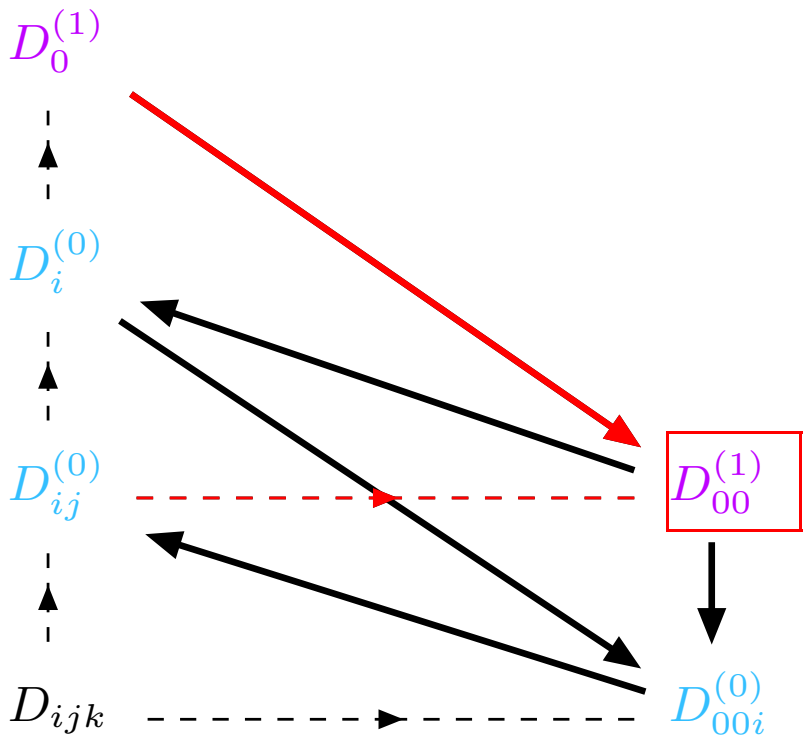


D_{ijkl} $\quad$ D_{00ij} $\quad$ D_{0000}

$\vdots$ $\quad$ $\vdots$ $\quad$ $\vdots$

Example: 4-point tensor integrals

Expansion for small Gram determinant: **step 2.1a**

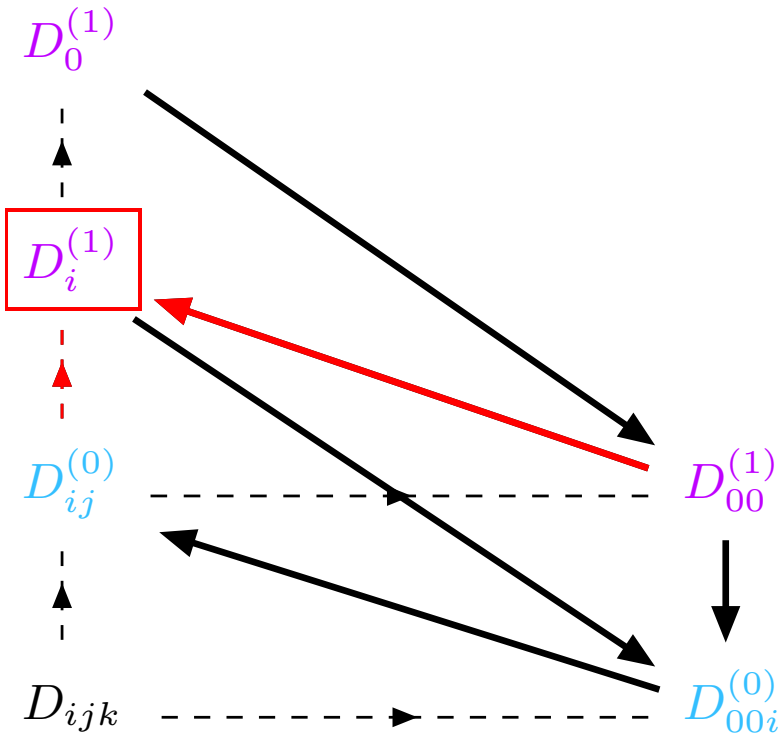


D_{ijkl} D_{00ij} D_{0000}

$\vdots$ $\vdots$ $\vdots$

Example: 4-point tensor integrals

Expansion for small Gram determinant: **step 2.1b**

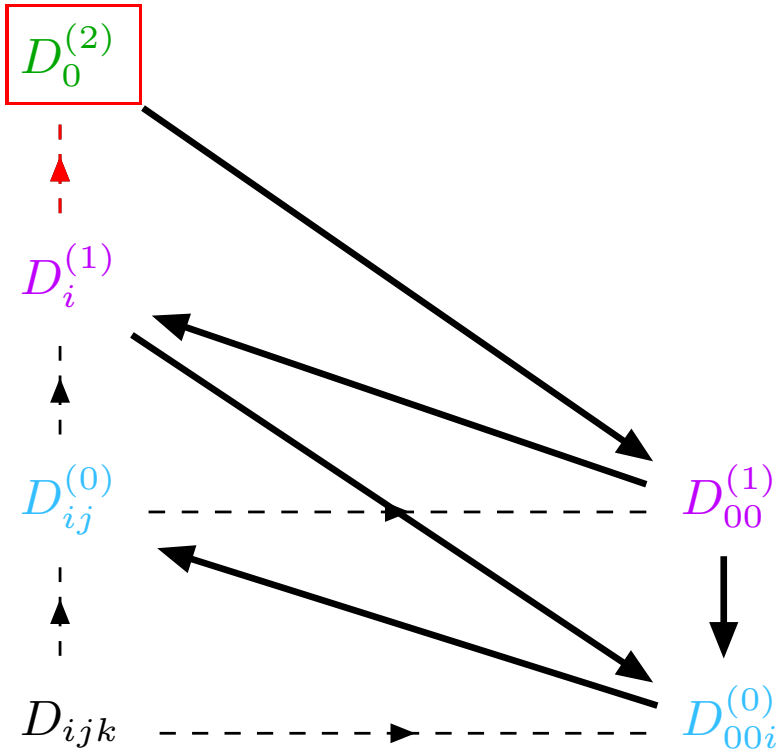


D_{ijkl} D_{00ij} D_{0000}

$\vdots$ $\vdots$ $\vdots$

Example: 4-point tensor integrals

Expansion for small Gram determinant: **step 2.0**

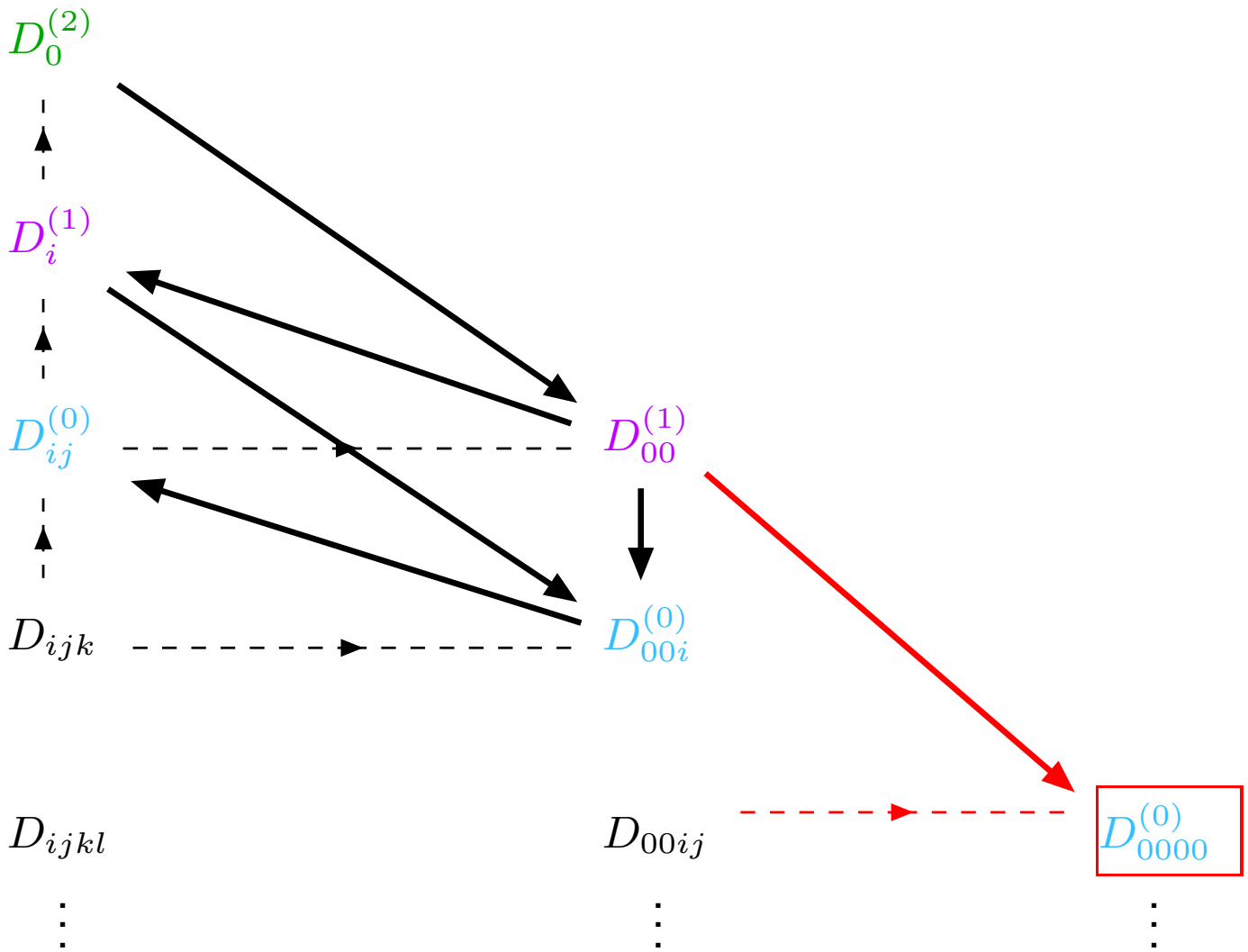


D_{ijkl} $\quad$ D_{00ij} $\quad$ D_{0000}

$\vdots$ $\quad$ $\vdots$ $\quad$ $\vdots$

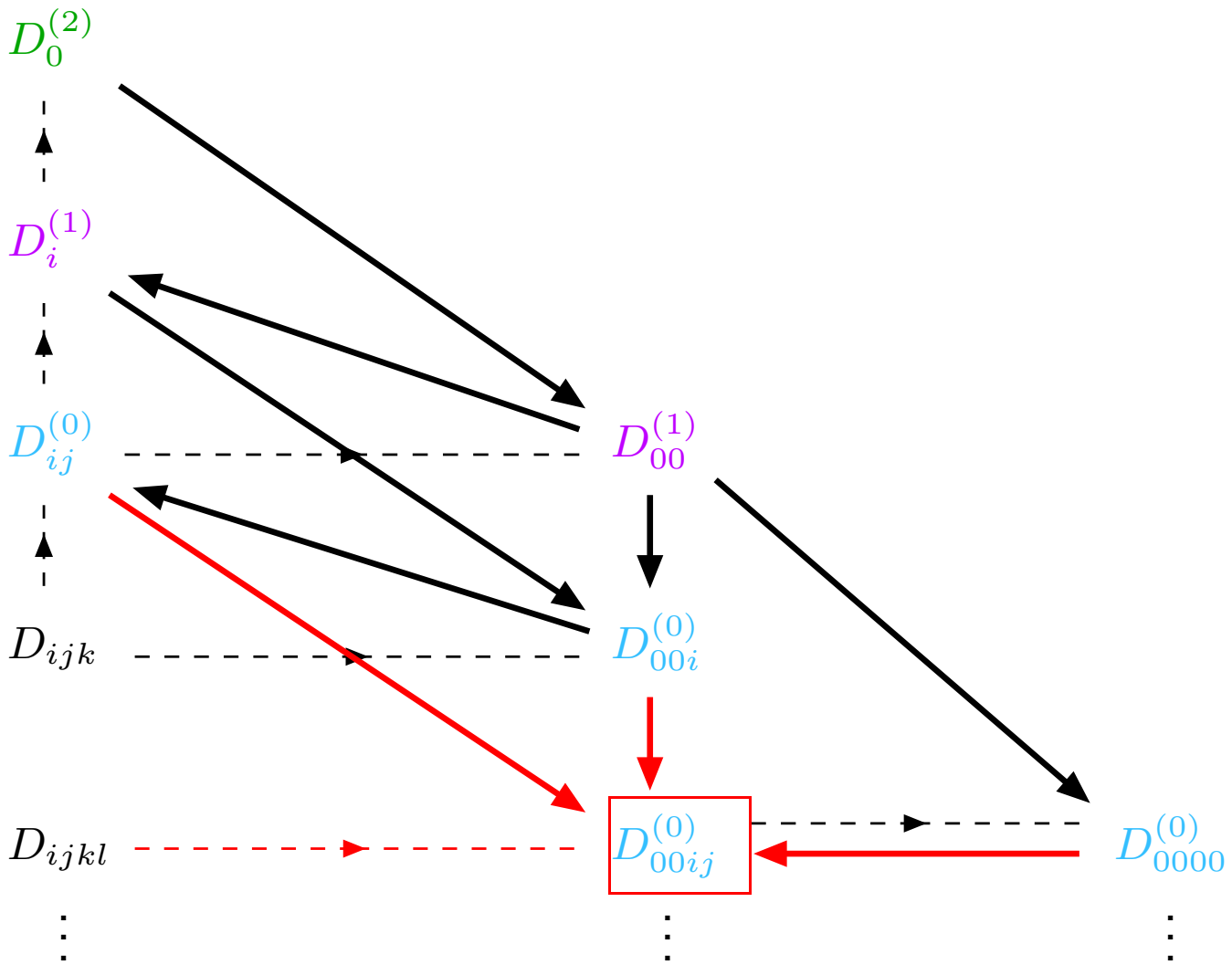
Example: 4-point tensor integrals

Expansion for small Gram determinant: **step 3a**



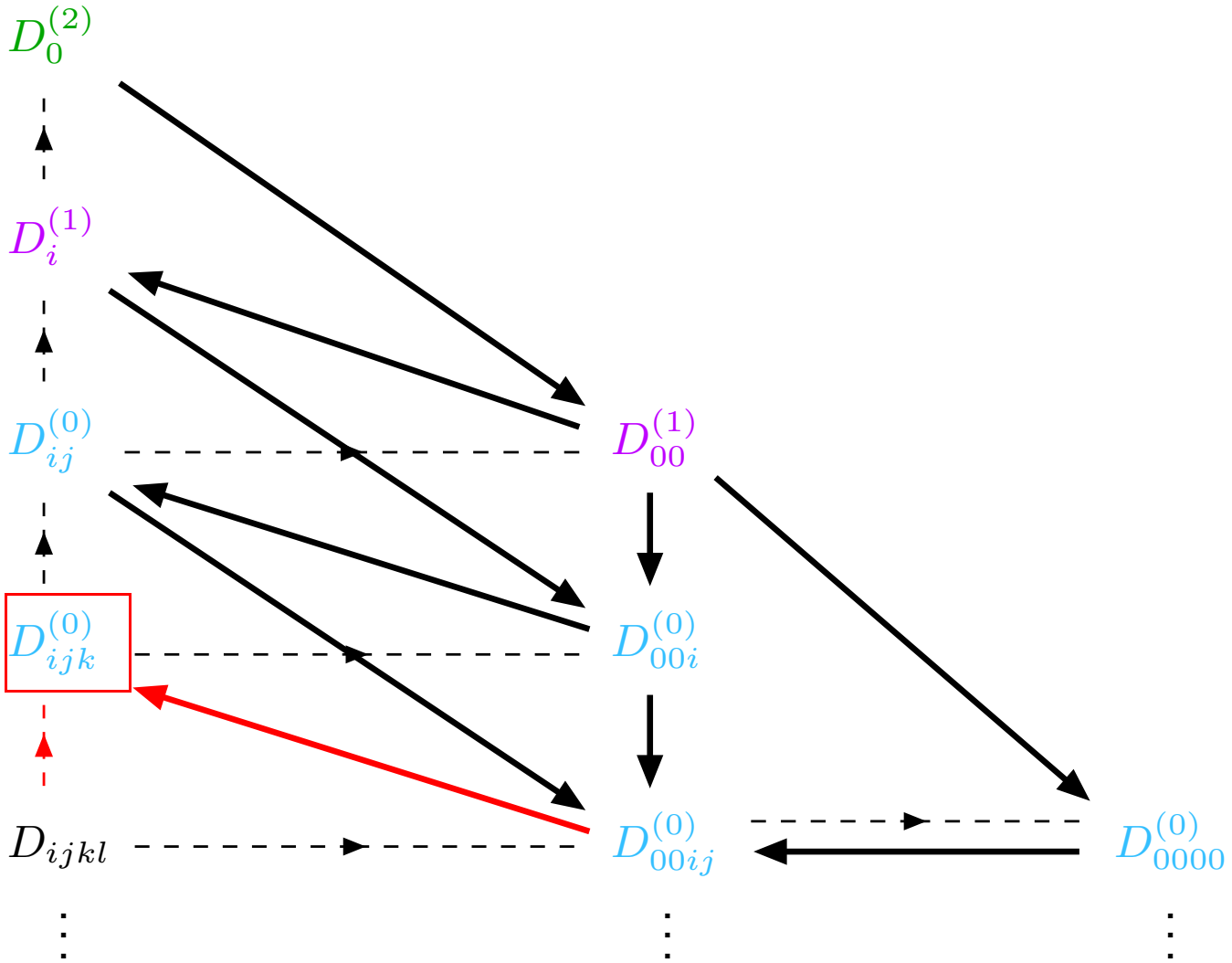
Example: 4-point tensor integrals

Expansion for small Gram determinant: **step 3b**



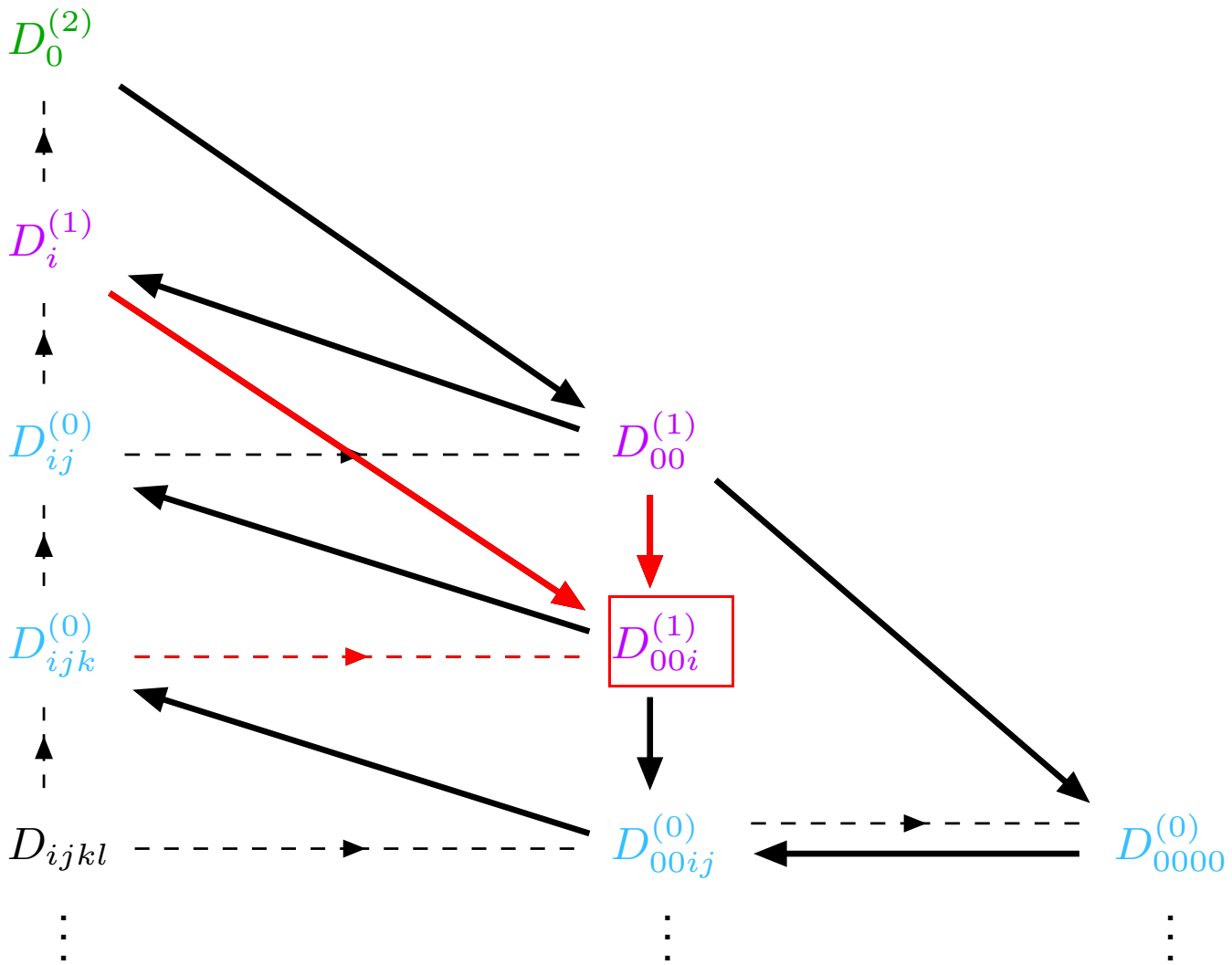
Example: 4-point tensor integrals

Expansion for small Gram determinant: **step 3c**



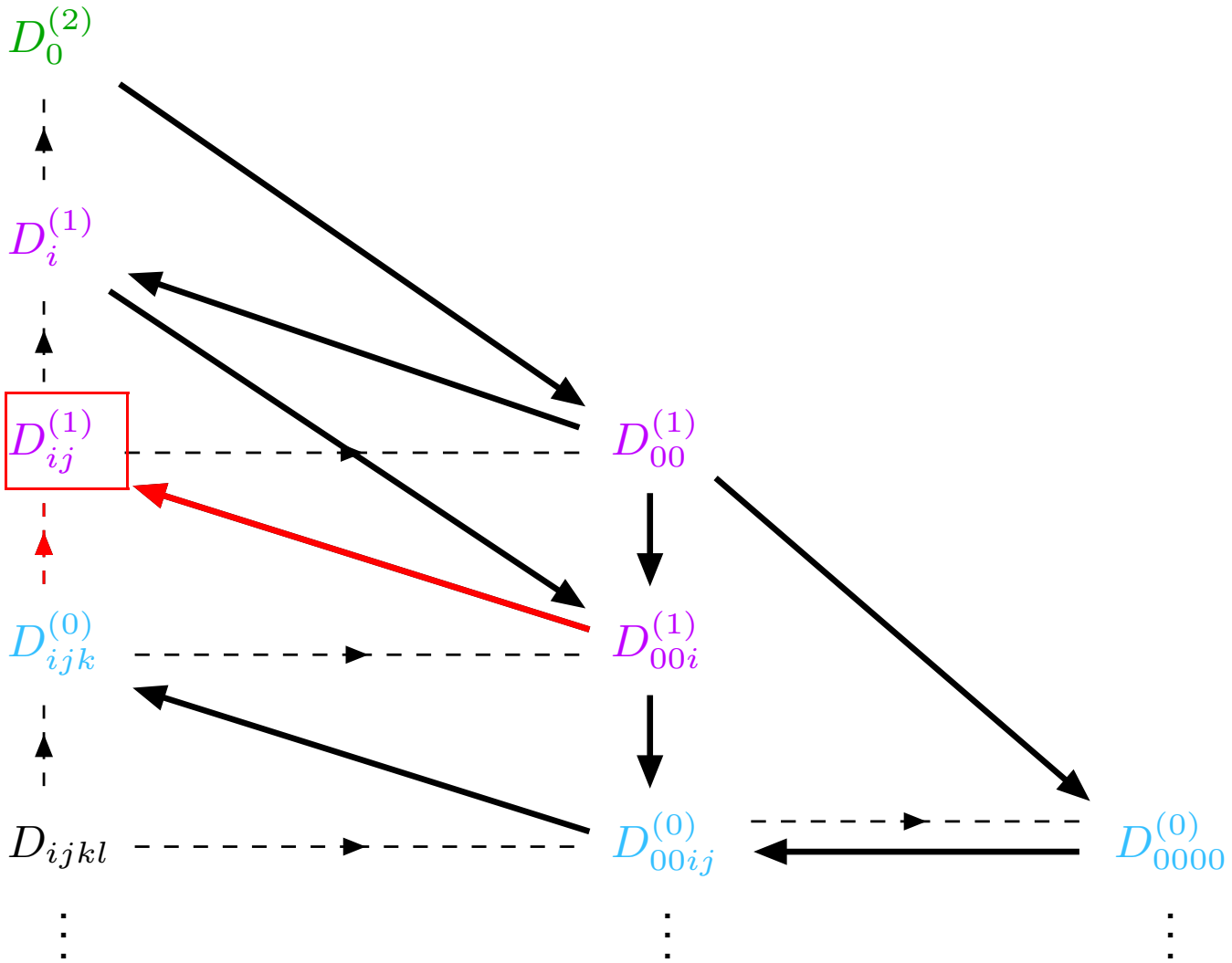
Example: 4-point tensor integrals

Expansion for small Gram determinant: **step 3.2a**



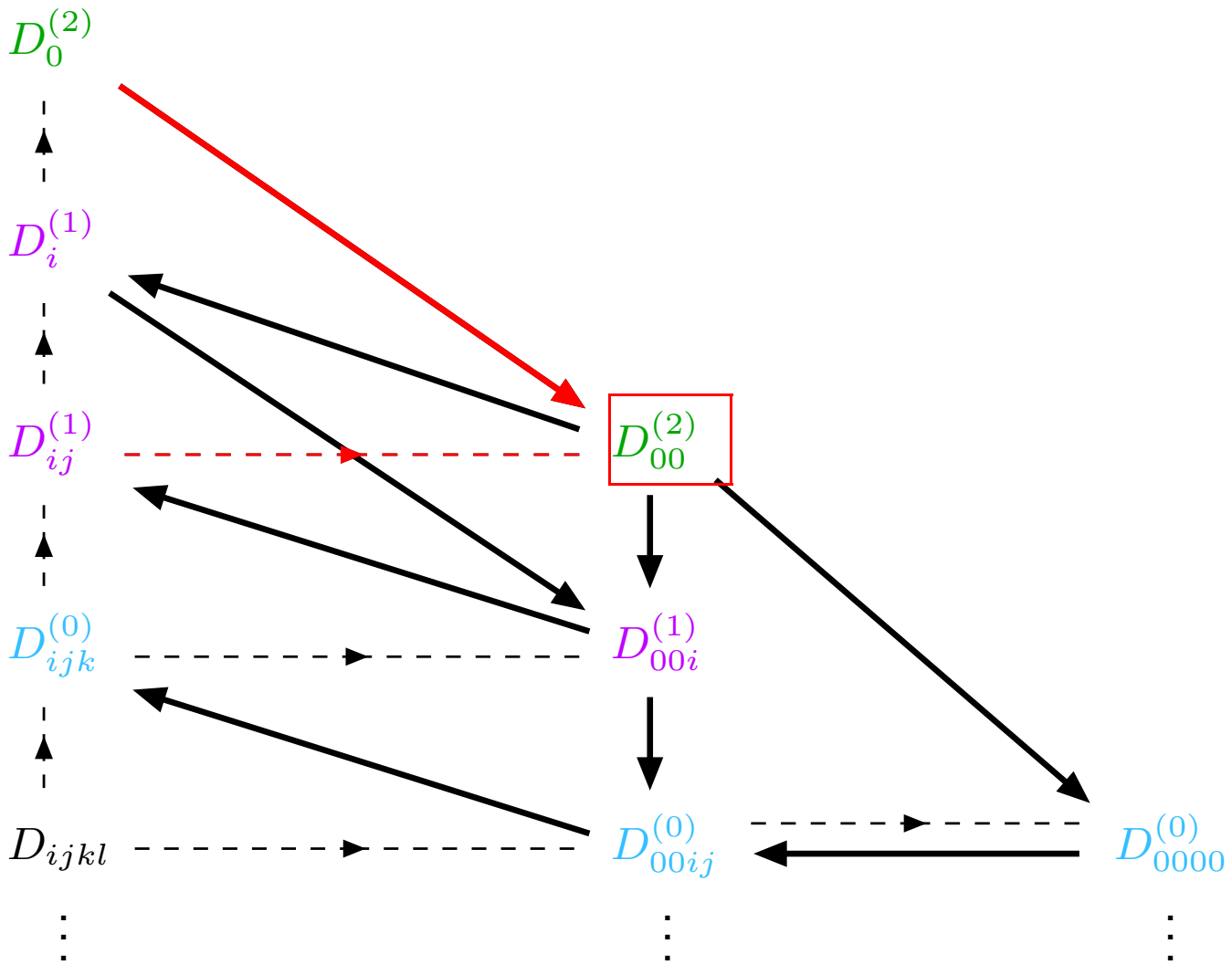
Example: 4-point tensor integrals

Expansion for small Gram determinant: **step 3.2b**



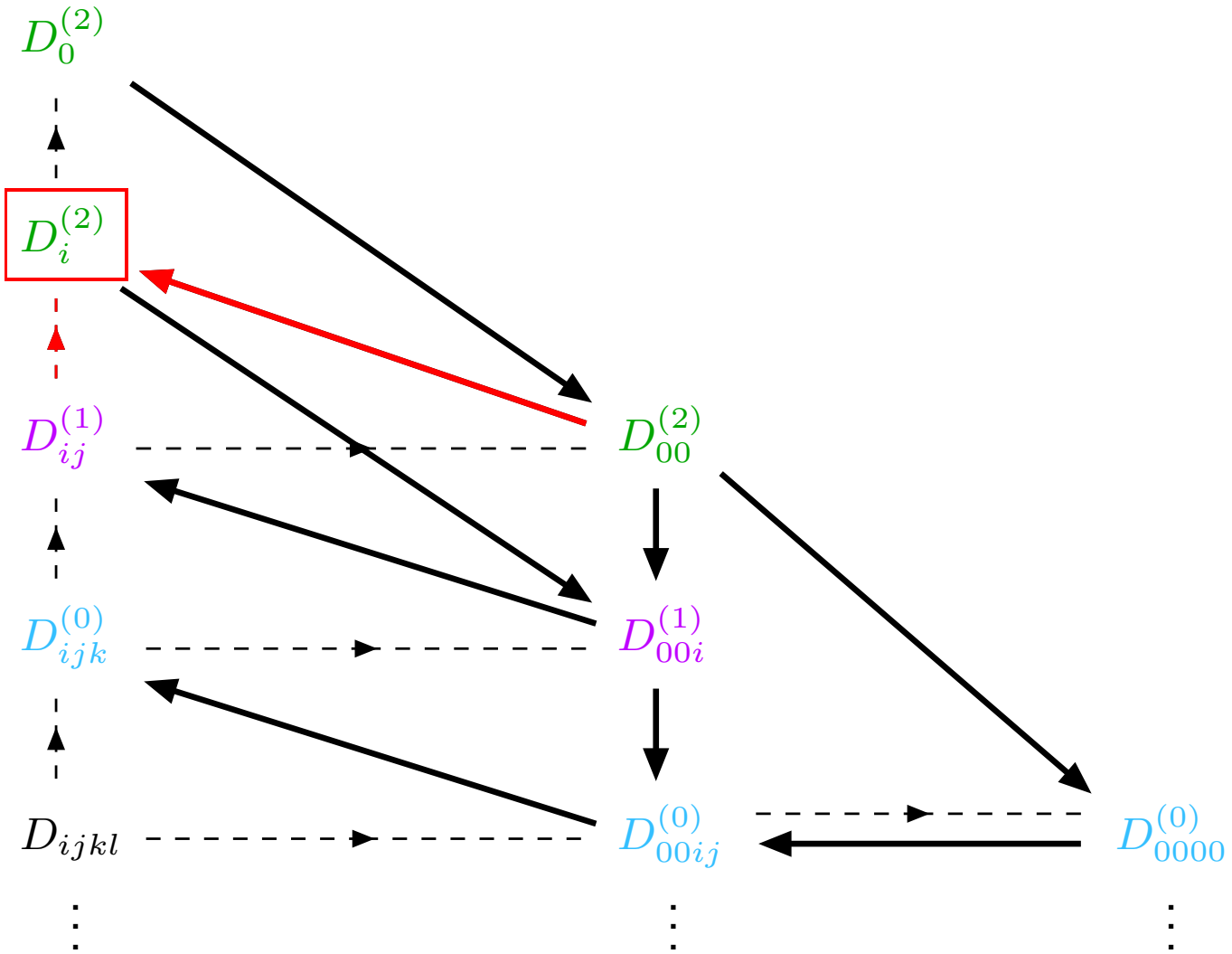
Example: 4-point tensor integrals

Expansion for small Gram determinant: **step 3.1a**



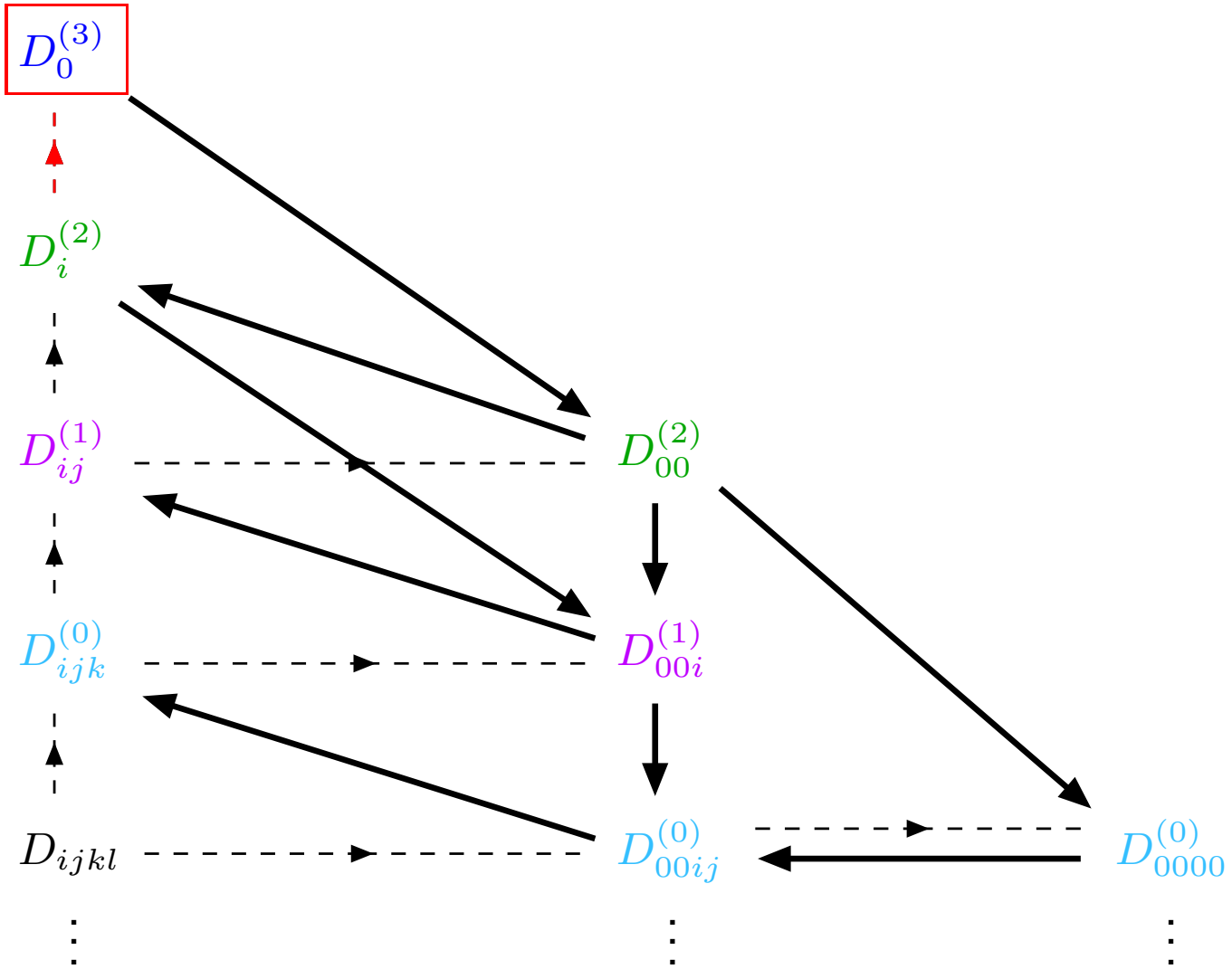
Example: 4-point tensor integrals

Expansion for small Gram determinant: **step 3.1b**



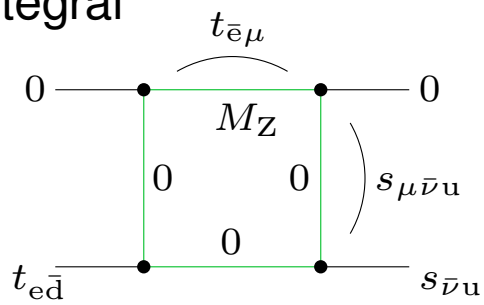
Example: 4-point tensor integrals

Expansion for small Gram determinant: **step 3.0**

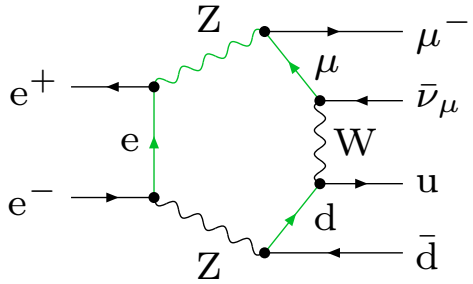


A typical example with small Gram determinant

box integral

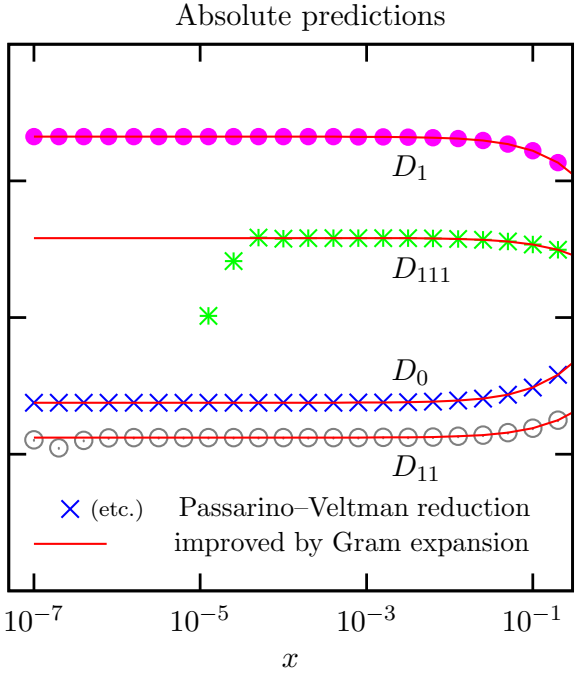


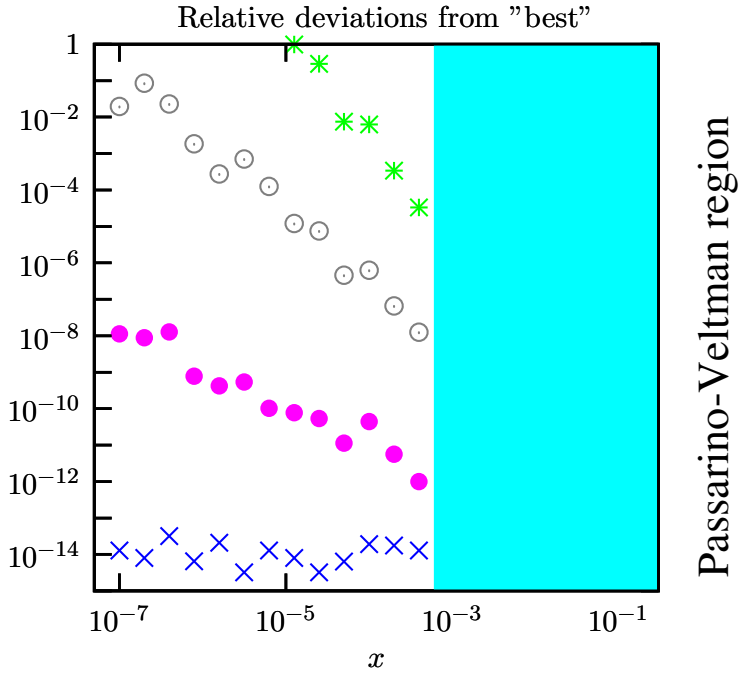
appears, e.g., in subgraph of diagram



Gram det.: $\det(Z) \to 0$ if $t_{e\bar{d}} \to t_{\rm crit} \equiv \dfrac{s_{\mu\bar{\nu}u}(s_{\mu\bar{\nu}u} - s_{\bar{\nu}u} + t_{\bar{e}\mu})}{s_{\mu\bar{\nu}u} - s_{\bar{\nu}u}}$

numerical comparison: maximal tensor rank $= 6$ (similar to $ee \to 4f$ application)





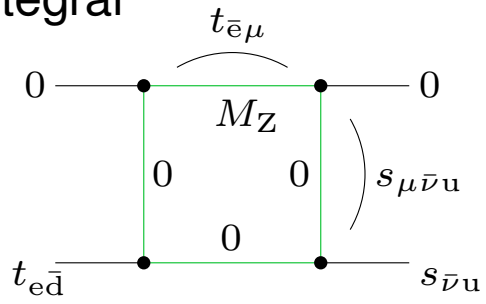
$$x \equiv \frac{t_{e\bar{d}}}{t_{\rm crit}} - 1$$

$$\begin{aligned} s_{\mu\bar{\nu}u} &= +2\times10^4\ \mathrm{GeV}^2 \\ s_{\bar{\nu}u} &= +1\times10^4\ \mathrm{GeV}^2 \\ t_{\bar{e}\mu} &= -4\times10^4\ \mathrm{GeV}^2 \\ t_{\rm crit} &= -6\times10^4\ \mathrm{GeV}^2 \end{aligned}$$

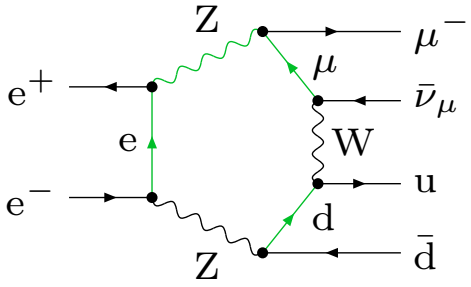
PV reduction breaks down, but Gram exp. stable for $\det(Z) \to 0$!

A typical example with small Gram determinant

box integral

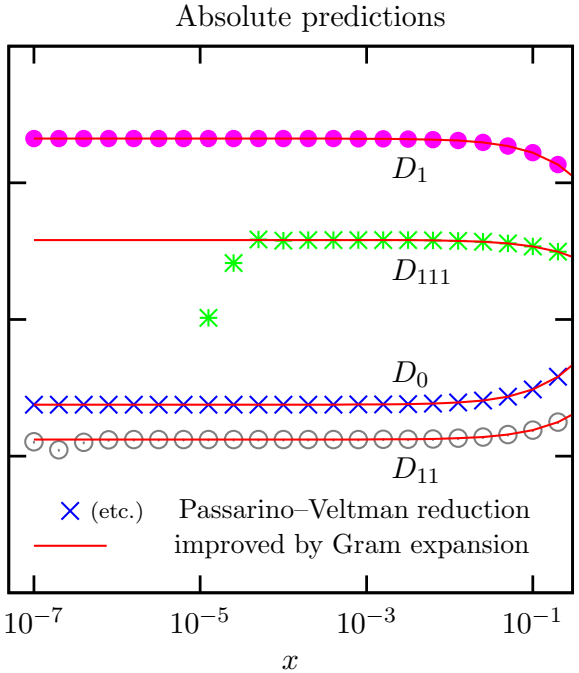


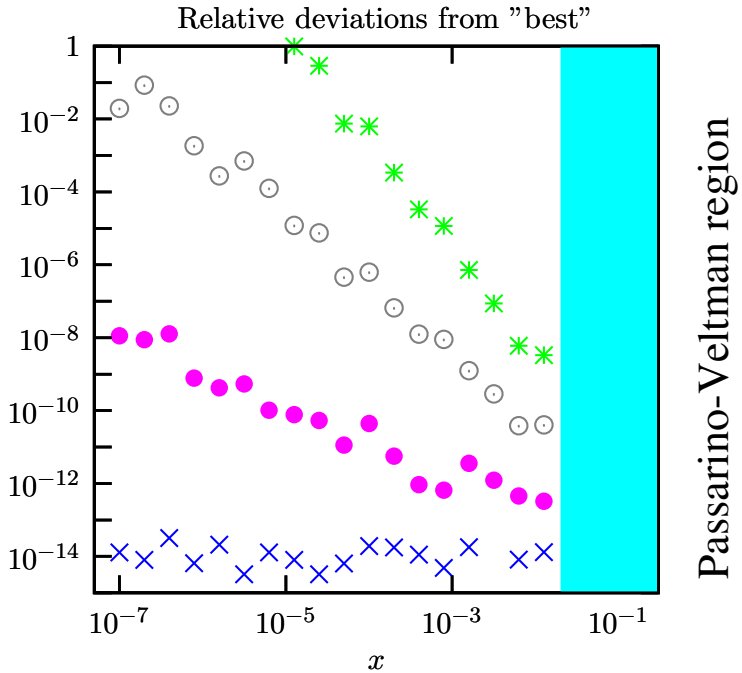
appears, e.g., in subgraph of diagram



Gram det.: $\det(Z) \to 0$ if $t_{e\bar{d}} \to t_{\rm crit} \equiv \dfrac{s_{\mu\bar{\nu}u}(s_{\mu\bar{\nu}u} - s_{\bar{\nu}u} + t_{\bar{e}\mu})}{s_{\mu\bar{\nu}u} - s_{\bar{\nu}u}}$

numerical comparison: maximal tensor rank $= 12$





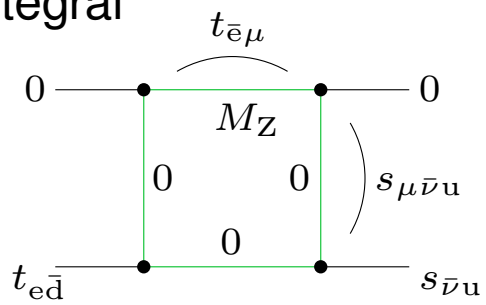
$$x \equiv \frac{t_{e\bar{d}}}{t_{\rm crit}} - 1$$

$$\begin{aligned} s_{\mu\bar{\nu}u} &= +2\times10^4\ \mathrm{GeV}^2 \\ s_{\bar{\nu}u} &= +1\times10^4\ \mathrm{GeV}^2 \\ t_{\bar{e}\mu} &= -4\times10^4\ \mathrm{GeV}^2 \\ t_{\rm crit} &= -6\times10^4\ \mathrm{GeV}^2 \end{aligned}$$

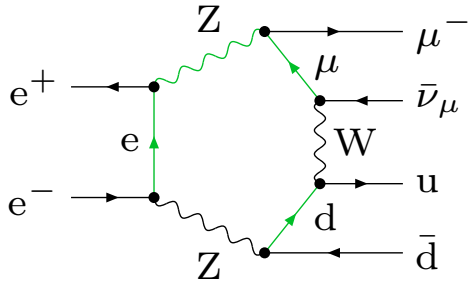
PV reduction breaks down, but Gram exp. stable for $\det(Z) \to 0$!

A typical example with small Gram determinant

box integral

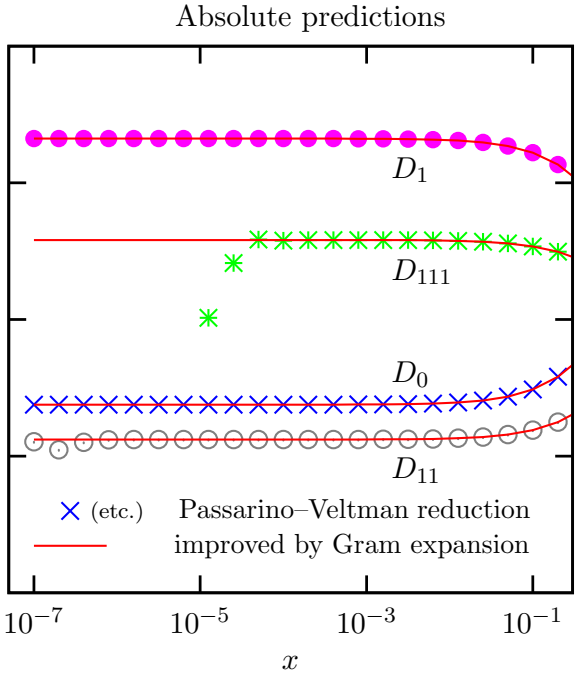


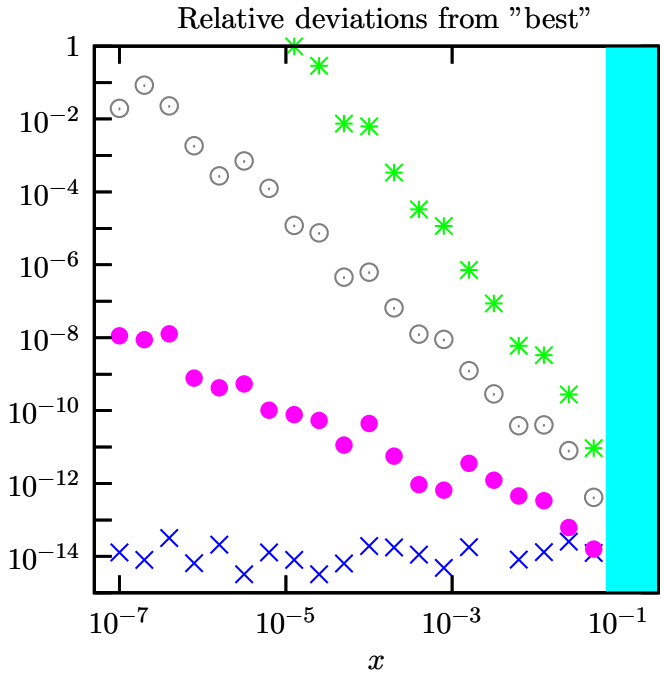
appears, e.g., in subgraph of diagram



Gram det.: $\det(Z) \to 0$ if $t_{e\bar{d}} \to t_{\rm crit} \equiv \dfrac{s_{\mu\bar{\nu}u}(s_{\mu\bar{\nu}u} - s_{\bar{\nu}u} + t_{\bar{e}\mu})}{s_{\mu\bar{\nu}u} - s_{\bar{\nu}u}}$

numerical comparison: maximal tensor rank $= 25$





$$x \equiv \frac{t_{e\bar{d}}}{t_{\rm crit}} - 1$$

$$\begin{aligned} s_{\mu\bar{\nu}u} &= +2\times10^4\ \mathrm{GeV}^2 \\ s_{\bar{\nu}u} &= +1\times10^4\ \mathrm{GeV}^2 \\ t_{\bar{e}\mu} &= -4\times10^4\ \mathrm{GeV}^2 \\ t_{\rm crit} &= -6\times10^4\ \mathrm{GeV}^2 \end{aligned}$$

PV reduction breaks down, but Gram exp. stable for $\det(Z) \to 0$!

Reduction of tensor integrals (summary)

- PV Veltman reduction also valid for 2-point and 1-point functions
  $\Rightarrow$ all one-loop tensor integrals can be reduced to scalar 1-, 2-, 3-, and 4-point functions D_0, C_0, B_0, A_0
  $\Rightarrow$ all one-loop amplitudes can be expressed in terms of D_0, C_0, B_0, A_0
  basis for unitarity methods
- small Gram determinants: D_0, C_0, B_0, A_0 become dependent
  $\Rightarrow$ calculation of corresponding coefficients becomes unstable
- 1- and 2-point integrals: direct calculation of tensor integrals possible and numerically stable (avoids inverse Gram determinants)
- expansion methods effectively eliminate dependent scalar integrals from basis, e.g. D_0 expressed in terms of C_0, B_0, A_0
  $\Rightarrow$ stable results for small Gram determinants
- for small $\tilde{X}_{0j}$ or $\tilde{Z}_{kl}$ alternative expansions required
- semi-numerical method: slower, useful for checks
  numerical calculation of logarithmic Feynman-parameter integral and algebraic reduction to this basis integral

Step 4: evaluation of scalar integrals

Use explicit analytic results

- non-singular cases with finite masses

  't Hooft, Veltman '79; Denner, Nierste, Scharf '91, van Oldenborgh '91, Dao, Le '09

- infrared-singular cases in mass regularization

  Beenakker, Denner '90

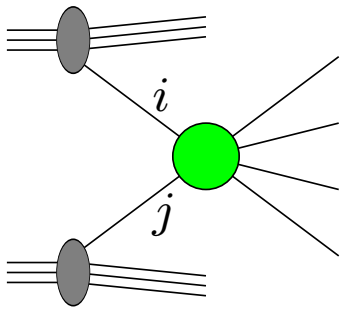
- mass-singular cases in dimensional regularization

  Bern, Dixon, Kosower '94; Duplancic, Nizic '01; van Hameren, Vollinga, Weinzierl '03, Ellis, Zanderighi '07

Calculational techniques II

real corrections

Hadronic cross section



$$i, j = \mathrm{u}, \mathrm{d}, \mathrm{s}, \mathrm{c}, \mathrm{b}, \mathrm{g}, \gamma$$

$$\sigma_{\mathrm{pp}}(P_1, P_2) = \int_0^1 \mathrm{d}x_1 \int_0^1 \mathrm{d}x_2 \sum_{i,j} f_i(x_1, \mu_{\mathrm{F}}^2) f_j(x_2, \mu_{\mathrm{F}}^2) \hat{\sigma}_{ij}(x_1 P_1, x_2 P_2)$$

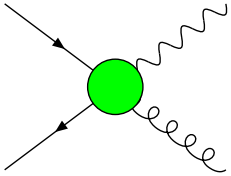
P_i : proton momenta
x_k : momentum fractions
$f_i(x_1, \mu_{\mathrm{F}}^2)$: parton distribution functions
μ_{F} : factorization scale
$\hat{\sigma}_{ij}$: partonic cross sections

many partonic processes, some appear only in higher orders

Example: partonic processes for $\mathrm{pp} \to \mathrm{Z} + \mathrm{jet}$

LO partonic processes:

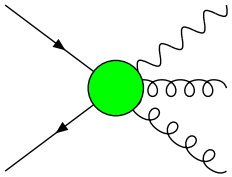
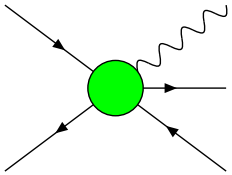
$q\bar{q} \to \mathrm{Z}g,\ qg \to \mathrm{Z}q,\ \bar{q}g \to \mathrm{Z}\bar{q}$ related by crossing

basic amplitude  $\mathcal{O}(\sqrt{\alpha\alpha_\mathrm{s}})$

partonic processes for real NLO QCD: ($\mathrm{pp} \to \mathrm{Z} + 2\mathrm{jets}$)

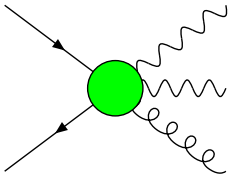
$q\bar{q} \to \mathrm{Z}gg,\ qg \to \mathrm{Z}qg,\ \bar{q}g \to \mathrm{Z}\bar{q}g,\ \ gg \to \mathrm{Z}q\bar{q}$ related by crossing

$q\bar{q} \to \mathrm{Z}q'\bar{q}',\ qq' \to \mathrm{Z}qq',\ \bar{q}\bar{q}' \to \mathrm{Z}\bar{q}\bar{q}'$ related by crossing

basic amplitudes  $\mathcal{O}(\sqrt{\alpha\alpha_\mathrm{s}^2})$,  $\mathcal{O}(\sqrt{\alpha\alpha_\mathrm{s}^2}), \mathcal{O}(\sqrt{\alpha^3})$

partonic processes for real NLO EW: ($\mathrm{pp} \to \mathrm{Z} + \mathrm{jet} + \gamma$)

$$\left.\begin{array}{l} q\bar{q} \to \mathrm{Z}g\gamma,\ qg \to \mathrm{Z}q\gamma,\ \bar{q}g \to \mathrm{Z}\bar{q}\gamma \\ q\gamma \to \mathrm{Z}qg,\ \bar{q}\gamma \to \mathrm{Z}\bar{q}g,\ g\gamma \to \mathrm{Z}q\bar{q} \end{array}\right\} \text{related by crossing}$$

basic amplitude  $\mathcal{O}(\sqrt{\alpha^2\alpha_\mathrm{s}})$

Factorization of electromagnetic initial-state singularities

Analogy to QCD-improved parton model:

collinear splittings $q \to q\gamma$, $\gamma \to q\bar{q}$ lead to quark mass singularities

$\hookrightarrow$ factorize $\alpha \ln m_q$ singularities and absorb into redefined PDFs

MRST2004QED set of $\mathcal{O}(\alpha)$-corrected PDFs Martin, Roberts, Stirling, Thorne '04

$\hookrightarrow$ should be used if EW $\mathcal{O}(\alpha)$ corrections are included

- DIS factorization scheme for $\mathcal{O}(\alpha)$ corrections
- additional real corrections from photons in initial state

recipe for factorization of mass singularities

$$\mathrm{d}\hat{\sigma}_{ij} \to \mathrm{d}\hat{\sigma}_{ij}^{\mathrm{fact}} = \mathrm{d}\hat{\sigma}_{ij} - \mathrm{d}\hat{\sigma}_{ij}^{\mathrm{fact,sing}}$$

$$\mathrm{d}\hat{\sigma}_{ij}^{\mathrm{fact,sing}}(p_1,p_2) = \int_0^1 \mathrm{d}z \left[f_i^{\mathrm{sing}}(z,\mu_{\mathrm{F}}^2)\, \mathrm{d}\hat{\sigma}_{ij,0}(zp_1,p_2) + f_j^{\mathrm{sing}}(z,\mu_{\mathrm{F}}^2)\, \mathrm{d}\hat{\sigma}_{ij,0}(p_1,zp_2) \right]$$

p_i : parton momenta

$f_i^{\mathrm{sing}}(z,\mu_{\mathrm{F}}^2)$: singular part of parton distribution functions

$\hat{\sigma}_{ij,0}$: LO partonic cross sections

Treatment of mass singularities

Real and virtual corrections involve soft and collinear singularities

$$\underbrace{\mathrm{d}\hat{\sigma}_{ij}^{\mathrm{NLO}}}_{\text{finite}} = \underbrace{\mathrm{d}\hat{\sigma}_{ij}^{\mathrm{virt}}}_{\text{divergent}} + \underbrace{\mathrm{d}\hat{\sigma}_{ij}^{\mathrm{real}}}_{\text{divergent}} - \underbrace{\mathrm{d}\hat{\sigma}_{ij}^{\mathrm{fact}}}_{\text{divergent}}$$

regularization:

- dimensionally (typically for NLO QCD)
- with infinitesimal quark, photon and gluon masses (typically for EW)

stable numerical integration $\Rightarrow$ careful treatment of singularities

two methods

- phase-space slicing method
  Giele, Glover '92; Giele et al. '93; Keller, Laenen '98; Harris, Owens '01
- subtraction method
  Frixione, Kunszt, Signer '96; Catani, Seymour '96; Dittmaier '99; Weinzierl, Phaf '01; Catani, Seymour, Dittmaier, Trocsanyi '02; Diener, Dittmaier, Hollik '05; Dittmaier, Kabelschacht, Kasprzik '08

Phase-space slicing method

Integrate real corrections in soft and collinear regions analytically using soft and collinear approximations

$$\int \mathrm{d}\hat{\sigma}_{ij}^{\mathrm{real}} = \underbrace{\int_{\substack{E>\delta_{\mathrm{s}}\sqrt{\hat{s}}/2 \\ \cos\theta<1-\delta_{\mathrm{c}}}} \mathrm{d}\hat{\sigma}_{ij}^{\mathrm{real}}}_{\substack{\mathrm{finite} \\ \mathrm{cut-dependent}}} + \underbrace{F(\delta_{\mathrm{s}},\delta_{\mathrm{c}}) \otimes \mathrm{d}\hat{\sigma}_{ij}^{\mathrm{LO}}}_{\substack{\mathrm{divergent} \\ \mathrm{cut-dependent} \\ \mathrm{approximative}}}$$

add to virtual corrections

$$\underbrace{\mathrm{d}\hat{\sigma}_{ij}^{\mathrm{NLO}}}_{\mathrm{finite}} = \underbrace{\int_{\substack{E>\delta_{\mathrm{s}}\sqrt{\hat{s}}/2 \\ \cos\theta<1-\delta_{\mathrm{c}}}} \mathrm{d}\hat{\sigma}_{ij}^{\mathrm{real}}}_{\substack{\mathrm{finite} \\ \mathrm{cut-dependent}}} + \underbrace{F(\delta_{\mathrm{s}},\delta_{\mathrm{c}}) \otimes \mathrm{d}\hat{\sigma}_{ij}^{\mathrm{LO}} + \mathrm{d}\hat{\sigma}_{ij}^{\mathrm{virt}} - \mathrm{d}\hat{\sigma}_{ij}^{\mathrm{fact}}}_{\substack{\mathrm{finite} \\ \mathrm{cut-dependent} \\ \mathrm{approximative}}}$$

large cut-off $\Rightarrow$ soft and collinear approximation not valid

small cut-offs $\Rightarrow$ large numerical cancellations, large errors

$\Rightarrow$ need to find a plateau for intermediate cut-offs (cut-off independence)

CPU-consuming but “simple” and important check

Dipole subtraction method

Subtract singular part of real corrections, integrate analytically over soft and collinear region (n final state particles in LO)

$$\int \mathrm{d}\hat{\sigma}_{ij}^{\mathrm{real}} = \underbrace{\int \left[\mathrm{d}\hat{\sigma}_{ij}^{\mathrm{real}} - \sum_{\substack{k,l=1\\k\neq l}}^{n+2} \mathrm{d}\hat{\sigma}_{ij}^{\mathrm{dipole},kl} \right]}_{\substack{\mathrm{finite}\\ \mathrm{regulator-independent}}} + \underbrace{\sum_{\substack{k,l=1\\k\neq l}}^{n+2} \mathcal{F}_{kl} \otimes \mathrm{d}\hat{\sigma}_{ij}^{\mathrm{LO}}}_{\mathrm{divergent}}$$

add back to virtual corrections

$$\underbrace{\mathrm{d}\hat{\sigma}_{ij}^{\mathrm{NLO}}}_{\mathrm{finite}} = \underbrace{\int \left[\mathrm{d}\hat{\sigma}_{ij}^{\mathrm{real}} - \sum_{\substack{k,l=1\\k\neq l}}^{n+2} \mathrm{d}\hat{\sigma}_{ij}^{\mathrm{dipole},kl} \right]}_{\mathrm{finite}} + \underbrace{\sum_{\substack{k,l=1\\k\neq l}}^{n+2} \mathcal{F}_{kl} \otimes \mathrm{d}\hat{\sigma}_{ij}^{\mathrm{LO}} + \mathrm{d}\hat{\sigma}_{ij}^{\mathrm{virt}} - \mathrm{d}\hat{\sigma}_{ij}^{\mathrm{fact}}}_{\mathrm{finite}}$$

one dipole for each pair of charged external particles, no approximation

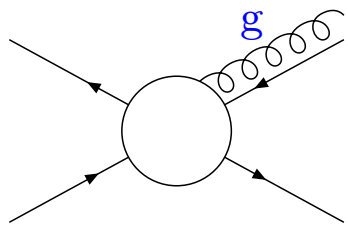
numerically stable/efficient, but many subtraction terms [up to $(n+2)(n+1)$]

Backup

Mass singularities and physical observables

Massless gauge theories

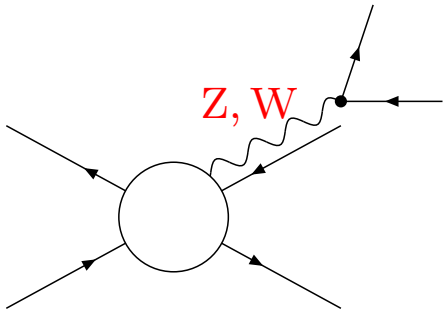
- real emission of soft and collinear massless particles cannot be detected



- $1/\varepsilon$ mass singularities cancel in sum of real and virtual corrections (KLN theorem)

Electroweak theory

- real emission of massive W and Z boson can be detected

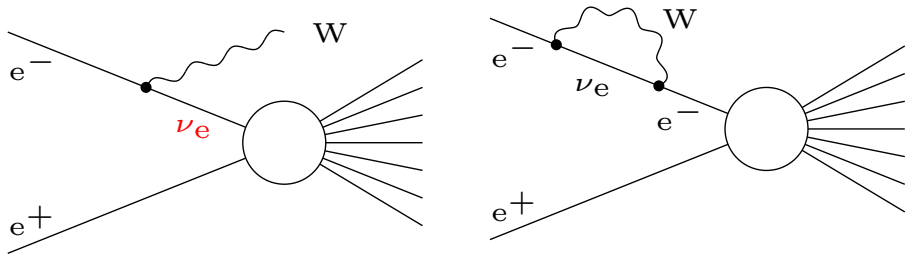


- physical observables receive large $\ln\left(\frac{s}{M_{\mathrm{W}}^2}\right)$ corrections

Non-cancellation of leading logarithms

Electroweak leading (double) logarithms do not cancel in inclusive observables between real and virtual effects! M. Ciafaloni, P. Ciafaloni, Comelli '00

example: W emission from incoming electron



⇒ different hard matrix elements in real and virtual corrections

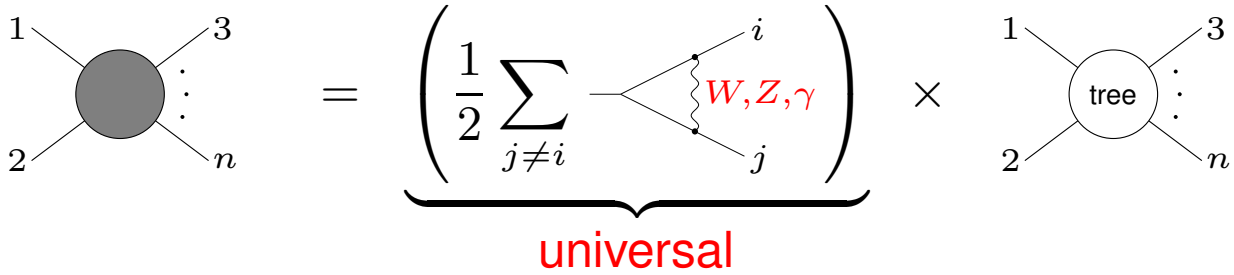
$$\sigma^{\mathrm{LL}}_{\mathrm{inclusive,e^+e^-}} = (\sigma^{\mathrm{LL}}_{0,\mathrm{e^+}\nu_\mathrm{e}} - \sigma^{\mathrm{LL}}_{0,\mathrm{e^+e^-}})\frac{\alpha}{4\pi s_\mathrm{w}^2}\ln^2\frac{s}{M^2} = -\sigma^{\mathrm{LL}}_{\mathrm{inclusive,e^+}\nu_\mathrm{e}}$$

⇒ electroweak correction of 5% at $1\,\mathrm{TeV}$
(strong corrections for $\mathrm{e^+e^-} \to \mathrm{hadrons} \approx 3\%$)

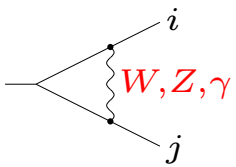
cancellation recovered by summing over complete multiplets
(colour average in QCD!)

Factorization and universality of one-loop EW logarithms

For arbitrary processes $(\mathrm{e}, \nu, \mathrm{u}, \mathrm{d}, \mathrm{t}, \mathrm{b}, \gamma, \mathrm{Z}, \mathrm{W}^{\pm}, \mathrm{H}, \mathrm{g})$ Denner, Pozzorini '01



proven with collinear Ward identities for spontaneously broken gauge theories



$$= \frac{\alpha}{4\pi}\left\{\sum_{V=\gamma,Z,W} I_i^V I_j^V \ln^2\frac{r_{ij}}{M_W^2} + 2 I_i^Z I_j^Z \ln\frac{r_{ij}}{M_W^2}\ln\frac{M_W^2}{M_Z^2} + \gamma_{ij}^{\mathrm{ew}} \ln\frac{s}{M_W^2}\right.$$

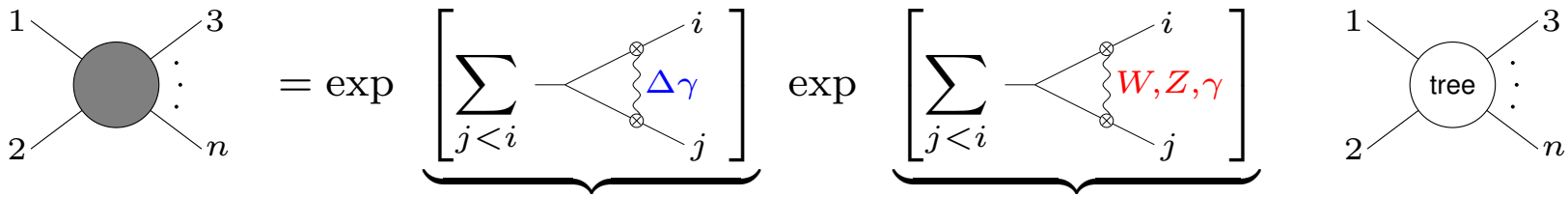
$$\left. + Q_i Q_j \sum_{k=i,j}\left[\ln\frac{r_{ij}}{m_k^2}\ln\frac{M_W^2}{\lambda^2} - \frac{1}{2}\ln^2\frac{M_W^2}{m_k^2} - \ln\frac{M_W^2}{\lambda^2} - \frac{1}{2}\ln\frac{M_W^2}{m_k^2}\right]\right\}$$

simple and general recipe for LL and NLL

Resummation formula based on IREE approach

IREE= infrared evolution equation

double exponentiation resulting from $M_\gamma \ll M_W \sim M_Z$



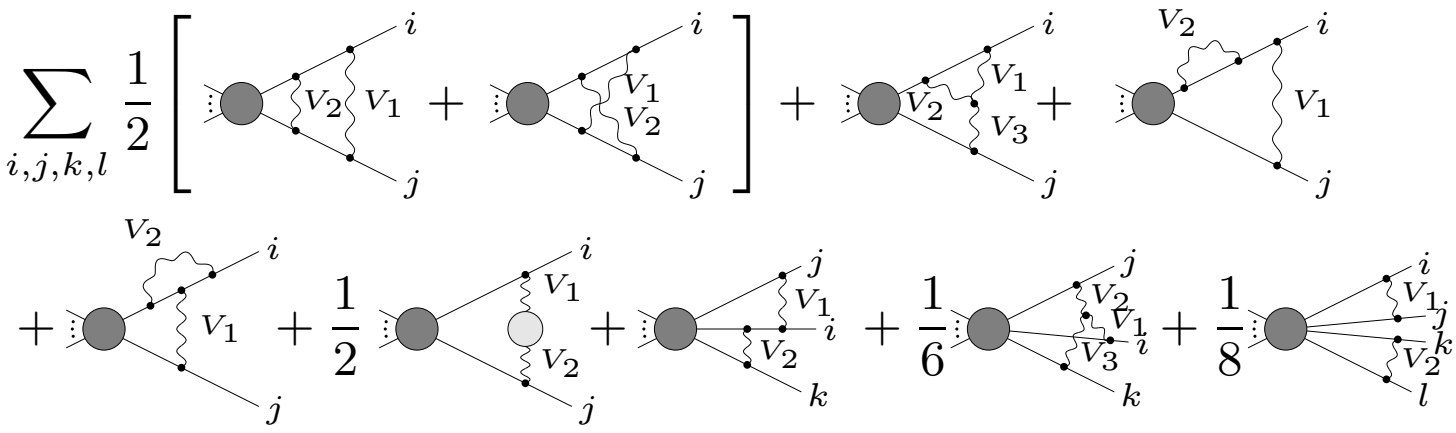
$\ln(M_W/M_\gamma) \equiv$ QED IR sing.

$\ln(s/M_W^2) \equiv$ SU(2)×U(1) theory with $M_\gamma = M_W$

existing results

$$\alpha^2 \Bigg[\underbrace{\underbrace{C_4 \ln^4\left(\frac{s}{M_W^2}\right)}_{\text{Fadin, Lipatov, Martin, Melles (2000)}} + \underbrace{C_3 \ln^3\left(\frac{s}{M_W^2}\right)}_{\text{Melles (2001–2004)}}}_{\text{arbitrary processes}} + \underbrace{\underbrace{C_2 \ln^2\left(\frac{s}{M_W^2}\right)}_{\text{Kühn, Moch, Penin, Smirnov (2000–2003)}} + \underbrace{C_1 \ln^1\left(\frac{s}{M_W^2}\right)}_{\text{Jantzen, Kühn, Penin, Smirnov (2004,2005)}}}_{\text{massless } f\bar{f} \to f'\bar{f}' \text{ processes}} \Bigg]$$

Two-loop calculations based on EW Lagrangian



results agree with IREE + $\alpha^2 \ln(M_Z^2/M_W^2) \ln^3(s/M_W^2)$ terms from mixing

$$\alpha^2 \left[\underbrace{C_4 \ln^4\left(\frac{s}{M_W^2}\right)}_{\text{Melles; Hori,Kawamura, Kodaira '00}} + \underbrace{C_3^{\text{ang}} \ln\left(\frac{t}{s}\right) \ln^3\left(\frac{s}{M_W^2}\right)}_{\text{Denner, Melles,Pozzorini '03}} + \underbrace{C_3 \ln^3\left(\frac{s}{M_W^2}\right)}_{\text{Pozzorini'04}} \right]$$

Beenakker, Werthenbach '00,'02 — arbitrary processes involving $\mathrm{Z, W, H, b, t, \ldots}$

Denner, Jantzen, Pozzorini '06 — massless $f_1 f_2 \to f_3 \ldots f_n$

Effects of real W- and Z-boson emission

- Virtual EW corrections enhanced by large logarithms at high energies
- electromagnetic and strong logarithms cancelled by adding real gluon/photon emission (necessary for IR-safe predictions)
- $M_{\mathrm{W}}, M_{\mathrm{Z}} > 0 \Rightarrow$ no theoretical need to include real W/Z-boson radiation
- for non-exclusive final states real massive gauge-boson radiation contributes to experimental observables $\Rightarrow$ partially cancels virtual EW corrections

Effect of real radiation investigated for various LHC processes for typical cuts Baur '06

- in many cases weak-boson radiation moderately reduces the effect of the $\mathcal{O}(\alpha)$ virtual corrections
- in some processes weak-boson radiation large unless a jet veto is imposed typically for processes where leading order is suppressed ($\mathrm{W}\gamma$, WZ production)
- electroweak corrections including weak-boson radiation are relevant for many processes at the LHC