23 February 2016

Master Integrals technique for multi-loop computations: the Differential Equations approach.

HiggsTools Journal Club

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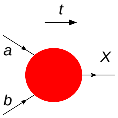
[a] *Nikhef*,

Perturbative QFT: higher-order corrections

Interactions in the SM can be described by means of Perturbative QFT.



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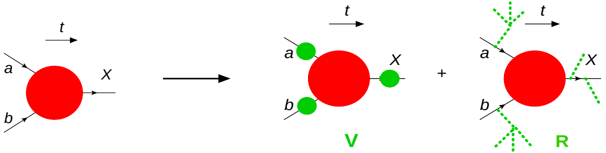

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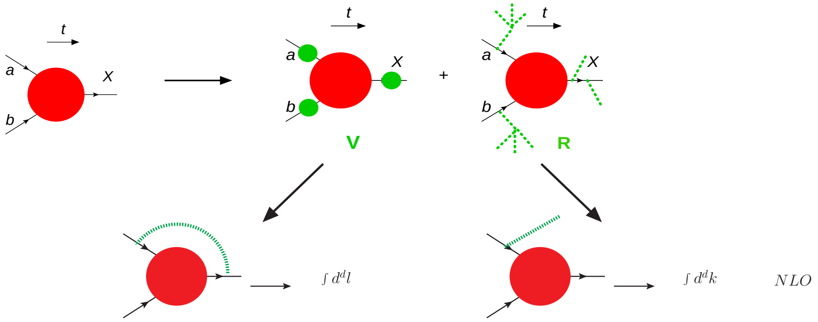

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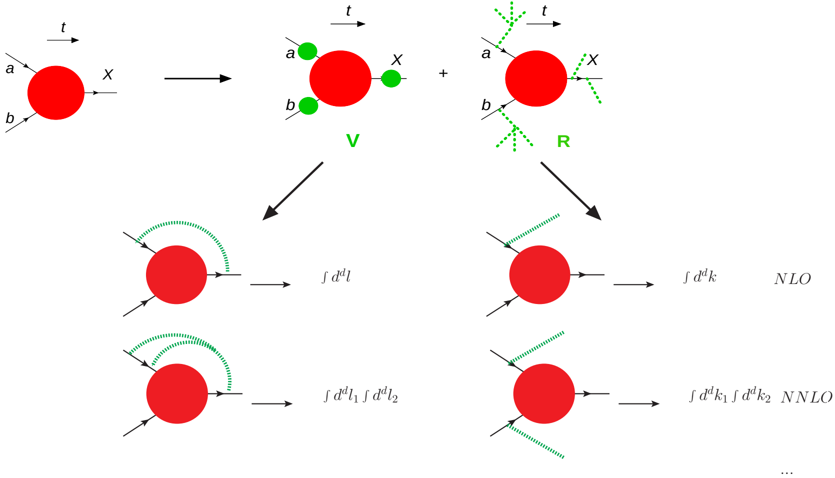
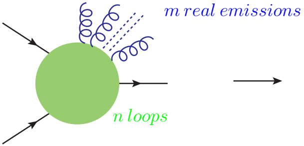


The amount of integrals over real and virtual radiation becomes huge very quickly with increasing number of loops/legs.

From Feynman Rules to Feynman Integrals

1)



$$\longrightarrow \quad \propto \int d^dl_1...d^dl_n A\left(l_1^{\mu_1},..,l_n^{\mu_n},k_1^{\nu_1},..,k_m^{\nu_m},p_a^{\rho},p_b^{\sigma}\right)$$

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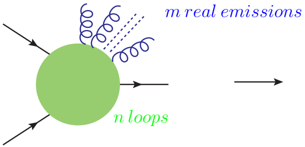

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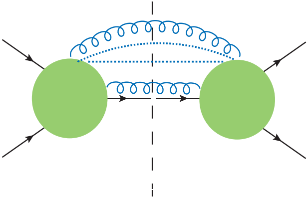

$$\longrightarrow \quad \propto \int d^dl_1...d^dl_n \int d^dk_1...d^dk_m |A|^2 \left(l_i \cdot k_j, l_i \cdot p_j, k_i \cdot p_j\right)$$

$$\times \prod_i \delta(k_i^2 - m_i^2) \times \delta((p_a + p_b - k_1 - ... - k_m)^2 - m_X^2)$$

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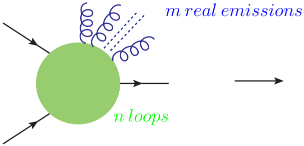

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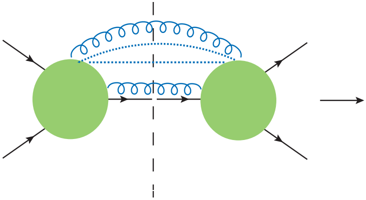

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3) $\sigma \propto \int d^dl_1...d^dl_n \int d^dk_1...d^dk_m |A|^2\left(l_i\cdot k_j, l_i\cdot p_j, k_i\cdot p_j\right)\times \prod_i \frac{1}{(k_i^2-m_i^2)_c}\times\frac{1}{((p_a+p_b-k_1-...-k_m)^2-m_X^2)_c}$

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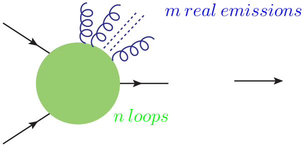

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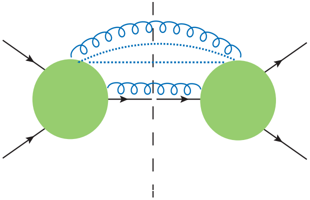

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4) $\sigma(\vec{s}) \propto \sum_i c_i(\vec{s})\, \mathcal{I}\left(\vec{s}, \{a_1^i, \ldots, a_{N_P}^i\}\right)$, $\quad \vec{s} = \{p_a \cdot p_b, m_a^2, m_b^2, m_X^2, p_j \cdot p_X\}$

$$\mathcal{I}\left(\vec{s}, \{a_1, \ldots, a_{N_P}\}\right) = \int d^d l_1 \ldots d^d l_n \int d^d k_1 \ldots d^d k_m \frac{1}{P_1^{a_1} \times \ldots \times P_{N_P}^{a_{N_P}}}, \qquad E_j = E_j(l_i \cdot k_j, l_i \cdot p_j, k_i \cdot p_j, m_{int_i}^2)$$

Master Integrals : Why and How?

In the calculation of multi-loop/multi-leg processes, the bottleneck is the huge amount of loop- and phase space- integrals.

$$\Downarrow$$

MASTER INTEGRALS TECHNIQUE

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Why?

It applies to Loop and Phase Space integrals
Part of the computation is automatized

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MASTER INTEGRALS TECHNIQUE

Why?

It applies to Loop and Phase Space integrals.

Part of the computation is automatized.

How?

1) Reduction of scalar matrix elements to a set of Master Integrals. $\rightarrow$ `FIRE`

2) Explicit evaluation of Masters *Feynman Calculus*.

3) "Cosmetics" and Renormalization of final expressions.

Integration by Parts Identities

$$\sigma(\vec{s}) \propto \sum_{j}^{N_{\mathcal{I}}} c_j(\vec{s}) \; \mathcal{I}\left(\vec{s}; a_1^j, \ldots, a_{n_P}^j\right) \overset{?}{\rightarrow} \begin{cases} \sigma \propto \sum_{j'}^{N'_{\mathcal{I}}} c_{j'} \; \mathcal{I}\left(a_1^{j'}, \ldots, a_{n_P}^{j'}\right), N' \ll N \\ \downarrow \\ \mathcal{I}\left(a_1^{j_1}, \ldots, a_{n_P}^{j_1}\right) + \mathcal{I}\left(a_1^{j_2}, \ldots, a_{n_P}^{j_2}\right) + \ldots \overset{?}{=} 0 \end{cases}$$

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IBPs for Feynman integrals: $\int d^d k_1 \ldots \int^d k_L \left[\frac{\partial}{\partial k_j} \cdot k_i \frac{1}{(k_1^2 - m_1^2)^{a_1} \ldots (k_L^2 - m_L^{a_L})}\right] = 0$

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2) Relation between integrals of the family: $I(a) = \frac{d - 2a + 2}{2(a-1)m^2} I(a-1)$

3) Solve the recursion: $I(a) = \frac{(-1)^a (1 - d/2)_{(a-1)}}{(a-1)!(m^2)^{a-1}} I_1, \quad (x)_a \equiv \frac{\Gamma(x+a)}{\Gamma(x)} = x(x+1)\dots(x+a-1)$

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$$\sigma \propto \sum_m^{N_{\mathcal{M}}} c_m\ \mathcal{M}\left(a_1^m, \ldots, a_{n_P}^m\right), N_{\mathcal{M}} \ll N$$

Solving the Masters: possible strategies

Set of MIs $\{\mathcal{M}_i\}$ is a **basis in the space of Feynman integrals** describing a certain process.

$\Rightarrow$ They cannot be further reduced and **must be solved explicitely**.

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- α/Feynman Parameters: $E(p, m^2) = \frac{i}{(p^2-m^2+i0)^{a_l}} \rightarrow \int_0^\infty d\alpha_l \; \alpha_l^{a_l-1} e^{i(p^2-m^2)a_l}$

- Mellin-Barnes: $\frac{1}{(X+Y)^\lambda} \rightarrow \frac{1}{\Gamma(\lambda)} \int_{-\infty}^{\infty} dz \frac{Y^z}{X^{\lambda+z}} \Gamma(\lambda+z)\Gamma(-z)$

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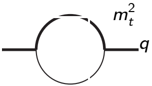
- ~~Mellin-Barnes: $\frac{1}{(X+Y)^\lambda} \rightarrow \frac{1}{\Gamma(\lambda)} \int_{-\infty}^{\infty} dz \, \frac{Y^z}{X^{\lambda+z}} \Gamma(\lambda + z) \Gamma(-z)$~~
- Differential Equations:
  1. MIs are functions of the external scales $\vec{s}$ governing the process
  $$\vec{M} = (\ldots M_i, \ldots), \Rightarrow \vec{M} = \vec{M}(\vec{s})$$
  2. They happen to satisfy coupled systems of (P)DEs, which set their dependence on such scales.

Generation of DE for a generic Feynman integral: 1-loop example

$$F^{ad}(x,1,1,\epsilon) = m^{2\epsilon} \int \frac{d^{4-2\epsilon}k}{(k^2-m^2)(q-k)^2} =$$



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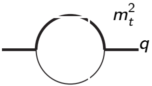

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$$\begin{aligned}
\mathbf{\frac{d}{dx}F^{ad}(x,1,1,\epsilon)} &= -\frac{m^2}{x}\frac{d}{dm^2}\left[(m^2)^{\epsilon}\int \frac{d^{4-2\epsilon}k}{(k^2-m^2)(q-k)^2}\right] \\
&= -\frac{1}{x}\left[\epsilon(m^2)^{\epsilon}\int \frac{d^{4-2\epsilon}k}{(k^2-m^2)(q-k)^2} + (m^2)^{1+\epsilon}\int \frac{d^{4-2\epsilon}k}{(k^2-m^2)^2(q-k)^2}\right] \\
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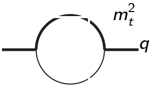

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$$\frac{\mathbf{d}}{\mathbf{dx}}\mathbf{F^{ad}}(\mathbf{x},\mathbf{1},\mathbf{1},\epsilon) = -\frac{m^2}{x}\frac{d}{dm^2}\left[(m^2)^\epsilon \int \frac{d^{4-2\epsilon}k}{(k^2-m^2)(q-k)^2}\right]$$

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DE $\quad \frac{d}{dx}F^{ad}(x,1,1,\epsilon) = -\frac{1}{x}\left[F^{ad}(x,1,1,\epsilon)\left(\epsilon + \frac{1-2\epsilon}{1-x}\right) - \frac{(1-\epsilon)}{1-x}F^{ad}(x,1,0,\epsilon)\right]$

..solvable in a bottom-up approach!

(Partial) Differential Equations for a set of Master Integrals

Given a set of masters $\vec{M}$ describing a certain process governed by $\vec{s}$

1) $\vec{M}_{\mathcal{T}} = \{M^{(1)}, ..., M^{(i)}, ...M^{(m)}\}, \qquad M^{(i)} = M^{(i)}(\vec{s};\epsilon), \qquad \vec{s} = \{s_1, ..., s_n\}.$

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we can write a system of coupled (partial) D.E. for them

$$\begin{cases} \frac{\partial}{\partial s_i} M^{(1)}(\vec{s};\epsilon) = d^j_{i1}(\vec{s};\epsilon)\,M^{(1)}(\vec{s};\epsilon) + \sum_{r=1,r\neq 1}^m d^r_{i1}(\vec{s};\epsilon)\,M^{(r)}(\vec{s};\epsilon) \\ \vdots \\ \frac{\partial}{\partial s_i} M^{(j)}(\vec{s};\epsilon) = d^j_{ij}(\vec{s};\epsilon)\,M^{(j)}(\vec{s};\epsilon) + \sum_{r=1,r\neq j}^m d^r_{ij}(\vec{s};\epsilon)\,M^{(r)}(\vec{s};\epsilon) \qquad , \quad \forall s_i \in \vec{s} \\ \vdots \\ \frac{\partial}{\partial s_i} M^{(m)}(\vec{s};\epsilon) = d^j_{im}(\vec{s};\epsilon)\,M^{(m)}(\vec{s};\epsilon) + \sum_{r=1,r\neq j}^m d^r_{im}(\vec{s};\epsilon)\,M^{(m)}(\vec{s};\epsilon) \end{cases}$$

Differential Equations: expansion around $\epsilon \to 0$

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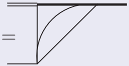
Example from $[b + W^* \to t]_{2L}$

$M_1 =$  $=?$

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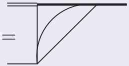
$M_1 =$  $=?$ D.E. $\frac{d}{dy} M_1(y) = \frac{4(3-(4-2\epsilon)(1-y)-4y)M_1(y)}{4y(1+y)} + \frac{(-8+3(4-2\epsilon))\mathbf{M_2(y)}}{4y(1+y)}$

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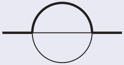
$\mathbf{M_2(y)} =$  $= \frac{m_2^{(-2)}(y)}{\epsilon^2} + \frac{m_2^{(1)}(y)}{\epsilon} + m_2^0(y) + m_2^{(1)}\epsilon(y) + m_2^{(2)}\epsilon^2(y) + \mathcal{O}(\epsilon^3)$

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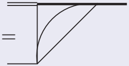
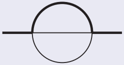
$M_1 =$  $=?$ D.E. $\frac{d}{dy} M_1(y) = \frac{4(3-(4-2\epsilon)(1-y)-4y)M_1(y)}{4y(1+y)} + \frac{(-8+3(4-2\epsilon))\mathbf{M_2(y)}}{4y(1+y)}$ ← known!

$\mathbf{M_2(y)} =$  $= \frac{m_2^{(-2)}(y)}{\epsilon^2} + \frac{m_2^{(1)}(y)}{\epsilon} + m_2^0(y) + m_2^{(1)}\epsilon(y) + m_2^{(2)}\epsilon^2(y) + \mathcal{O}(\epsilon^3)$

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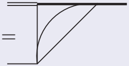
$M_1 =$  $= ?$

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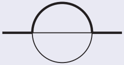
$\mathbf{M_2(y)} =$  $= \frac{m_2^{(-2)}(y)}{\epsilon^2} + \frac{m_2^{(1)}(y)}{\epsilon} + m_2^0(y) + m_2^{(1)}\epsilon(y) + m_2^{(2)}\epsilon^2(y) + \mathcal{O}(\epsilon^3)$

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$\mathcal{O}(\epsilon^0)$:...

... integrable with appropriate boundary conditions!

Differential Equations: Problematics and Bottlenecks

Masters can be computed as Laurent series in ϵ

$$M_i(\vec{s};\epsilon) = \sum_{k=n_0}^{n} \epsilon^k M_i^{(k)}(\vec{s}) + \mathcal{O}(\epsilon^{n+1})$$

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CANONICAL BASIS

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EXPANSION BY REGION

Canonical basis: philosophy (arXiV:1304.1806, J.Henn)

a) Basis $\vec{M}$ of MIs can be freely chosen.

b) Quality of D.E. can sensibly change with the basis choice.

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Solution can be written in terms of **iterated integrals**,

and under certain conditions in terms of **Multiple Polylogarithms**.

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$$\mathcal{O}(\epsilon^{-4}) : \quad \begin{cases} \partial_z m_1^{(-4)} = 0 \\ \partial_z m_2^{(-4)} = 0 \end{cases} \Rightarrow \begin{cases} m_1^{(-4)} = cost \\ m_2^{(-4)} = cost \end{cases}$$

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and so on ...$\Rightarrow$ The system decouples order by order in ϵ

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- $\Rightarrow$ ad-hoc techniques are required
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A powerful technique for analytic Evaluation is represented by Differential Equations.

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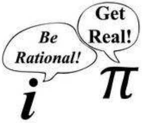

... thank you for your attention.

Back-up slides

NUMERICAL METHOD for Feynman Integrals

Sector Decomposition - Philosophy

- Counterterms are needed to integrate numerically divergent integrals
- How to construct them when there are overlapping singularities?

$$I = \int_0^1 dx \int_0^1 dy\ x^{-1-a\epsilon} y^{-b\epsilon} (x + (1-x)y)^{-1} =$$

$$= \int_0^1 dx \int_0^1 dy x^{-1-a\epsilon} y^{-b\epsilon} (x + (1-x)y)^{-1} [\Theta(x-y) + \Theta(y-x)] =$$

$$= \int_0^1 dx\ x^{-1-(a+b)\epsilon} \int_0^1 dt t^{-b\epsilon} (1 + (1-x)t)^{-1} +$$

$$+ \int_0^1 dy y^{-1-(a+b)\epsilon} \int_0^1 dt\ t^{-1-a\epsilon} (1 + (1-y)t)^{-1} (\text{ Ask Stephen for more info.. })$$