• X, Y, Z the new revolution

• Charmonium a golden system

• Our present understanding of charmonium below threshold: EFTs and lattice

• Charmonium at and above threshold

• EFT for charmonium hybrids: results and comparison to the data
TABLE 10: Quarkonium-like states at the open flavor thresholds. For charged states, the $C$-parity is given for the neutral members of the corresponding isotriplets.

<table>
<thead>
<tr>
<th>State</th>
<th>$M$, MeV</th>
<th>$\Gamma$, MeV</th>
<th>$J^{PC}$</th>
<th>Process (mode)</th>
<th>Experiment (#$\sigma$)</th>
<th>Year</th>
<th>Status</th>
</tr>
</thead>
<tbody>
<tr>
<td>$X(3872)$</td>
<td>3871.68 ± 0.17</td>
<td>&lt; 1.2</td>
<td>1++</td>
<td>$B \rightarrow K(\pi^+\pi^- J/\psi)$</td>
<td>Belle [772, 992] (10), BaBar [993] (8.6)</td>
<td>2003</td>
<td>Ok</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$p\bar{p} \rightarrow (\pi^+\pi^- J/\psi)$...</td>
<td>CDF [994, 995] (11.6), D0 [996] (5.2)</td>
<td>2003</td>
<td>Ok</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$pp \rightarrow (\pi^+\pi^- J/\psi)$...</td>
<td>LHCb [997, 998] (np)</td>
<td>2012</td>
<td>Ok</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$B \rightarrow K(\pi^+\pi^- \pi^0 J/\psi)$</td>
<td>Belle [999] (4.3), BaBar [1000] (4.0)</td>
<td>2005</td>
<td>Ok</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$B \rightarrow K(\gamma J/\psi)$</td>
<td>Belle [1001] (5.5), BaBar [1002] (3.5)</td>
<td>2005</td>
<td>Ok</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$B \rightarrow K(\gamma \psi(2S))$</td>
<td>LHCb [1003] (4.4)</td>
<td>2008</td>
<td>NC!</td>
</tr>
<tr>
<td>$Z_c(3885)^+$</td>
<td>3883.9 ± 4.5</td>
<td>25 ± 12</td>
<td>1++</td>
<td>$Y(4260) \rightarrow \pi^-(D\bar{D}^*)^+$</td>
<td>Belle [1004] (6.4), BaBar [1005] (4.9)</td>
<td>2006</td>
<td>Ok</td>
</tr>
<tr>
<td>$Z_c(3900)^+$</td>
<td>3891.2 ± 3.3</td>
<td>40 ± 8</td>
<td>??-</td>
<td>$Y(4260) \rightarrow \pi^- (\pi^+ J/\psi)$</td>
<td>BES III [1006] (np)</td>
<td>2013</td>
<td>NC!</td>
</tr>
<tr>
<td>$Z_c(4020)^+$</td>
<td>4022.9 ± 2.8</td>
<td>7.9 ± 3.7</td>
<td>??-</td>
<td>$Y(4260, 4360) \rightarrow \pi^-(\pi^+ h_c)$</td>
<td>BES III [1010] (8.9)</td>
<td>2013</td>
<td>NC!</td>
</tr>
<tr>
<td>$Z_c(4025)^+$</td>
<td>4026.3 ± 4.5</td>
<td>24.8 ± 9.5</td>
<td>??-</td>
<td>$Y(4260) \rightarrow \pi^- (D^<em>\bar{D}^</em>)^+$</td>
<td>BES III [1011] (10)</td>
<td>2013</td>
<td>NC!</td>
</tr>
<tr>
<td>$Z_b(10610)^+$</td>
<td>10607.2 ± 2.0</td>
<td>18.4 ± 2.4</td>
<td>1++</td>
<td>$Y(10860) \rightarrow \pi^-(\pi^+ T(1S, 2S, 3S))$</td>
<td>Belle [1012–1014] (&gt;10)</td>
<td>2011</td>
<td>Ok</td>
</tr>
<tr>
<td></td>
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<td></td>
<td></td>
<td>$Y(10860) \rightarrow \pi^-(\pi^+ h_b(1P, 2P))$</td>
<td>Belle [1013] (16)</td>
<td>2011</td>
<td>Ok</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$Y(10860) \rightarrow \pi^- (BB^*)^+$</td>
<td>Belle [1015] (8)</td>
<td>2012</td>
<td>NC!</td>
</tr>
<tr>
<td></td>
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<td></td>
<td></td>
<td>$Y(10860) \rightarrow \pi^- (\pi^+ Y(1S, 2S, 3S))$</td>
<td>Belle [1012, 1013] (&gt;10)</td>
<td>2011</td>
<td>Ok</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$Y(10860) \rightarrow \pi^- (\pi^+ h_b(1P, 2P))$</td>
<td>Belle [1013] (16)</td>
<td>2011</td>
<td>Ok</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$Y(10860) \rightarrow \pi^- (B^<em>\bar{B}^</em>)^+$</td>
<td>Belle [1015] (6.8)</td>
<td>2012</td>
<td>NC!</td>
</tr>
</tbody>
</table>

arXiv:1404.3723v1
### TABLE 12: Quarkonium-like states above the corresponding open flavor thresholds. For charged states, the $C$-parity is given for the neutral members of the corresponding isospinets.

<table>
<thead>
<tr>
<th>State</th>
<th>$M$, MeV</th>
<th>$\Gamma$, MeV</th>
<th>$J^{PC}$</th>
<th>Process (mode)</th>
<th>Experiment (#$\sigma$)</th>
<th>Year</th>
<th>Status</th>
</tr>
</thead>
<tbody>
<tr>
<td>$Y(3915)$</td>
<td>3918.4 ± 1.9</td>
<td>20 ± 5</td>
<td>0/2+</td>
<td>$B \rightarrow K^{0}(\omega J/\psi)$</td>
<td>Belle [1050] (8), BaBar [1000, 1051] (19)</td>
<td>2004</td>
<td>Ok</td>
</tr>
<tr>
<td>$\chi_{c2}(2P)$</td>
<td>3927.2 ± 2.6</td>
<td>24 ± 6</td>
<td>2++</td>
<td>$e^+e^- \rightarrow e^+e^- (\omega J/\psi)$</td>
<td>Belle [1052] (7.7), BaBar [1053] (7.6)</td>
<td>2009</td>
<td>Ok</td>
</tr>
<tr>
<td>$X(3940)$</td>
<td>3942.0+8-7</td>
<td>37.0-17</td>
<td>?+</td>
<td>$e^+e^- \rightarrow e^+e^- (D\bar{D})$</td>
<td>BaBar [1054] (5.3), BaBar [1055] (5.8)</td>
<td>2005</td>
<td>Ok</td>
</tr>
<tr>
<td>$Y(4008)$</td>
<td>3891 ± 42</td>
<td>255 ± 42</td>
<td>1−</td>
<td>$e^+e^- \rightarrow (\pi^+\pi^- J/\psi)$</td>
<td>Belle [1048, 1049] (6)</td>
<td>2005</td>
<td>NC!</td>
</tr>
<tr>
<td>$\psi(4040)$</td>
<td>4039 ± 1</td>
<td>80 ± 10</td>
<td>1−</td>
<td>$e^+e^- \rightarrow (D^<em>(\bar{D}^</em>)^*)(\pi)$</td>
<td>Belle [1008, 1056] (7.4)</td>
<td>2007</td>
<td>NC!</td>
</tr>
<tr>
<td>$Z(4050)^+$</td>
<td>4051.0+24-34</td>
<td>82.0+113-35</td>
<td>?+</td>
<td>$B^0 \rightarrow K^- (\pi^+ \chi_{c1})$</td>
<td>BaBar [1058] (5.0), BaBar [1059] (1.1)</td>
<td>2008</td>
<td>NC!</td>
</tr>
<tr>
<td>$Y(4140)$</td>
<td>4145.8 ± 2.6</td>
<td>18 ± 8</td>
<td>?+</td>
<td>$B^+ \rightarrow K^+(\phi J/\psi)$</td>
<td>CDF [1060] (5.0), Belle [1061] (1.9),</td>
<td>2009</td>
<td>NC!</td>
</tr>
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<td></td>
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<td></td>
<td>LHCb [1062] (1.4), CMS [1063] (5)</td>
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</tr>
<tr>
<td>$\psi(4160)$</td>
<td>4153 ± 3</td>
<td>103 ± 8</td>
<td>1−</td>
<td>$e^+e^- \rightarrow (D^<em>(\bar{D}^</em>)^*)(\pi)$</td>
<td>PDG [1]</td>
<td>1978</td>
<td>Ok</td>
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<tr>
<td>$X(4160)$</td>
<td>4156.0+29-25</td>
<td>139.0+113-65</td>
<td>?+</td>
<td>$e^+e^- \rightarrow (\eta J/\psi)$</td>
<td>Belle [1057] (6.5)</td>
<td>2013</td>
<td>NC!</td>
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<tr>
<td>$Z(4200)^+$</td>
<td>4196.0+35-30</td>
<td>370.0+99-130</td>
<td>1−</td>
<td>$B^0 \rightarrow K^- (\pi^+ J/\psi)$</td>
<td>Belle [1049] (5.5)</td>
<td>2007</td>
<td>NC!</td>
</tr>
<tr>
<td>$Z(4250)^+$</td>
<td>4248.0+185-45</td>
<td>177.0+321-72</td>
<td>?+</td>
<td>$B^0 \rightarrow K^- (\pi^+ \chi_{c1})$</td>
<td>Belle [1058] (5.0), BaBar [1059] (2.0)</td>
<td>2008</td>
<td>NC!</td>
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<tr>
<td>$Y(4260)$</td>
<td>4250 ± 9</td>
<td>108 ± 12</td>
<td>1−</td>
<td>$e^+e^- \rightarrow (\pi^+ J/\psi)$</td>
<td>BaBar [1066, 1067] (8), CLEO [1068, 1069] (11)</td>
<td>2005</td>
<td>Ok</td>
</tr>
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<td></td>
<td></td>
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<td></td>
<td>Belle [1008, 1056] (15), BES III [1007] (np)</td>
<td></td>
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</tr>
<tr>
<td>$X(4350)$</td>
<td>4350.0+4.6-5</td>
<td>13.0+18-16</td>
<td>0/2+</td>
<td>$e^+e^- \rightarrow e^+e^- (\phi J/\psi)$</td>
<td>BaBar [1067] (np), Belle [1008] (np)</td>
<td>2012</td>
<td>Ok</td>
</tr>
<tr>
<td>$Y(4360)$</td>
<td>4354 ± 11</td>
<td>78 ± 16</td>
<td>1−</td>
<td>$e^+e^- \rightarrow (\pi^+\pi^- J/\psi)$</td>
<td>BES III [1007] (8), Belle [1008] (5.2)</td>
<td>2013</td>
<td>Ok</td>
</tr>
<tr>
<td>$Z(4430)^+$</td>
<td>4458.0+15-16</td>
<td>166.0+37-32</td>
<td>1−</td>
<td>$B^0 \rightarrow K^- (\pi^+\pi^- J/\psi)$</td>
<td>BES III [1070] (5.3)</td>
<td>2013</td>
<td>NC!</td>
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<td></td>
<td>CMS [1063] (&gt;3), D0 [1064] (np)</td>
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<tr>
<td>$X(4630)$</td>
<td>4634.0+9-11</td>
<td>92.0+41-32</td>
<td>1−</td>
<td>$e^+e^- \rightarrow (\Lambda_c^+ \bar{\Lambda}_c^-)$</td>
<td>Belle [1071] (3.2)</td>
<td>2009</td>
<td>NC!</td>
</tr>
<tr>
<td>$Y(4660)$</td>
<td>4665 ± 10</td>
<td>53 ± 14</td>
<td>1−</td>
<td>$e^+e^- \rightarrow (\pi^+\pi^- J/\psi)$</td>
<td>Belle [1072] (8), BaBar [1073] (np)</td>
<td>2007</td>
<td>Ok</td>
</tr>
<tr>
<td>$\Upsilon(10860)$</td>
<td>10876 ± 11</td>
<td>55 ± 28</td>
<td>1−</td>
<td>$e^+e^- \rightarrow (B^{(<em>)}(\bar{B}^{(</em>)})^*)(\pi)$</td>
<td>Belle [1074, 1075] (6.4), BaBar [1076] (2.4)</td>
<td>2007</td>
<td>Ok</td>
</tr>
<tr>
<td>$\Upsilon_b(10888)$</td>
<td>10888.4 ± 3.0</td>
<td>30.7+8.9-7.7</td>
<td>1−</td>
<td>$e^+e^- \rightarrow (\pi^+\pi^- J/\psi)$</td>
<td>LHCb [1077] (13.9)</td>
<td>2014</td>
<td>NC!</td>
</tr>
<tr>
<td></td>
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<td></td>
<td></td>
<td></td>
<td>Belle [1065] (4.0)</td>
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<td></td>
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<td>Belle [1078] (8.2)</td>
<td>2007</td>
<td>NC!</td>
</tr>
<tr>
<td></td>
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<td></td>
<td></td>
<td></td>
<td>Belle [1072] (5.8), BaBar [1073] (5)</td>
<td>2007</td>
<td>Ok</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>PDG [1]</td>
<td>1985</td>
<td>Ok</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>Belle [1013, 1014, 1079] (&gt;10)</td>
<td>2007</td>
<td>Ok</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>Belle [1013, 1014] (&gt;5)</td>
<td>2011</td>
<td>Ok</td>
</tr>
<tr>
<td></td>
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<td></td>
<td></td>
<td></td>
<td>Belle [1013, 1014] (&gt;10)</td>
<td>2011</td>
<td>Ok</td>
</tr>
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<td></td>
<td></td>
<td>Belle [948] (10)</td>
<td>2012</td>
<td>Ok</td>
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<td></td>
<td>Belle [948] (9)</td>
<td>2012</td>
<td>Ok</td>
</tr>
<tr>
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<td></td>
<td></td>
<td></td>
<td>Belle [1080] (2.3)</td>
<td>2008</td>
<td>NC!</td>
</tr>
</tbody>
</table>
bottomonium: the present revolution

$B_s^* B_s^* \quad Z'_b \quad \Upsilon(4S)$

$B^* B \quad Z_b \quad h_b(3P) \quad \Upsilon(3S)$

$B_s B_s \quad h_b(2P)$

$\eta_b(3S) \quad h_b(2P)$

$\eta_b(2S) \quad h_b(1P)$

$\chi_{bJ}(1P)$

$\chi_{bJ}(3P) \quad \Upsilon(2^3D_J) \quad 2^1D_2$

$\Upsilon(1^3D_J) \quad 1^1D_2$

$\Upsilon(5S)$

$\Upsilon(1S)$

M GeV $J^{PC}$: $0^{-+}$ $1^{++}$ $1^{--}$ $0^{++}$ $1^{++}$ $2^{++}$ $1^{--}, 2^{--}, 3^{--}, 2^{++}$
The November revolution in 1974: the $J/\psi$ discovery

$\Gamma \sim 90$ KeV

Aubert et al. BNL 74
The November revolution in 1974: the $J/\psi$ discovery

Samuel Ting: “It is like to stumble on a village where people live 70000 years”

$\Gamma \sim 90 \text{ KeV}$

Aubert et al. BNL 74
The November revolution in 1974: the $J/\psi$ discovery

Samuel Ting: “It is like to stumble on a village where people live 70000 years”

$\Gamma \sim 90$ KeV

it has been the confirmation of the charmed quark prediction and of QCD (strong int theory) foundations

narrow width and asymptotic freedom

annihilation at large scale controlled by small $\alpha_s$

first discovery of a quark of large mass moving “slowly”
Heavy quarks offer a privileged access

A large scale \( m_Q \gg \Lambda_{\text{QCD}} \) \[ \alpha_s(m_Q) \ll 1 \]

with \( Q, \bar{Q} = c, b, t \)

\( m_c \sim 1.5 \text{GeV} \)

\( m_b \sim 5 \text{GeV} \)

\( m_t \sim 170 \text{GeV} \)
Heavy quarks offer a privileged access to the strong sector of the Standard Model. The heavy quarks, $Q$, and their antiquarks, $ar{Q}$, can be $c$, $b$, or $t$. The masses of these quarks are:

- $m_c \sim 1.5\text{GeV}$
- $m_b \sim 5\text{GeV}$
- $m_t \sim 170\text{GeV}$

A large scale $m_Q \gg \Lambda_{\text{QCD}}$ and $\alpha_s(m_Q) \ll 1$.

Heavy quarkonia are nonrelativistic bound systems: multiscale systems.

Many scales: a challenge and an opportunity.

Electromagnetic bound states: atoms, molecules, etc.
Quarkonium scales

Normalized with respect to $\chi_b(1P)$ and $\chi_c(1P)$
Quarkonium scales

\begin{align*}
\text{MeV} & \\
Y(4S) & \text{BB threshold} \\
Y(3S) & \psi(3S) & X_b(2P) \\
\psi(3770) & \text{DD threshold} \\
Y(2S) & \psi(2S) & \eta_c(2S) & \eta_c(1S) & \eta_c(1S) & X_c(1P) & h_c(1P) \\
Y(1S) & J/\psi & \chi_b(1P) \\
\eta_c(1S) & \chi_c(1P)
\end{align*}

Normalized with respect to \( \chi_b(1P) \) and \( \chi_c(1P) \)

NR bound states have at least 3 scales:
\( m \gg mv \gg mv^2 \quad v \ll 1 \)

\( mv \sim r^{-1} \)

\( \text{and } \Lambda_{QCD} \)

The system is nonrelativistic (NR):
\[ \Delta E \sim mv^2, \Delta f_s E \sim mv^4 \]
\[ v_b^2 \sim 0.1, v_c^2 \sim 0.3 \]

The mass scale is perturbative:
\[ m_Q \gg \Lambda_{QCD} \]
\[ m_b \sim 5 \text{ GeV}; m_c \sim 1.5 \text{ GeV} \]
Quarkonium as a confinement and deconfinement probe

The rich structure of separated energy scales makes QQbar an ideal probe.

At zero temperature

- The different quarkonium radii provide different measures of the transition from a Coulombic to a confined bound state.

At finite temperature

- Quarkonia dissociate at different temperatures in dependences of their radius: they are a QGP thermometer.
QCD theory of Quarkonium: a very hard problem

Close to the bound state $\alpha_s \sim \nu$

\[
\frac{g^2}{p^2} (1 + \frac{m\alpha_s}{p}) \sim \frac{1}{E - (\frac{p^2}{m} + V)}
\]
QCD theory of Quarkonium: a very hard problem

Close to the bound state $\alpha_s \sim \nu$

$p \sim m\alpha_s$

\[
\frac{g^2}{p^2} \left(1 + \frac{m\alpha_s}{p}\right)
\]

\[
\sim \frac{1}{E - \left(\frac{p^2}{m} + V\right)}
\]

- From $(\frac{p^2}{m} + V)\phi = E\phi \rightarrow p \sim mv$ and $E = \frac{p^2}{m} + V \sim mv^2$. 
QCD theory of Quarkonium: a very hard problem

Close to the bound state \( \alpha_s \sim \nu \)

\[
p \sim m\alpha_s + \ldots
\]

\[
\frac{g^2}{p^2} \left(1 + \frac{m\alpha_s}{p}\right)
\]

\[
E \sim mv^2
\]

multiscale diagrams have a complicate power counting and contribute to all orders in the coupling

\[
L^{-1} \ll \lambda \ll \Lambda \ll a^{-1}
\]

Difficult also for the lattice!
Quarkonium with Non relativistic Effective Field Theories

\[ \mathcal{L}_{\text{EFT}} = \sum_n c_n \left( \frac{E_\Lambda}{\mu} \right) \frac{O_n(\mu, \lambda)}{E_\Lambda} \]

\[ \langle O_n \rangle \sim E_\lambda^n \]
Quarkonium with Non relativistic Effective Field Theories

\[ \mathcal{L}_{\text{EFT}} = \sum_n c_n \frac{E_\Lambda}{\mu} \frac{O_n(\mu, \lambda)}{E_\Lambda} \]

\[ \langle O_n \rangle \sim E_\lambda^n \]

Color degrees of freedom 3X3=1+8 singlet and octet QQbar

Hard

Soft (relative momentum)

Ultrasoft (binding energy)

\[ \frac{E_\lambda}{E_\Lambda} = \frac{mv}{m} \]
Quarkonium with Non-relativistic Effective Field Theories

\[ \mathcal{L}_{\text{EFT}} = \sum_n c_n \frac{E_\Lambda / \mu}{E_\Lambda} \frac{O_n(\mu, \lambda)}{E_\Lambda} \]

\[ \langle O_n \rangle \sim E_\Lambda^n \]

Color degrees of freedom
3X3=1+8
singlet and octet QQbar

Hard

Soft
(relative momentum)

Ultrasoft
(binding energy)

\[ \frac{E_\lambda}{E_\Lambda} = \frac{mv}{m} \]

\[ \frac{E_\lambda}{E_\Lambda} = \frac{mv^2}{mv} \]
Quarkonium with NR EFT: Non Relativistic QCD (NRQCD)

QCD

\text{perturbative matching} \quad \text{perturbative matching}

\rightarrow

\mu

\mu^l

\rightarrow

pNRQCD

\text{nonperturbative matching} \quad \text{(long-range quarkonium)} \quad \text{perturbative matching} \quad \text{(short-range quarkonium)}

m

mv

mv^2

E \sim mv^2

p \sim mv

\sim m
Quarkonium with NR EFT: Non Relativistic QCD (NRQCD)
Quarkonium with NR EFT: Non Relativistic QCD (NRQCD)

\[ \mathcal{L}_{\text{NRQCD}} = \sum_n c(\alpha_s(m/\mu)) \times \frac{O_n(\mu, \lambda)}{m^n} \]
Quarkonium with NR EFT: potential NonRelativistic QCD (pNRQCD)
Quarkonium with NR EFT: potential NonRelativistic QCD (pNRQCD)
Quarkonium with NR EFT: potential NonRelativistic QCD (pNRQCD)

\[
\mathcal{L}_{\text{pNRQCD}} = \sum_k \sum_n \frac{1}{m^k} c_k(\alpha_s(m/\mu)) \times V(r\mu', r\mu) \times O_n(\mu', \lambda) r^n
\]
Quarkonium with NR EFT: potential NonRelativistic QCD (pNRQCD)

\[ \mathcal{L}_{pNRQCD} = \sum_k \sum_n \frac{1}{m_k} c_k(\alpha_s(m/\mu)) \times V(r\mu', r\mu) \times O_n(\mu', \lambda) r^n \]
In QCD another scale is relevant: \( \Lambda_{QCD} \)

Quarkonium with NR EFT: pNRQCD

Weakly coupled pNRQCD

Strongly coupled pNRQCD


A potential picture arises at the level of pNRQCD:

- the potential is perturbative if \( mv \gg \Lambda_{QCD} \)
- the potential is non-perturbative if \( mv \sim \Lambda_{QCD} \)
today we have a complete theory description for quarkonia states away from the strong decay threshold
today we have a complete theory description for quarkonia states away from the strong decay threshold

The EFT has been constructed pNRQCD

* Work at calculating higher order perturbative corrections in $v$ and $\alpha_s$
* Resumming the log
* Calculating/extracting nonperturbatively the low energy quantities
* Extending the theory (electromagnetic effect, 3 bodies)
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The issue here is precision physics and the study of confinement
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The EFT has been constructed

- Work at calculating higher order perturbative corrections in $\nu$ and $\alpha_s$
- Resumming the log
- Calculating/extracting nonperturbatively the low energy quantities
- Extending the theory (electromagnetic effect, 3 bodies)

The issue here is precision physics and the study of confinement

- Precise and systematic high order calculations allow the extraction of precise determinations of standard model parameters like the quark masses and $\alpha_s$
- The EFT has allowed to systematically factorize and to study the low energy nonperturbative contributions
Weakly coupled pNRQCD $r \ll \Lambda_{\text{QCD}}^{-1}$

\[
\mathcal{L} = -\frac{1}{4} F_{\mu \nu}^a F^{\mu \nu} a + \text{Tr} \left\{ S^\dagger \left( i \partial_0 - \frac{p^2}{m} - V_s \right) S \right. \\
+ O^\dagger \left( i D_0 - \frac{p^2}{m} - V_o \right) O \left\} \\
+ V_A \text{Tr} \left\{ O^\dagger r \cdot g E S + S^\dagger r \cdot g E O \right\} \\
+ \frac{V_B}{2} \text{Tr} \left\{ O^\dagger r \cdot g E O + O^\dagger O r \cdot g E \right\} \\
+ \ldots
\]

Singlet static potential

Octet static potential

Singlet propagator

Octet propagator

Pineda, Soto 97; Brambilla, Pineda, Soto, Vairo 99-
strongly coupled pNRQCD

\[ r \sim \Lambda_{QCD}^{-1} \quad mv \sim \Lambda_{QCD} \]

- integrate out all scales above
- gluonic excitations develop a gap and are integrated out

⇒ The singlet quarkonium field \( S \) of energy \( mv^2 \) is the only the degree of freedom of pNRQCD (up to ultrasoft light quarks, e.g. pions).

\[ \mathcal{L} = \text{Tr} \left\{ \bar{S} \left( i\partial_0 - \frac{P^2}{m} - V_s \right) S \right\} \]
• A potential description emerges from the EFT

• The potentials \( V = \text{Re} V + \text{Im} V \) from QCD in the matching: get spectra and decays

• \( V \) to be calculated on the lattice or in QCD vacuum models

\[
\mathcal{L} = \text{Tr} \left\{ S^\dagger \left( i\partial_0 - \frac{\mathbf{p}^2}{m} - V_s \right) S \right\}
\]

\( r \sim \Lambda_{QCD}^{-1} \quad mv \sim \Lambda_{QCD} \)

\( mv^2 \quad \Lambda_{QCD} \)

\( \kappa = 0.1575 \)

\( mb \sim \Lambda_{QCD} \)

\( \text{quenched} \)

\( \Rightarrow \) The singlet quarkonium field \( S \) of energy \( mv^2 \) is the only the degree of freedom of pNRQCD (up to ultrasoft light quarks, e.g. pions).
Quarkonium singlet static potential

\[ V = V_0 + \frac{1}{m} V_1 + \frac{1}{m^2} (V_{SD} + V_{VD}) \]

\[ V_s^{(0)} = \lim_{T \to \infty} \frac{i}{T} \ln \langle W(r \times T) \rangle = \lim_{T \to \infty} \frac{i}{T} \ln \langle \square \rangle \]

\[ W = \langle \exp\{ig \oint A^\mu dx_\mu\} \rangle \]
QCD Spin dependent potentials

\[ V_{SD}^{(2)} = \frac{1}{r} \left( c_F \epsilon^{k ij} \frac{2 \tau^k}{r} i \int_0^\infty dt \, t \left( \langle \begin{array}{c} 1 \\ 3 \end{array} \rangle - \frac{1}{2} V_s^{(0)} \right) (S_1 + S_2) \cdot (L - \frac{1}{3} \langle \begin{array}{c} 1 \\ 3 \end{array} \rangle - \delta_{ij} \langle \begin{array}{c} 2 \\ 4 \end{array} \rangle) \times \left( S_1 \cdot S_2 - 3 (S_1 \cdot \hat{r}) (S_2 \cdot \hat{r}) \right) \right) \]

\[ - c_F \frac{2}{3} \hat{r}_i \hat{r}_j i \int_0^\infty dt \left( \langle \begin{array}{c} 1 \\ 3 \end{array} \rangle - \delta_{ij} \langle \begin{array}{c} 2 \\ 4 \end{array} \rangle \right) \times \left( \langle \begin{array}{c} 2 \\ 4 \end{array} \rangle \right) \]

\[ + \left( \frac{2}{3} c_F \frac{2}{3} i \int_0^\infty dt \langle \begin{array}{c} 1 \\ 3 \end{array} \rangle - 4 (d_2 + C_F d_4) \delta^{(3)}(r) \right) S_1 \cdot S_2 \]
QCD Spin dependent potentials

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\[ -c_F \hat{r}_i \hat{r}_j i \int_0^\infty dt \left( \langle \begin{array}{c} 1 \\ 3 \end{array} \rangle - \frac{\delta_{ij}}{3} \langle \begin{array}{c} 1 \\ 2 \end{array} \rangle \right) \times \left( S_1 \cdot S_2 - 3(S_1 \cdot \hat{r})(S_2 \cdot \hat{r}) \right) \]

\[ + \left( \frac{2}{3} c_F i \int_0^\infty dt \langle \begin{array}{c} 1 \\ 2 \end{array} \rangle - 4(d_2 + C_F d_4) \delta^{(3)}(r) \right) (S_1 \cdot S_2) \]

- factorization between high energy part in matching coefficient and low energy part in wilson loops; power counting;

Eichten, Feinberg 81, Gromes 84, Chen et al. 95 Brambilla, Vairo 99 Pineda, Vairo 00
Spin dependent potentials

\[ V_{LS}, \quad V_{LS}, \quad V_T, \quad V_{SS} \]

Koma Koma Wittig 05, Koma Koma 06

Terrific advance in the data precision with Lüscher multilevel algorithm!

N. B., Martinez, Vairo 2014
Spin dependent potentials

Such data can distinguish different models for the dynamics of low energy QCD e.g. effective string model

N. B., Martinez, Vairo 2014
For states close or above the strong decay threshold the situation is much more complicated.

there is no mass gap between quarkonium and the creation of a heavy-light mesons couple

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Many phenomenological models exist
States made of two heavy and light quarks

- Pairs of heavy-light mesons: $D\bar{D}, B\bar{B}, \ldots$

- Molecular states, i.e. states built on the pair of heavy-light mesons.
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- The usual quarkonium states, built on the static potential, may also give rise to molecular states through the interaction with light hadrons (hadro-quarkonium).
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Maiani, Piccinini, Polosa et al. 2005--
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Vijande, Valcarce, Richard

Having the spectrum of tetraquark potentials, like we have for the gluonic excitations, would allow us to build a plethora of states on each of the tetraquark potentials, many of them developing a width due to decays through pion (or other light hadrons) emission. Diquarks have been recently investigated on the lattice.
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Choosing one of these degrees of freedom and an interaction originates a model for exotics.
$X(3872)$: interpretations

4-quark state with $J^{PC} = 1^{++}$
Two neutral states made of quark rearrangements. Predictions based on the phenomenological Hamiltonian:

\[ H = \sum_{ij} C_{ij} T^a \otimes T^a \sigma \otimes \sigma; \]

Høgassen et al 05

\[ X \sim (c\bar{c})_{S=1}^8 \otimes (q\bar{q})_{S=1}^8 \]
\[ \sim (c\bar{q})_{S=1}^1 \otimes (q\bar{c})_{S=1}^1 + (c\bar{q})_{S=1}^1 \otimes (\bar{q}c)_{S=0}^1 \]

Törnqvist 93, Swanson 04

\[ X \sim (c\bar{q})_{S=0}^1 \otimes (q\bar{c})_{S=1}^1 + (c\bar{q})_{S=1}^1 \otimes (q\bar{c})_{S=0}^1 \]
\[ \sim D \bar{D}^* + D^* \bar{D} \]

This is assumed to be the dominant long-range Fock component; short-range components of the type \((c\bar{c})_{S=1}^1 \otimes (q\bar{q})_{S=1}^1 \)
\[ \sim J/\psi \rho, \omega \text{ are assumed as well.} \]

Maiani et al 04

\[ X \sim (cq)_{S=1}^3 \otimes (\bar{c}\bar{q})_{S=0}^3 + (cq)_{S=0}^3 \otimes (\bar{c}\bar{q})_{S=1}^3 \]

the dynamical assumption (there is no scale separation like in the doubly heavy baryons) is that quark pair cluster in tightly bound color triplet diquarks (see 1-gluon exchange); the difficulty in breaking the system explains the narrow width.

4-quark state with \( J^{PC} = 1^{++} \)

Molecular model

Tetraquark model
In some cases it is possible to develop an EFT owing to special dynamical condition. This happens if the state is sufficiently close to a threshold and if it has $S$-wave coupling to the threshold $\rightarrow$ loosely bound molecule with universal properties.

An example is the $X(3872)$ interpreted as a $D^0 \bar{D}^* \ 0$ or $\bar{D}^0 D^* 0$ molecule. In this case, one may take advantage of the hierarchy of scales:

$\Lambda_{\text{QCD}} \gg m_\pi \gg \frac{m^2_\pi}{M_{D^0}} \approx 10 \text{ MeV} \gg E_{\text{binding}}$

$\approx M_X - (M_{D^* 0} + M_{D^0}) = (0.1 \pm 1.0) \text{ MeV}$

Systems with a short-range interaction and a large scattering length have universal properties that may be exploited: in particular, production and decay amplitudes factorize in a short-range and a long-range part, where the latter depends only on one single parameter, the scattering length. An universal property that fits well with the observed large branching fraction of the $X(3872)$ decaying into $D^0 \bar{D}^0 \pi^0$ is $B(X \rightarrow D^0 \bar{D}^0 \pi^0) \approx B(D^* 0 \rightarrow D^0 \pi^0) \approx 60\%$. 

Pakvasa Suzuki 03, Voloshin 03, Braaten Kusunoki 03
We need a description of states close or above threshold from QCD

Already the case of QCD without light quark is very interesting. The degrees of freedom are heavy quarkonium, heavy hybrids and glueballs.
We need a description of states close or above threshold from QCD

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\[
\begin{align*}
(Q\bar{Q})_1 & \quad (Q\bar{Q})_1 + \text{Glueball} & (Q\bar{Q})_8 G \\
\text{hybrid} & & \end{align*}
\]
We need a description of states close or above threshold from QCD

Already the case of QCD without light quark is very interesting. The degrees of freedom are heavy quarkonium, heavy hybrids and glueballs

**Static Lattice energies**  Juge Kuti Morningstar 2003

Symmetries

Static states classified by symmetry group $D_{\infty h}$

Representations labeled $\Lambda_\eta$

- $\Lambda$ rotational quantum number
  
  $|\hat{n} \cdot K| = 0, 1, 2 \ldots$ corresponds to

  $\Lambda = \Sigma, \Pi, \Delta \ldots$

- $\eta$ eigenvalue of CP:

  $g \overset{\Delta}{=} +1$ (gerade), $u \overset{\Delta}{=} -1$ (ungerade)

- $\sigma$ eigenvalue of reflections

- $\sigma$ label only displayed on $\Sigma$ states (others are degenerate)

- The static energies correspond to the irreducible representations of $D_{\infty h}$

- In general it can be more than one state for each irreducible representation $D_{\infty h}$, usually denoted by primes, e.g. $\Pi_u, \Pi'_u, \Pi''_u \ldots$
even the case without light quark is difficult

static Lattice energies

- $\Sigma_g^+$ is the ground state potential that generates the standard quarkonium states.
- The rest of the static energies correspond to excited gluonic states that generate hybrids.
- The two lowest hybrid static energies are $\Pi_u$ and $\Sigma_u^-$, they are nearly degenerate at short distances.
- The static energies have been computed in quenched lattice QCD, the most recent data by Juge, Kuti, Morningstar, 2002 and Bali and Pineda 2003.
- Quenched and unquenched calculations for $\Sigma_g^+$ and $\Pi_u$ were compared in Bali et al 2000 and good agreement was found below string breaking distance.

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In the limit $r \to 0$ more symmetry: $D_{\infty h} \to O(3) \times C$

- Several $\Lambda^\sigma_\eta$ representations contained in one $J^{PC}$ representation:
- Static energies in these multiplets have same $r \to 0$ limit.

<table>
<thead>
<tr>
<th>$L = 1$</th>
<th>$L = 2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Sigma_g^+$</td>
<td>$r \cdot (E)$</td>
</tr>
<tr>
<td>$\Sigma_g^-'$</td>
<td>$r \times (E)$</td>
</tr>
<tr>
<td>$\Pi_g$</td>
<td>$(r \times D)^i(r \times B)^j + (r \times D)^j(r \times B)^i$</td>
</tr>
<tr>
<td>$\Delta_g$</td>
<td>$(r \cdot D)(r \cdot E)$</td>
</tr>
<tr>
<td>$\Sigma_u^+$</td>
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Gluonic excitations in pNRQCD: one can determine the form of the potential

- At lowest order in the multipole expansion, the singlet decouples while the octet is still coupled to gluons.

- Static hybrids at short distance are called gluelumps and are described by a static adjoint source ($O$) in the presence of a gluonic field ($H$):

$$H(R, r, t) = \text{Tr}\{OH\}$$

$$H = e^{-iT^*E_H}$$

$$E_H = V_o + \frac{i}{T} \ln \langle H^a \left( \frac{T}{2} \right) \phi_{ab}^{\text{adj}} H^b \left( -\frac{T}{2} \right) \rangle$$

$$\langle H^a \left( \frac{T}{2} \right) \phi_{ab}^{\text{adj}} H^b \left( -\frac{T}{2} \right) \rangle_{np} \sim h e^{-iT^*H}$$

$$E_H(r) = V_o(r) + \Lambda_H + O(r^2)$$
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**Diagram notes:**
- octet potential
- gluclump mass
- correction softly breaking the symmetry
We define symmetries and states in NRQCD

We match the energy and the states to pNRQCD at order 1/m in the expansion (but no spin for now) and identify coupled Schroedinger equations for Sigma_u and Pi_u

These are nonperturbative and would require lattice calculations of matrix elements

Lacking the lattice calculation, we identify the potentials with a multipole expansion in pNRQCD, solve the coupled equations and get the lowest ccbar, bbar and bcbar multiplets
The two lowest laying hybrid static energies are $\Pi_u$ and $\Sigma_u^-$.

They are generated by a gluelump with quantum numbers $1^{+-}$ and thus are degenerate at short distances.

The kinetic operator mixes them but not with other multiplets.

Well separated by a gap of $\sim 1 \text{ GeV}$ from the next multiplet with the same CP.
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$\Lambda_H$ and $b_H$ are nonperturbative and should be obtained from lattice calculations.
Lowest energy multiplet $\Sigma_u^- - \Pi_u$

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$E_H(r) = V_O(r) + \Lambda_H + b_H r^2$

$\Lambda_H$ and $b_H$ are nonperturbative and should be obtained from lattice calculations.

$\Lambda_H$

- It is a non-perturbative quantity.
- It depends on the particular operator $H^a$, however it is the same for operators corresponding to different projections of the same gluonic operators.
- The gluelump masses have been determined in the lattice. Foster et al 1999; Bali, Pineda 2004; Marsh Lewis 2014
- At the subtraction scale $\nu_F = 1$ GeV: $\Lambda_{1+}^{RS} = 0.87(15)$ GeV.
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**$b_H$**
- It is a non-perturbative quantity.
- Proportional to $r^2$ due to rotational invariance and the multipole expansion.
- We are going to fix it through a fit to the static energies lattice data.
- Breaks the degeneracy of the potentials.

$$E_H(r) = V_O(r) + \Lambda_H + b_H r^2$$
Lowest energy multiplet $\Sigma_u^- - \Pi_u$

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### Coupled radial Schrödinger equations

Projection vectors in matrix elements allow for two different solutions (coupled or uncoupled) for the $\Sigma_u^-$ and $\Pi_u$ radial wave functions:

1st solution

\[
\begin{bmatrix}
-\frac{1}{2\mu r^2} \partial_r r^2 \partial_r + \frac{1}{2\mu r^2} \left( l(l+1) + 2 \frac{2\sqrt{l(l+1)}}{l(l+1)} \right) + \begin{pmatrix} E^{(0)}_\Sigma & 0 \\ 0 & E^{(0)}_\Pi \end{pmatrix} \\
\end{bmatrix}
\begin{pmatrix}
\psi_\Sigma \\
\psi_\Pi \\
\end{pmatrix}
= \mathcal{E}
\begin{pmatrix}
\psi_\Sigma \\
\psi_\Pi \\
\end{pmatrix}
\]

2nd solution

\[
\begin{bmatrix}
-\frac{1}{2\mu r^2} \partial_r r^2 \partial_r + \frac{l(l+1)}{2\mu r^2} + E^{(0)}_\Pi \\
\end{bmatrix}
\psi_\Pi = \mathcal{E} \psi_\Pi
\]

- energy eigenvalue $\mathcal{E}$ gives hybrid mass: $m_H = m_Q + m_{\bar{Q}} + \mathcal{E}$
- $l(l+1)$ is the eigenvalue of angular momentum $L^2 = (L_Q \bar{Q} + L_g)^2$
- the two solutions correspond to opposite parity states: $(-1)^l$ and $(-1)^{l+1}$
- corresponding eigenvalues under charge conjugation: $(-1)^{l+s}$ and $(-1)^{l+s+1}$
- Schrödinger equations can be solved numerically
The Lambda-doubling effect breaks the degeneracy between opposite parity spin-symmetry multiplets and lowers the mass of the multiplets that get mixed contributions of different static energies.

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$$\left[ -\frac{1}{2\mu r^2} \partial_r r^2 \partial_r + \frac{1}{2\mu r^2} \left( \frac{l(l+1) + 2}{2\sqrt{l(l+1)}} \right) \right] \begin{bmatrix} \psi_{\Sigma} \\ \psi_{\Pi} \end{bmatrix} = \mathcal{E} \begin{bmatrix} \psi_{\Sigma} \\ \psi_{\Pi} \end{bmatrix}$$

2nd solution

$$\left[ -\frac{1}{2\mu r^2} \partial_r r^2 \partial_r + \frac{l(l+1)}{2\mu r^2} + E_{\Pi}^{(0)} \right] \psi_{\Pi} = \mathcal{E} \psi_{\Pi}$$

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- corresponding eigenvalues under charge conjugation: $(-1)^{l+s}$ and $(-1)^{l+s+1}$
- Schrödinger equations can be solved numerically
Lattice data: Bali, Pineda 2004; Juge, Kuti, Morningstar 2003, dashed line $V(0.5)$, solid line $V(0.25)$

$V(0.25)$

1. $r \leq 0.25$ fm: pNRQCD potential.
   - Lattice data fitted for the $r = 0 - 0.25$ fm range with the same energy offsets as in $V(0.5)$.
     
     $$b^{(0.25)}_\Sigma = 1.246 \text{ GeV/fm}^2, \quad b^{(0.25)}_\Pi = 0.000 \text{ GeV/fm}^2. $$

2. $r > 0.25$ fm: phenomenological potential.
   - $V'(r) = \frac{a_1}{r} + \sqrt{a_2 r^2 + a_3} + a_4$.
   - Same energy offsets as in $V(0.25)$.
   - **Constraint:** Continuity up to first derivatives.
Hybrid state masses from $V^{(0.25)}$

Solving the coupled Schrödinger equations we obtain

<table>
<thead>
<tr>
<th>$GeV$</th>
<th>$c\bar{c}$</th>
<th></th>
<th>$b\bar{c}$</th>
<th></th>
<th>$b\bar{b}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$m_H$</td>
<td>$\langle 1/r \rangle$</td>
<td>$E_{kin}$</td>
<td>$P_\Pi$</td>
<td>$m_H$</td>
<td>$\langle 1/r \rangle$</td>
</tr>
<tr>
<td>$H_1$</td>
<td>4.15</td>
<td>0.42</td>
<td>0.16</td>
<td>0.82</td>
<td>7.48</td>
</tr>
<tr>
<td>$H_1'$</td>
<td>4.51</td>
<td>0.34</td>
<td>0.34</td>
<td>0.87</td>
<td>7.76</td>
</tr>
<tr>
<td>$H_2$</td>
<td>4.28</td>
<td>0.28</td>
<td>0.24</td>
<td>1.00</td>
<td>7.58</td>
</tr>
<tr>
<td>$H_2'$</td>
<td>4.67</td>
<td>0.25</td>
<td>0.42</td>
<td>1.00</td>
<td>7.89</td>
</tr>
<tr>
<td>$H_3$</td>
<td>4.59</td>
<td>0.32</td>
<td>0.32</td>
<td>0.00</td>
<td>7.85</td>
</tr>
<tr>
<td>$H_4$</td>
<td>4.37</td>
<td>0.28</td>
<td>0.27</td>
<td>0.83</td>
<td>7.65</td>
</tr>
<tr>
<td>$H_5$</td>
<td>4.48</td>
<td>0.23</td>
<td>0.33</td>
<td>1.00</td>
<td>7.73</td>
</tr>
<tr>
<td>$H_6$</td>
<td>4.57</td>
<td>0.22</td>
<td>0.37</td>
<td>0.85</td>
<td>7.82</td>
</tr>
<tr>
<td>$H_7$</td>
<td>4.67</td>
<td>0.19</td>
<td>0.43</td>
<td>1.00</td>
<td>7.89</td>
</tr>
</tbody>
</table>

**Consistency test:**

1. The multipole expansion requires $\langle 1/r \rangle > E_{kin}$.

As expected our approach works better in bottomonium than charmonium.

**Spin symmetry multiplets**

<table>
<thead>
<tr>
<th>$H_i$</th>
<th>{spin, $j$, $J$}</th>
<th>$\Sigma_u^-, \Pi_u$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$H_1$</td>
<td>${1^{--}, (0, 1, 2)^{--}}$</td>
<td>$\Sigma_u^-, \Pi_u$</td>
</tr>
<tr>
<td>$H_2$</td>
<td>${1^{++}, (0, 1, 2)^{+-}}$</td>
<td>$\Pi_u$</td>
</tr>
<tr>
<td>$H_3$</td>
<td>${0^{++}, 1^{-+}}$</td>
<td>$\Sigma_u^-$</td>
</tr>
<tr>
<td>$H_4$</td>
<td>${2^{++}, (1, 2, 3)^{+-}}$</td>
<td>$\Sigma_u^-, \Pi_u$</td>
</tr>
<tr>
<td>$H_5$</td>
<td>${2^{--}, (1, 2, 3)^{+-}}$</td>
<td>$\Pi_u$</td>
</tr>
<tr>
<td>$H_6$</td>
<td>${3^{--}, (2, 3, 4)^{+-}}$</td>
<td>$\Sigma_u, \Pi_u$</td>
</tr>
<tr>
<td>$H_7$</td>
<td>${3^{++}, (2, 3, 4)^{+-}}$</td>
<td>$\Pi_u$</td>
</tr>
</tbody>
</table>

Berwein, N.B., Tarrus, Vairo arXiv:1510.04299
Identification with experimental states

Most of the candidates have $1^{−−}$ or $0^{++}/2^{++}$ since the main observation channels are production by $e^+e^−$ or $\gamma\gamma$ annihilation respectively.

- **Charmonium states** (Belle, CDF, BESIII, Babar):

  ![Graph showing mass distribution of charmonium states](image)

  - Error bands come from the uncertainty on the gluclump mass.

  - **Bottomonium states**: $Y_b(10890)[1^{−−}]$, $m = 10.8884 \pm 3.0$ (Belle). Possible $H_1$ candidate, $m_{H_1} = 10.79 \pm 0.15$.

  **However**, except for $Y(4220)$, all other candidates observed decay modes violate Heavy Quark Spin Symmetry.

Berwein, N.B., Tarrus, Vairo arXiv:1510.04299
Comparison to direct lattice calculations

FIG. 5. Comparison of the results from direct lattice computations of the masses for charmonium hybrids [48] with our results using the $V^{(0.25)}$ potential. The direct lattice mass predictions are plotted in solid lines with error bars corresponding the mass uncertainties. Our results for the $H_1$, $H_2$, $H_3$, and $H_4$ multiplets have been plotted in error bands corresponding to the gluelump mass uncertainty of ±0.15 GeV.

We observe the same Lambda-doubling pattern in lattice calculations, multiplets that receive mixed contributions from Sigma_u and Pi_u have lower masses then those that remain pure Pi_u states.
Comparison to direct lattice calculations

We observe the same Lambda-doubling pattern in lattice calculations, multiplets that receive mixed contributions from Sigma_u and Pi_u have lower masses than those that remain pure Pi_u states.

We have considered the mass splittings between hybrids with our results using the TUM-EFT 48/14.

Support the result of the pNRQCD and BO approaches that the hybrid states appear in three distinct multiplets (H_2, H_3, and H_4) as compared to the constituent gluon picture, where they are assumed to form one supermultiplet together (cf. also the discussion in [34]).

L. Liu et al. [Hadron Spectrum Collaboration], JHEP 1207, 126 (2012) [arXiv:1204.5425]

The ground states of the corresponding to hybrid operators. They identified three hybrid states, which couple to the degenerate multiplets, that coexist in the mass spectrum. The masses of the degenerate multiplets, which couple to the degenerate multiplets, are given in units of GeV.

We have compared with the spin averages from the direct lattice calculations, multiplets that receive mixed contributions from Sigma_u and Pi_u have lower masses than those that remain pure Pi_u states.

The direct lattice mass predictions are compared with the spin averages from the direct lattice calculations, multiplets that receive mixed contributions from Sigma_u and Pi_u have lower masses than those that remain pure Pi_u states.
Comparison with direct lattice computations

**Bottomonium sector**

- Calculations done by Juge, Kuti, Morningstar 1999 and Liao, Manke 2002 using quenched lattice QCD.
- Juge, Kuti, Morningstar 1999 included no spin or relativistic effects.
- Liao, Manke 2002 calculations are fully relativistic.

![Graph showing mass splittings between bottomonium states](image)

Error bands take into account the uncertainty on the gluelump mass ±0.15 GeV

<table>
<thead>
<tr>
<th>Split (GeV)</th>
<th>JKM</th>
<th>$\sqrt{V(0.25)}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\delta m_{H_2-H_1}$</td>
<td>0.04</td>
<td>0.05</td>
</tr>
<tr>
<td>$\delta m_{H_3-H_1}$</td>
<td>0.33</td>
<td>0.27</td>
</tr>
<tr>
<td>$\delta m_{H_3-H_2}$</td>
<td>0.30</td>
<td>0.22</td>
</tr>
<tr>
<td>$\delta m_{H'_1-H_1}$</td>
<td>0.42</td>
<td>0.19</td>
</tr>
</tbody>
</table>

- Our masses are 0.15 – 0.25 GeV lower except the for the $H'_1$ multiplet, which is larger by 0.36 GeV.
- Good agreement with the mass gaps between multiplets, in particular the $\Lambda$-doubling effect ($\delta m_{H_2-H_1}$).
Comparison to sum rules

FIG. 7. Comparison of the mass predictions for charmonium hybrids in the upper figure and for bottomonium hybrids in the lower figure, obtained using QCD sum rules [68], with our results using the $V^{(0.25)}$ potential. The solid lines correspond to the QCD sum rules masses with error bars corresponding to their uncertainties. Our results for the $H_1$, $H_2$, $H_3$, and $H_4$ multiplets have been plotted in error bands corresponding to the gluelump mass uncertainty of ±0.15 GeV.
Conclusions

Quarkonium is a golden system to study strong interactions. For states below threshold, non-relativistic EFTs provide a systematic tool to investigate a wide range of observables in the realm of QCD.

For states close or above the strong decay threshold, the situation is much more complicated. Many degrees of freedom show up, and the absence of a clear systematic is an obstacle to a universal picture.

We have presented results obtained for the hybrid masses in pNRQCD that show a very rich structure of multiplets.

We are working to obtain decays and transitions, to obtain a version of strongly coupled pNRQCD including hybrids and to include light quark degrees of freedom.

Lattice input needed: gluelump masses, tetra quark potentials

- Fundamental experimental input (like confirmation, quantum numbers, widths and masses) is still crucially missing for some of these states.
<table>
<thead>
<tr>
<th>State</th>
<th>$M$ (MeV)</th>
<th>$\Gamma$ (MeV)</th>
<th>$J^{PC}$</th>
<th>Decay modes</th>
<th>1st observation</th>
</tr>
</thead>
<tbody>
<tr>
<td>$X(3823)$</td>
<td>$3823.1 \pm 1.9$</td>
<td>$&lt; 24$</td>
<td>$?^{2-}$</td>
<td>$\chi_{c1}\gamma$</td>
<td>Belle 2013</td>
</tr>
<tr>
<td>$X(3872)$</td>
<td>$3871.68 \pm 0.17$</td>
<td>$&lt; 1.2$</td>
<td>$1^{++}$</td>
<td>$J/\psi\pi^+\pi^-, J/\psi\pi^+\pi^-\pi^0$</td>
<td>Belle 2003</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$D^0\bar{D}^0\pi^0, D^0\bar{D}^0\gamma$</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$J/\psi\gamma, \psi(2S)\gamma$</td>
<td></td>
</tr>
<tr>
<td>$X(3915)$</td>
<td>$3917.5 \pm 1.9$</td>
<td>$20 \pm 5$</td>
<td>$0^{++}$</td>
<td>$J/\psi\omega$,</td>
<td>Belle 2004</td>
</tr>
<tr>
<td>$\chi_{c2}(2P)$</td>
<td>$3927.2 \pm 2.6$</td>
<td>$24 \pm 6$</td>
<td>$2^{++}$</td>
<td>$D\bar{D}$</td>
<td>Belle 2005</td>
</tr>
<tr>
<td>$X(3940)$</td>
<td>$3942^{+9}_{-8}$</td>
<td>$37^{+27}_{-17}$</td>
<td>$?^{+}$</td>
<td>$D^<em>\bar{D}, D\bar{D}^</em>$</td>
<td>Belle 2007</td>
</tr>
<tr>
<td>$G(3900)$</td>
<td>$3943 \pm 21$</td>
<td>$52 \pm 11$</td>
<td>$1^{--}$</td>
<td>$D\bar{D}$</td>
<td>Babar 2007</td>
</tr>
<tr>
<td>$Y(4008)$</td>
<td>$4008^{+121}_{-49}$</td>
<td>$226 \pm 97$</td>
<td>$1^{--}$</td>
<td>$J/\psi\pi^+\pi^-$</td>
<td>Belle 2007</td>
</tr>
<tr>
<td>$Y(4140)$</td>
<td>$4144.5 \pm 2.6$</td>
<td>$15^{+11}_{-7}$</td>
<td>$?^{+}$</td>
<td>$J/\psi\phi$</td>
<td>CDF 2009</td>
</tr>
<tr>
<td>$X(4160)$</td>
<td>$4156^{+29}_{-25}$</td>
<td>$139^{+113}_{-65}$</td>
<td>$?^{+}$</td>
<td>$D^<em>\bar{D}^</em>$</td>
<td>Belle 2007</td>
</tr>
<tr>
<td>$Y(4220)$</td>
<td>$4216 \pm 7$</td>
<td>$39 \pm 17$</td>
<td>$1^{--}$</td>
<td>$h_c(1P)\pi^+\pi^-$</td>
<td>BESIII 2013</td>
</tr>
<tr>
<td>$Y(4230)$</td>
<td>$4230 \pm 14$</td>
<td>$38 \pm 14$</td>
<td>$1^{--}$</td>
<td>$\chi_{c0}\omega$,</td>
<td>BESIII 2014</td>
</tr>
<tr>
<td>$Y(4260)$</td>
<td>$4263^{+8}_{-9}$</td>
<td>$95 \pm 14$</td>
<td>$1^{--}$</td>
<td>$J/\psi\pi^+\pi^-, J/\psi\pi^0\pi^0$</td>
<td>Babar 2005</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$Z_c(3900)\pi$,</td>
<td></td>
</tr>
<tr>
<td>$Y(4274)$</td>
<td>$4293 \pm 20$</td>
<td>$35 \pm 16$</td>
<td>$?^{+}$</td>
<td>$J/\psi\phi$</td>
<td>CDF 2010</td>
</tr>
<tr>
<td>$X(4350)$</td>
<td>$4350.6^{+4.6}_{-5.1}$</td>
<td>$13.3^{+18.4}_{-10.0}$</td>
<td>$0/2^{++}$</td>
<td>$J/\psi\phi$,</td>
<td>Belle 2009</td>
</tr>
<tr>
<td>$Y(4360)$</td>
<td>$4354 \pm 11$</td>
<td>$78 \pm 16$</td>
<td>$1^{--}$</td>
<td>$\psi(2S)\pi^+\pi^-$</td>
<td>Babar 2007</td>
</tr>
<tr>
<td>$X(4630)$</td>
<td>$4634^{+9}_{-11}$</td>
<td>$92^{+41}_{-32}$</td>
<td>$1^{--}$</td>
<td>$\Lambda_c^+\Lambda_c^-, \Lambda_c^-\Lambda_c^-$</td>
<td>Babar 2007</td>
</tr>
<tr>
<td>$Y(4660)$</td>
<td>$4665 \pm 10$</td>
<td>$53 \pm 14$</td>
<td>$1^{--}$</td>
<td>$\psi(2S)\pi^+\pi^-$</td>
<td>Babar 2007</td>
</tr>
<tr>
<td>$Y_b(10890)$</td>
<td>$10888.4 \pm 3.0$</td>
<td>$30.7^{+8.9}_{-7.7}$</td>
<td>$1^{--}$</td>
<td>$\Upsilon(nS)\pi^+\pi^-$</td>
<td>Belle 2010</td>
</tr>
</tbody>
</table>

TABLE V: Neutral mesons above open flavor threshold excluding isospin partners of charged states.
Results I: Comparison to BO approximation

For Hybrid multiplets:

<table>
<thead>
<tr>
<th></th>
<th>$l$</th>
<th>$J^{PC}$ (s = 0, s = 1)</th>
<th>$E_{n}^{(0)}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$H_1$</td>
<td>1</td>
<td>(1^{--}, (0, 1, 2)^{--})</td>
<td>$\Sigma_\bar{u}, \Pi_u$</td>
</tr>
<tr>
<td>$H_2$</td>
<td>1</td>
<td>(1^{++}, (0, 1, 2)^{+-})</td>
<td>$\Pi_u$</td>
</tr>
<tr>
<td>$H_3$</td>
<td>0</td>
<td>(0^{++}, 1^{+-})</td>
<td>$\Sigma_{\bar{u}}$</td>
</tr>
<tr>
<td>$H_4$</td>
<td>2</td>
<td>(2^{++}, (1, 2, 3)^{+-})</td>
<td>$\Sigma_{\bar{u}}, \Pi_u$</td>
</tr>
<tr>
<td>$H_5$</td>
<td>2</td>
<td>(2^{--}, (1, 2, 3)^{--})</td>
<td>$\Pi_u$</td>
</tr>
</tbody>
</table>

- no distinction between opposite parity states in BO
- mixed states lie lower than pure
- discrepancy for $H_2$, $H_3$ and $H_5$ due to different potential fits

Braaten, Langmack and Smith 2014
Comparison with direct lattice computations

Charmonium sector

- Calculations done by the Hadron Spectrum Collaboration using unquenched lattice QCD with a pion mass of 400 MeV. Liu et al. 2012
- They worked in the constituent gluon picture, which consider the multiplets $H_2$, $H_3$, $H_4$ as part of the same multiplet.
- Their results are given with the $\eta_c$ mass subtracted.

Error bands take into account the uncertainty on the gluelump mass ±0.15 GeV

<table>
<thead>
<tr>
<th>Split (GeV)</th>
<th>Liu</th>
<th>$V^{(0.25)}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\delta m_{H_2-H_1}$</td>
<td>0.10</td>
<td>0.13</td>
</tr>
<tr>
<td>$\delta m_{H_4-H_1}$</td>
<td>0.24</td>
<td>0.22</td>
</tr>
<tr>
<td>$\delta m_{H_4-H_2}$</td>
<td>0.13</td>
<td>0.09</td>
</tr>
<tr>
<td>$\delta m_{H_3-H_1}$</td>
<td>0.20</td>
<td>0.44</td>
</tr>
<tr>
<td>$\delta m_{H_3-H_2}$</td>
<td>0.09</td>
<td>0.31</td>
</tr>
</tbody>
</table>

- Our masses are 0.1 – 0.14 GeV lower except the for the $H_3$ multiplet, which is the only one dominated by $\Sigma_u^-$. 
- Good agreement with the mass gaps between multiplets, in particular the $\Lambda$-doubling effect ($\delta m_{H_2-H_1}$).
FIG. 7. Comparison of the mass predictions for charmonium hybrids in the upper figure and for bottomonium hybrids in the lower figure, obtained using QCD sum rules [68], with our results using the $V^{(0.25)}$ potential. The solid lines correspond to the QCD sum rules masses with error bars corresponding to their uncertainties. Our results for the $H_1$, $H_2$, $H_3$, and $H_4$ multiplets have been plotted in error bands corresponding to the gluebump mass uncertainty of $\pm 0.15$ GeV.
We have computed the heavy hybrid masses using a QCD analog of the Born-Oppenheimer approximation including the Λ–doubling terms by using coupled Schrödinger equations.

The static energies have been obtained combining pNRQCD for short distances and lattice data for long distances.

A large set of masses for spin symmetry multiplets for $c\bar{c}$, $b\bar{c}$ and $b\bar{b}$ has been obtained.

Λ–doubling effect lowers the mass of the multiplets generated by a mix of static energies, the same pattern is observed in direct lattice calculations and QCD sum rules.

Mass gaps between multiplets in good agreement with direct lattice computations, but the absolute values are shifted.

Several experimental candidates for Charmonium hybrids belonging to the $H_1$, $H_2$, $H_4$ and $H_1'$ multiplets.

One experimental candidate to the bottomonium $H_1$ multiplet.