

Super-horizon growth of perturbations in globally adiabatic inflationary models

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Outline

- 1 The standard cosmological model and inflation
- 2 Adiabaticity
- 3 Cosmological perturbations
- 4 Gravity independent conservation laws for adiabatic perturbations
 - The case of general relativity
- 5 Globally adiabatic models

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- One the most successful mechanisms to explain the origin of anisotropies and inhomogeneities in the observed Universe is **inflation**
- Primordial curvature perturbations are the seeds from which **large scale structure** has formed
- Cosmic microwave background (**CMB**) radiation **anisotropies** can also be explained in the same way
- Inflation explains why CMB is isotropic(horizon problem) and why the spatial curvature today is so small(flatness problem)

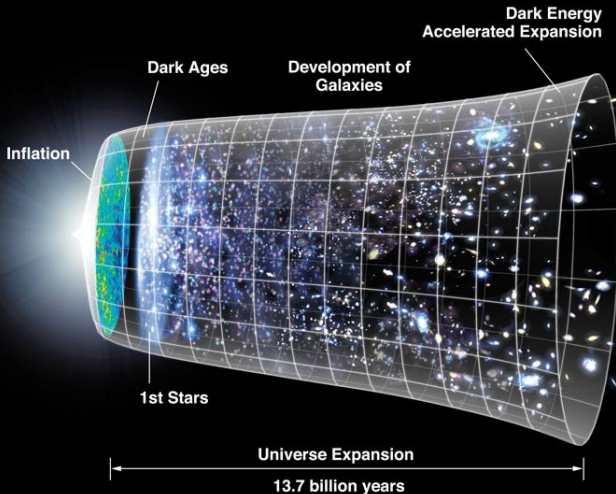
The standard cosmological model and inflation

Adiabaticity

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NASA/WMAP



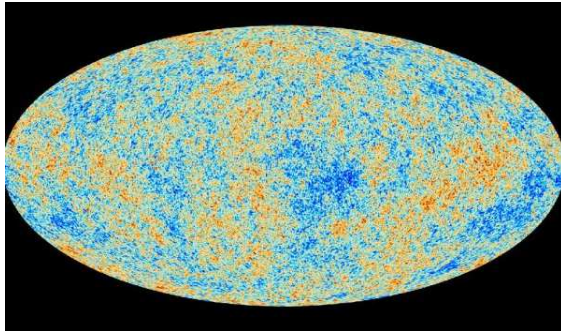


Figure: All-sky image of the Cosmic microwave background as seen by Planck. Hotter (colder) regions of the sky are represented by red (blue). Image credit: ESA and the Planck Collaboration.

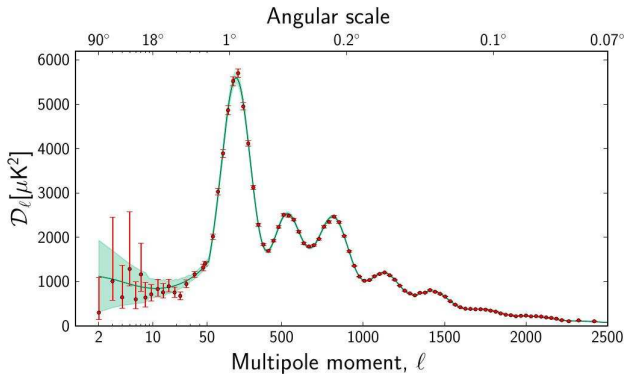


Figure: The temperature angular power spectrum of the CMB based on the Planck satellite data. The green line is the theoretical prediction. Image credit: Ade et al. Planck 2013 results.

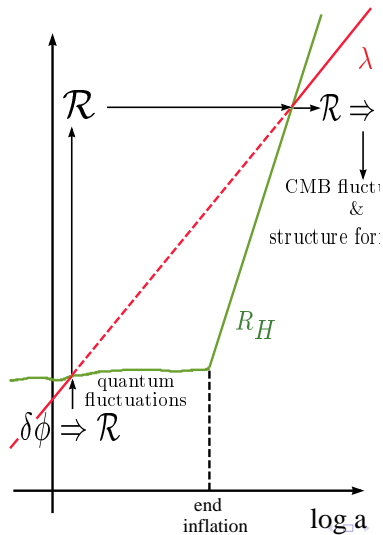
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- The original definition of adiabaticity comes from thermodynamics, and is based on the definition of adiabatic fluid, also known as barotropic or isentropic fluid, which admits an equation of state $P(\rho)$ so that

$$\delta P_{ad} = c_w^2 \delta \rho \quad , \quad c_w^2 \equiv \frac{P'}{\rho'} \quad (1)$$

where the prime denotes derivative respect to time.

Adiabatic perturbations move with the speed of sound c_w , also called adiabatic speed of sound.

- It is natural to define the non-adiabatic pressure as

$$\delta P_{nad} \equiv \delta P - c_w^2 \delta \rho = \delta P_{ud}, \quad (2)$$

which is gauge invariant and vanishes by construction for a perfect fluid. This is the definition used in the derivation of theorems for the **conservation of adiabatic curvature**

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Scalar metric perturbation

- Metric perturbation can be classified according to their behavior under spatial rotations as scalar, vector, and tensor
- According to inflation theory primordial scalar perturbations the seeds of the CMB temperature anisotropy and of large structure formation
- We set the perturbed metric as

$$ds^2 = a^2 \left[-(1 + 2A)d\eta^2 + 2\partial_j B dx^j d\eta + \left\{ \delta_{ij}(1 + 2\mathcal{R}) + 2\partial_i \partial_j E \right\} dx^i dx^j \right], \quad (3)$$

Perturbed energy-momentum tensor

- The perturbed energy-momentum tensor can be written as

$$T_0^0 = -(\rho + \delta\rho), \quad T_j^0 = (\rho + P)u^0 u_j = \frac{\rho + P}{a} u_j,$$

$$T_j^i = (P + \delta P)\delta_j^i + \Pi_j^i; \quad \Pi^k_k \equiv 0. \quad (4)$$

For a scalar perturbations

$$u_j = -a\partial_j(v - B) \quad \rightarrow \quad T_j^0 = -(\rho + P)\partial_j(v - B) \quad (5)$$

Π^k_j in the form can be written as

$$\Pi_{ij} = \delta_{ik}\Pi^k_j = \left[\partial_i\partial_j - \frac{1}{3}\delta_{ij}\Delta^{(3)} \right] \Pi, \quad (6)$$

where $\Delta^{(3)} = \delta^{ij}\partial_i\partial_j$.

Gauge invariant variables

- The following gauge-invariant variables are relevant to set the initial conditions before inflation:

$$\mathcal{R}_c \equiv \mathcal{R} - \mathcal{H}(v - B), \quad (7)$$

$$\zeta \equiv \mathcal{R} - \frac{\mathcal{H}}{\rho'} \delta\rho = \mathcal{R} + \frac{\delta\rho}{3(\rho + P)}, \quad (8)$$

$$V_f \equiv (v - B) - \frac{\mathcal{R}}{\mathcal{H}}. \quad (9)$$

Geometrical meaning of gauge invariant variables

- \mathcal{R}_c is the curvature perturbation on comoving slices ($v - B = 0$)
- ζ is the curvature perturbation on uniform density slices ($\delta\rho = 0$)
- V_f is the velocity potential on flat slices ($\mathcal{R} = 0$)
- They are related to each other as

$$\mathcal{R}_c = -\mathcal{H}V_f, \quad (10)$$

$$\zeta \equiv \mathcal{R}_{ud} = \mathcal{R}_c + \frac{\delta\rho_c}{3(\rho + P)}. \quad (11)$$

Conservation of the perturbed energy momentum-tensor in the comoving gauge

- In the comoving slices gauge, $v - B = 0$ ($\Leftrightarrow T^0_j = 0$)
 $\delta(\nabla_\mu T^\mu_j) = 0$ implies

$$(\rho + P)A_c + \delta P_c + \frac{2}{3}\nabla^2\Pi = 0. \quad (12)$$

If the perturbation is isotropic, by definition $\Pi = 0$. Thus we find

$$\delta P_c = -(\rho + P)A_c. \quad (13)$$

- Note that this relation between δP_c and A_c is completely **independent of the theory of gravity.**

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Gravity theory independent synchronous, comoving, and uniform density gauge coincidence

- Combining the definition of δP_{nad} with $\delta P_c = -(\rho + P)A_c$ and defining the sound speed of perturbations as $\delta P_c = c_s^2 \delta \rho_c$ we get the following important relations

$$\delta P_{nad} = (c_s^2 - c_w^2) \delta \rho_c = \frac{c_w^2 - c_s^2}{c_s^2} (\rho + P) A_c. \quad (14)$$

- For adiabatic perturbations the synchronous ($A = 0$), uniform density ($\delta \rho = 0$) and comoving gauge coincide as long as $c_s \neq c_w$.

$$\{\delta P_{nad} \approx 0, c_s \neq c_w\} \Rightarrow \delta \rho_c \approx A_c \approx 0. \quad (15)$$

Implications for CMB anisotropy spectrum calculation

- This is why numerical codes such as CAMB which solve the perturbed Boltzmann equations to calculate the CMB anisotropy spectrum use the **synchronous gauge** but the initial conditions are set using the spectrum for primordial curvature **perturbations on comoving slices** \mathcal{R}_C . In fact for standard slow-roll attractor models we have $c_S \neq c_W$. For example for a minimally coupled scalar field one has

$$c_W^2 = -1 + \frac{2\epsilon}{3} - \frac{\eta}{3}, \quad c_S^2 = 1, \quad (16)$$

with ϵ, η the usual slow-roll parameters.

- **But what happens when $c_S = c_W$?**

Gravity independent relation between \mathcal{R}_c and ζ

- Combining the relation for $\delta\rho_c$, with the gauge transformation from uniform density to the comoving slices gauge we get another important relation which is independent of the gravity theory

$$\zeta = \mathcal{R}_c - \frac{H}{\dot{\rho}} \delta\rho_c = \mathcal{R}_c + \delta P_{nad} \frac{H}{\dot{\rho}(c_w^2 - c_s^2)} \quad (17)$$

- When $c_s \neq c_w$, as in attractor slow-roll models, adiabaticity implies that $\mathcal{R}_c = \zeta$.

General relativity case for generic matter field

- In general relativity the perturbed Einstein's equations in the comoving slices gauge give for the G_i^0 -component

$$\mathcal{R}'_c = \mathcal{H}A_c \quad (18)$$

which combined with the equation for δP_{nad} gives the important relations

$$\delta P_{nad} = \left[\left(\frac{c_w}{c_s} \right)^2 - 1 \right] (\rho + P) \frac{\dot{\mathcal{R}}_c}{H} \quad (19)$$

$$\zeta = \mathcal{R}_c - \frac{\dot{\mathcal{R}}_c}{3c_s^2 H}. \quad (20)$$

- Thus $\delta P_{nad} = 0$ if either $c_w^2 = c_s^2$ or $\dot{\mathcal{R}}_c = 0$. In particular in the latter case, $\dot{\mathcal{R}}_c = 0$, we have $\zeta = \mathcal{R}_c$.

Relation with previous results

- We can immediately deduce that in general relativity there are two possible scenarios for the non-conservation of \mathcal{R}_C ,

$$(1) \quad c_S^2 = c_W^2, \quad \delta P_{nad} = 0$$

$$(2) \quad c_S^2 \neq c_W^2, \quad \delta P_{nad} \neq 0$$

- The first case corresponds to **globally adiabatic (GA)** models because $c_S^2 = c_W^2 \leftrightarrow \delta P_{nad} = 0$ on **any scale**
- The second case is the standard slow-roll inflation for example, where on superhorizon scales $\zeta = \mathcal{R}_C$ and they are both conserved if $\delta P_{nad} = 0$, but **only on super-horizon scales**

Relation with previous results

- Demanding $\delta(\nabla_\mu T_0^\mu) = 0$ yields, in the uniform density slicing,

$$\zeta' = -\frac{\mathcal{H}\delta P_{nad}}{(\rho + P)} + \frac{1}{3} \Delta^{(3)} (v - E')_{ud} \quad (21)$$

- The usual interpretation of the above equation is that for **adiabatic perturbations**, ζ is conserved on **super-horizon scales**, as long as the **gradient terms can be neglected**.
- As we have proved, when $c_s = c_w$ this is not necessarily true anymore, and indeed neglecting the gradient terms may not be justified, as for example happens in ultra-slow inflation(USR).

Ultra-slow inflation

- Ultra-slow inflation is a single scalar field minimally coupled model with standard kinetic term and constant potential
The density and pressure perturbations become equal to each other, $\delta P = \delta \rho$, in arbitrary gauge.
- Therefore we have

$$c_w^2 - c_s^2 = \delta P_{nad} = 0, \dot{\phi} \propto a^{-3} \quad (22)$$

$$\epsilon \equiv -\frac{\dot{H}}{H^2} = \frac{\dot{\phi}^2}{2H^2} \propto a^{-6}, \quad \eta \equiv \frac{\dot{\epsilon}}{\epsilon} = -6 \quad (23)$$

Super-horizon growth of curvature perturbations for \mathcal{R}_c

- In this case adiabaticity does not imply the conservation of \mathcal{R}_c neither of ζ , and they can differ from each other because

$$\zeta = \mathcal{R}_c + \delta P_{nad} \frac{H}{\dot{\rho}(c_w^2 - c_s^2)} \quad (24)$$

$$\delta P_{nad} = \left[\left(\frac{c_w}{c_s} \right)^2 - 1 \right] (\rho + P) \frac{\dot{\mathcal{R}}_c}{H} \quad (25)$$

- On super horizon scales we get The superhorizon solution for \mathcal{R}_c is then

$$\mathcal{R}_c = c_1 \left(1 + \mathcal{O}(k^2) \right) + c_2 a^3 \left(1 + \frac{1}{2} \frac{k^2}{\mathcal{H}^2} + \mathcal{O}(k^4) \right). \quad (26)$$

Super-horizon growth of curvature perturbations for ζ

- Inserting this into the equation relating \mathcal{R}_c and ζ we get

$$\zeta = c_1 \left(1 + \mathcal{O}(k^2) \right) + \frac{c_2 a^3}{3} \left(\frac{k^2}{\mathcal{H}^2} + \mathcal{O}(k^4) \right). \quad (27)$$

Thus we see that the time-dependent solution grows like a even on superhorizon scales. More specifically, $\zeta(t) \approx \zeta(t_k) a(t)/a(t_k)$ where t_k is the horizon crossing time $a(t_k) = kH$ of the wavenumber k .

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Inversion method to find models violating super-horizon curvature perturbations

- We have developed an inversion method to find new models exhibiting super-horizon growth of curvature perturbations based on the equivalence between K-inflation globally adiabatic models (GA) and barotropic fluids.
- Imposing conditions on $\epsilon(a)$ we have found the corresponding equation of state for the barotropic fluid admitting such a background behavior and the correspondent equivalent scalar field Lagrangian, obtaining an infinite class of models which do not conserve curvature perturbations

Inversion method to find models violating super-horizon curvature perturbations

- These single scalar field models **do not satisfy the non-Gaussianity consistency** conditions
- They could have **large local shape** f_{NL} , while satisfying other observational constraints such the spectral index,

- From the equation for the curvature perturbation on comoving slices

$$\frac{\partial}{\partial t} \left(\frac{a^3 \epsilon}{c_s^2} \frac{\partial}{\partial t} \mathcal{R}_c \right) - a \epsilon \Delta \mathcal{R}_c = 0, \quad (28)$$

we can deduce, after re-expressing the time derivative in terms of the derivative respect to the scale factor a , that on superhorizon scales there is (apart from a constant solution) a solution of the form,

$$\mathcal{R}_c \propto \int^a \frac{da}{a} f(a); \quad f(a) \equiv \frac{c_s^2(a)}{Ha^3 \epsilon(a)}, \quad (29)$$

Constructing GA models



$$\epsilon = -\frac{\dot{H}}{H^2} = \frac{3}{2} \frac{\rho + P}{\rho}. \quad (30)$$

In terms of the scale factor and ϵ the energy conservation equation reads

$$\frac{d\rho}{da} + \frac{3}{a}(\rho + p) = \frac{d\rho}{da} + \frac{2\epsilon\rho}{a} = 0. \quad (31)$$

We may now define the quantity $b(a) = 2\epsilon\rho$. The function $f(a)$ can be re-written in terms of it as

$$f(a) \propto \frac{Hc_s^2}{a^3 b(a)}. \quad (32)$$

- It is possible to associate any barotropic perfect fluid with an equivalent K-inflation model such that $c_s = c_w$ according to

$$2 \int^P \frac{du}{F(u)} = \log(Y), \quad (33)$$

where $F(P) = \rho(P) + P$ and $Y = g(\phi)X$.

Examples

- Generalized Ultra-slow inflation : $2\epsilon\rho = b(a) \propto a^{-n}$

$$L = P(Y) = Y^{n/(2n-6)} - V_0. \quad (34)$$

- Lambert Inflation : $\epsilon \propto a^{-n}$

$$L = \rho_0 \left(\frac{n-3}{3} W(y) - 1 \right) \exp \left[\frac{n-3}{n} W(y) \right] - V_0$$

$$X(z) = ze^z, z = X^{-1}(ze^z) \equiv W(ze^z)$$

Globally adiabatic inflationary models

- They can be an alternative to attractor models
- These single scalar field models do not satisfy the non-Gaussianity consistency conditions and could have large local shape f_{NL} , while satisfying other observational constraints such the spectral index, and could be an alternative to attractor models
- **Adiabatic perturbations are not always conserved**, and on the contrary, GA models are good candidates for the violation of curvature perturbations on super-horizon scales together with non adiabatic models